and define a distance $\delta(x,y)$ for every two octogonally rational numbers x and y of $\mathcal{J}$ such that $x<y$. We set $\delta(y,x)=\delta(x,y)$ and $\delta(x,x)=0$. If two octogonally rational numbers differ by less than $1/8^n$, their distances differ by less than $1/2^n$. Hence it is easy to extend the definition of $\delta(x,y)$ to any two numbers x and y of $\mathcal{J}$. The length of each end-to-end polygon is 1. The absolute length of $\mathcal{J}$ is unbounded [7]).

[7]) A slight modification of the above construction leads to an arc having *the absolute length* ∞ *and the length* 0. We divide the interval $[0,1]$ into four instead of eight equal parts and define the distances from 0 to $\frac{1}{4}$, and from $\frac{1}{2}$ to $\frac{3}{4}$ to be $\frac{1}{2}$, and the distances from $\frac{1}{4}$ to $\frac{1}{2}$ and from $\frac{3}{4}$ to 1 to be $-\frac{1}{4}$. Iteration of this procedure leads to the indicated result.

Mr. Sheldon L. Levy pointed out that the original example (with divisions into eight parts) can be simplified. It is sufficient to divide the interval $[0,1]$ into three equal parts and to define the distance from 0 to $\frac{1}{3}$ as $\frac{2}{3}$, the distance from $\frac{1}{3}$ to $\frac{2}{3}$ as $-\frac{1}{3}$, and the distance from $\frac{2}{3}$ to 1 as $\frac{2}{3}$. Iteration of this procedure eads to an arc whose absolute length is ∞ and whose length is 1.

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Group invariant continua.

By

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1. We denote by X a compact (=bicompact) connected Hausdorff space. Let Z be a group which is also a topological space. It is not required that Z be a topological group. Let f be a map (=continuous transformation) of $Z\times X$ into X. Writing Z multiplicatively it is assumed that

$$f(e,x)=x \quad \text{for each } x\in X, \ e \text{ the neutral element}$$

and

$$f(z,f(z',x))=f(zz',x) \quad \text{for each } x\in X \text{ and } z,z'\in Z.$$

On setting $z(x)=f(z,x)$ it may easily be verified that z is a homeomorphism of X onto X and that z^{-1} is the inverse of z as a transformation. Accordingly we shall say (somewhat incorrectly) that *Z acts as a group of homeomorphisms on X.*

If A is any subset of X we define $Z(A)$ as the union of all the sets $z(A)$, $z\in Z$. It is an immediate consequence that

$$Z(A)=\bigcup_{z\in Z} z(A)=\bigcup_{a\in A} Z(a).$$

A subset A of X will be termed *Z-invariant* if $Z(A)=A$ or, equivalently $z(A)=A$ for each z in Z. Clearly X is Z-invariant.

In this note we prove among other things the

Theorem. *Let X be metric and locally connected. If Z is abelian then there exists a Z-invariant cyclic element.*

The first result of this character was proved by W. L. Ayres [1] who assumed that Z was generated by a single map, i. e., that Z was cyclic. For other results of this type see [6], Chap. XII, and [5] and the reference given here to G. E. Schweigert. In addition to extending this result from the case in which Z is cyclic to the case in which Z is merely abelian we remove the restrictions that X be metric and locally connected.

We remark finally that this proposition may fail if Z is not commutative. Let X be the closed unit interval and T the multiplicative group of all positive numbers. Let $t(x)=x^t$. Also write $u(x)=1-x$ so that u is a reflection of X in its midpoint. If Z is the group generated by u and T, one readily confirms the statement that no point of X is fixed under every element of Z.

2. We first state

Theorem 1. *If X is a compact connected Hausdorff space and Z is an abelian group of homeomorphisms acting on X then there is a Z-invariant subcontinuum of X having no cutpoint.*

Since X is a Z-invariant continuum it follows from the maximality theorem of Hausdorff (see, e. g., [4]) that there exists a collection G of subcontinua of X such that

(a) Each continuum in G is Z-invariant.

(b) For any pair K, K' of elements of G either $K\subset K'$ or $K'\subset K$.

(c) If G_0 is any other collection of continuua satisfying (a) and (b) and containing G then $G=G_0$.

If H is the intersection of all the elements of G it may be verified that H is a minimal Z-invariant subcontinuum of X. (See, for example, [3], section 3, and references given there). Since we may clearly regard Z as a group of homeomorphisms acting on H it follows that Theorem 1 will hold provided we can prove

Theorem 2. *If Z acts as an abelian group of homeomorphisms on X and no proper subcontinuum of X is Z-invariant, then X has no cutpoint.*

In the course of the proof of this result considerable use will be made of the following definitions and theorems, which constitute an analog of Whyburn's cyclic element theory for non-separable spaces. Proofs, further results and references to the work of Ayres, Kuratowski, Moore and others will be found in [3], section 2. See also in this connection [6], Chap. IV.

We recall first that X is a compact Hausdorff space and hence normal. A *chain* is a subcontinuum C of X with the property that if, a, b are distinct points of C and no point separates x from either a or b then x is in C. If X is metric and locally connected then „chain" and „A-set" are equivalent. The intersection of any family of chains is a chain and the meet of a chain and a subcontinuum of X is a continuum. Also if R is a component of $X-x$ then $R\cup x$ is a chain. Further, if $X-x=M\cup N$, then $M\cup x$ is a chain. Deviating slightly from the terminology of [3], a set P is a prime chain if it is (a) either a cutpoint or an endpoint or (b) a minimal non-degenerate chain. If X is metric and locally connected then „cyclic element" and „prime chain" are equivalent. Two prime chains meet, if at all, in a cutpoint of X. If a subcontinuum of X has no cutpoint it lies in a prime chain. If p is neither an endpoint nor a cutpoint it is contained in a non-degenerate prime chain of X.

3. Turning now to the proof of theorem 2 we assume that Z is an abelian group of homeomophisms acting on X, that no proper subcontinuum of X is Z-invariant and that Q, the set of all cutpoints of X, is not empty. Let p be some definite point of $X-Q$ (see [3], p. 491) which will be held fixed during the course of the proof.

For any two points $a, b \in Q-p$ we write aRb if b separates a and p in X. By a standard type of argument we see that

$(R\cdot 1)$ $\quad aRa$ is false for every a.

$(R\cdot 2)$ $\quad aRb$ implies that bRa is false.

$(R\cdot 3)$ $\quad aRb$ and bRc imply aRc.

Let $q \in Q$. If, for each $q' \in Q-q$, we have neither qRq' nor $q'Rq$ let $Q_0=q$. If not let Q_0 be a subset of Q such that

$(Q_0\cdot 1)$ $\quad q_1, q_2 \in Q_0$ imply q_1Rq_2 or q_2Rq_1.

$(Q_0\cdot 2)$ $\quad Q_0$ is maximal relative to $(Q_0\cdot 1)$.

The existence of Q_0 follows from the maximality principal of Hausdorff cited earlier.

The proof is divided in two parts according as (Case I) Q_0 has only one point q_0 or a point q_0 such that q_0Rq holds for each point in Q_0 or (Case II) Q_0 is non-degenerate and for each point q in Q_0 there is a point q' in Q_0 such that $q'Rq_0$ holds.

Case I. There is a separation $X-q_0=U\cup V$ with $p\in V$ and U non-void. Suppose that U contains a point q of Q. We have, by definition, qRq_0. It is then impossible that Q_0 contain only the point q_0. But for each point q' of Q_0-q_0 we would have qRq' contrary to the fact that Q_0 is maximal. Accordingly U contains no cutpoint of X and we have $Q\subset V\cup q_0=A$. Since each z is a homeomorphism it is clear that Q is Z-invariant. Thus $Q\subset z(A)$ for each z. Now A is a chain and consequently so is each set $z(A)$. Denote by A_0 the intersection of all the sets $z(A)$. Since an arbitrary intersections of chains is a chain it follows that A_0 is a chain and thus a continuum. But clearly A_0 is Z-invariant and a proper subcontinuum of X. This is a contradiction.

Case II. For each q in Q_0 let $L(q)$ be the set of all points q' of Q_0 such that we have $q'Rq$. No set $L(q)$ is empty.

If $q_1 \in Q_0$ and q'', q' are in $L(q_1)$ then q'' and q' lie in the same component, $C(q_1)$, of $X-q_1$. For we may assume that we have $q''Rq'$ by $(Q_0 \cdot 1)$, so that we have a separation

(*) $$X-q'=W \cup S \quad \text{with } q'' \in W \text{ and } p \in S.$$

Now q_1 must be in S since other wise we would have q_1Rq' contrary to $q'Rq_1$, in view of $(R \cdot 2)$. Also we have a separation

$$X-q_1=U \cup V \quad \text{with } q' \in U \text{ and } p \in V.$$

Now $W \cup q'$ is a continuum containing q'' but not q_1. Thus $W \cup q' \subset U$. Hence q' and q'' lie in the same component of $X-q_1$.

Hence $C(q_1) \cup q_1$ is a chain, $A(q_1)$, and $L(q_1)$ is a subset of $A(q_1)$.

If $q'Rq_1$ holds with $q', q_1 \in Q_0$ then $C(q') \in C(q_1)$ and hence $A(q') \subset A(q_1)$. For, there is a q'' in Q_0 with $q''Rq'$. With the notation as in (*) $q'' \in W \subset C(q_1)$. But $C(q')$ is a connected subset of $X-q'$ containing q''. Hence $C(q') \subset W$.

Let A be the intersection of all the sets $A(q)$, $q \in Q_0$. Suppose that G is an open set containing A. Since X is compact and the sets $A(q)$ are non-empty, closed and ordered by inclusion A is non-empty and there exists therefore a set $A(q_1) \subset G$. Let q', q'' be points of Q_0 with $q''Rq'Rq_1$. With the notation as in (*) we have $W \cup q' \subset A(q_1) \subset G$. Now W is open and its boundary $F(W)=\overline{W}-W$ is exactly the point q'. Also it follows readily that A contains no cutpoint of X and so is a non-degenerate prime chain or an end point. We may indeed argue as follows. If A is a point it is certainly an endpoint and so a prime chain containing no cutpoint of X. Next, A contains no point of Q_0. For let q_1 be such a point and q' a point of Q_0 for which we have $q'Rq_1$. Then $A(q')=C(q') \cup q' \subset C(q_1) \cup q'$. But q_1 and q' are distinct and q_1 is not in $C(q_1)$. Hence q_1 is not in $A(q')$, a contradiction. But this argument also shows that A is the intersection of the sets $C(q)$, $q \in Q_0$. Hence any cutpoint q in A is in some set $C(q')$ with q' in Q_0 and it is clear that then we have qRq' for each q' in Q_0. By $(Q_0 \cdot 2)$ we then have q in Q_0, a contradiction. Since A contains no cutpoint of X and is a continuum it is contained in a prime chain. But A is a chain and hence by definition a prime chain.

Since A is a continuum and X is irreducibly Z-invariant there is a z_0 in Z such that A and $z_0(A)$ do not intersect. Moreover, A and $z_0(A)$ are prime chains and meet in at most a cutpoint of X. As we have seen A contains no cutpoint of X and so the sets in question are disjoint. It follows (recall that X is normal) that there exists an open set G about A with the property that $\overline{G}$ and $z_0(\overline{G})=\overline{z_0(G)}$ have no point in common. There is then the neighborhood W of A constructed above with $A \subset W \subset G$ and $F(W)=q'$.

In applying Theorem I of [2] we observe that it is not necessary to assume that the space be metric, but merely that it be a compact connected Hausdorff space. The existence of an irreducibly z_0-invariant continuum may be shown using the Hausdorff maximality principal instead of separability. Accordingly there is at least one prime chain which is invariant under z_0.

Remark: Up to this point the commutativity of Z has not been used.

Let P be the union of all the prime chains invariant under z_0. If E is any one of these and $z \in Z$ we have $z_0z(E)=zz_0(E_0)=z(E)$. Thus $z(E) \subset P$ and so $Z(E) \subset P$. It follows that $Z(P) \subset P$. Also, since Z is a group, we have $P \subset Z(P)$. Hence P is Z-invariant. Let C be the smallest chain containing P. It follows quite readily that C is Z-invariant. Thus C being a continuum we get $C=X$.

Let E be any z_0-invariant prime chain. Then E cannot lie wholly in $\overline{W}$ since $\overline{W}$ and $z_0(\overline{W})$ are disjoint. Since q' is a cutpoint and E is connected E cannot contain points in both W and $X-\overline{W}$. Thus no such set meets W and hence $P \subset X-W=S \cup q'$ (notation as in (*)). But $S \cup q'$ is a chain containing P and so $C \subset X-W$. This is a contradiction and so X has no cutpoint.

In virtue of Theorem 1 and the results on chains we may state

Theorem 3. *If Z is an abelian group of homeomorphisms acting on X then there exists a Z-invariant prime chain.*

4. There are certain related problems of some interest. Is Theorem 3 valid if „abelian" is replaced by „compact" or even „compact and totally disconnected"? It is perhaps of interest to observe that no use was made of the topology in Z in any of the above proof. If Z is cyclic and X locally connected it is known [5] that the existence of a Z-invariant endpoint implies the existence of a second Z-invariant point. Is this proposition valid if „cyclic" is replaced by „abelian" or „compact"?

In the example given earlier the group T is connected and leaves only the endpoints invariant. But we have

Theorem 4. *If Z is connected and X metric then every endpoint and non-degenerate prime chain is invariant.*

To see this let p be a non-invariant endpoint so that for some z the points p and $z(p)$ are distinct. Then some point x separates p and $z(p)$ in X. But $Z(p)=f(Z\times p)$ is connected and so contains x. Thus x is the image of p under a homeomorphism so that x must be both and endpoint and a cutpoint, an absurdity. Let P be a non-invariant prime chain containing more than one point. Now P is a continuum and no point separates any two points of P in X. Accordingly P contains a non cutpoint, x, of X. But some point y of X separates $P-y$ and $z(P)-y$ and from an argument similar to the above we see that x must be a cut point.

Bibliography.

[1] W. L. Ayres, *Some generalizations of the Sherrer fixed-point theorem*, Fund. Math. **16** (1930), pp. 332-336.

[2] J. L. Kelley, *Fixed sets under homeomorphism*, Duke Mathematical Journal, **15** (1939), pp. 535-537.

[3] A. D. Wallace, *Monotone transformations*, Duke Mathematical Journal **9** (1942), pp. 487-506.

[4] —, *A substitute for the axiom of choice*, Bulletin of the American Mathematical Society **50** (1944), p. 278.

[5] —, *A fixed-point theorem*, Bulletin of the American Mathematical Society **51** (1945), pp. 413-416.

[6] G. T. Whyburn, *Analytic Topology*, New York 1942.

Sur certains espaces abstraits.

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1. Soit A l'ensemble des fonctions réelles $f(x)$ finies sur un ensemble donné G. On dira qu'une suite $f_n(x)$ $(n=1,2,...)$ de fonctions de A satisfait à la condition (K), lorsqu'il existe une fonction $g(x)\in A$, telle qu'on a sur G, quel que soit q naturel,

(1) $\qquad q|f_m(x)-f_n(x)|\leqslant g(x)$ pour m et n suffisamment grands.

Pareillement, on dira que $f_n(x)$ satisfait à la condition (K'), lorsque l'inégalité (1) a lieu presque partout dans G.

Cela posé, on a les théorèmes suivants:

Théorème 1. *Si l'ensemble A se compose de toutes les fonctions qui sont bornées sur chacun des ensembles d'une suite G_ν $(\nu=1,2,...)$ telle que $\sum_{\nu=1}^{\infty} G_\nu=G$, alors la condition (K) est nécessaire et suffisante pour que la suite $f_n(x)$ converge uniformément sur chacun des ensembles G_ν.*

Théorème 2. *Si l'ensemble A se compose de fonctions continues dans un ensemble ouvert G, la condition (K) est nécessaire et suffisante pour que la suite $f_n(x)$ converge uniformément dans l'intérieur de G, c'est-à-dire uniformément sur tout compact contenu dans G.*

Théorème 3. *Si l'ensemble A se compose de toutes les fonctions mesurables (L) et finies sur un ensemble G mesurable (L), la condition (K') est nécessaire et suffisante pour que la suite $f_n(x)$ converge presque partout dans G.*

Théorème 4. *Si l'ensemble A se compose des fonctions p-sommables $(p>0)$ sur un ensemble G, mesurable (L), la condition (K') est nécessaire et suffisante pour que la suite $f_n(x)$ converge presque partout dans G et que la suite des modules $|f_n(x)|$ soit bornée presque partout par une fonction p-sommable.*