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Closure algebras.

By

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This paper treats of σ-complete Boolean algebras on which there is defined a closure operation satisfying the well-known axioms of Kuratowski[1]). A σ-complete Boolean algebra with a closure operation is called a closure algebra.

Almost all topological theorems which can be expressed in terms of the theory of Boolean algebras hold also for closure algebras. The proof of these theorems on closure algebras is often the same as the proof of analogous theorems on topological spaces. C. Kuratowski has worked out a method for the proof of topological theorems, the so-called topological calculus[2]). This method is especially suitable for generalizing topological theorems to the case of closure algebras. In general, in order to obtain a proof of a theorem on closure algebras it is sufficient to replace the term: „a subset of a topological space" by the term „an element of a closure algebra" in Kuratowski's proof of an analogous theorem on topological spaces. Therefore I shall omit proofs of many theorems on closure algebras.

The specification of all topological theorems which hold for closure algebras is not the purpose of this paper. I shall show only the method and the direction of generalizing and I shall cite many examples of topological theorems (given in the work Kuratowski [1]) which can be generalized.

[1]) Finitely additive Boolean algebras and lattices with a closure operation were examined by many writers. See e. g. Mc Kinsey and Tarski [1]; Monteiro and Ribeiro [1]; Nöbeling [1]; Terasaka [1].

[2]) Kuratowski [1] and Kuratowski [2]. See also S. Janiszewski, *Thèse*, Journ. Ec. Polytechn. (1911).

As a generalization of the notion of a mapping between topological spaces, I shall consider the notion of a homomorphism of a closure algebra $\boldsymbol{A}$ in a closure algebra $\boldsymbol{B}$. A homomorphism f of $\boldsymbol{A}$ in $\boldsymbol{B}$ is said to be continuous provided $\overline{f(A)} \subset f(\bar{A})$ for every $A \in \boldsymbol{A}$. Two closure algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ are homeomorphic, if there exists an isomorphism h of $\boldsymbol{A}$ on $\boldsymbol{B}$ such that both h and h^{-1} are continuous. Besides the notion of a homeomorphism it is convenient to introduce the notion of weak homeomorphism. For instance, if $\mathfrak{X}$ is a non-enumerable metric separable space, the closure algebra $\mathfrak{S}(\mathfrak{X})$ of all subsets of $\mathfrak{X}$ and the closure algebra $\mathfrak{B}(\mathfrak{X})$ of all Borel subsets of $\mathfrak{X}$ are not homeomorphic. $\mathfrak{S}(\mathfrak{X})$ and $\mathfrak{B}(\mathfrak{X})$ possess however many common topological properties, and often the study of the topology of $\mathfrak{S}(\mathfrak{X})$ can be reduced to the study of properties of $\mathfrak{B}(\mathfrak{X})$. Therefore I shall introduce the following definition: two closure algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ are called weakly homeomorphic if the closure algebras $\mathfrak{B}(\boldsymbol{A})$ and $\mathfrak{B}(\boldsymbol{B})$ (of all Borel elements of $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively) are homeomorphic. In the case of topological spaces the notions of a homeomorphism and of a weak homeomorphism coincide.

The main problem which arises here is whether closure algebras represent an essential generalization of the notion of a topological space. The answer is affirmative. I shall give a general method for the construction of closure algebras which are not weakly homeomorphic to a topological space. The above-mentioned method is the well-known method of the division by an ideal: if $\boldsymbol{A}$ is a closure algebra and $\boldsymbol{I}$ is a σ-ideal of $\boldsymbol{A}$, the closure operation in $\boldsymbol{A}$ induces, in a very natural way, a closure operation in the quotient algebra $\boldsymbol{A}/\boldsymbol{I}$.

In the first part of this paper I shall study general closure algebras satisfying the four axioms of Kuratowski only. In the second part I shall examine closure algebras satisfying in addition a new fifth axiom. These closure algebras are called C-algebras. C-algebras represent a generalization of metric separable spaces and possess many properties of these spaces. If $\mathfrak{X}$ is a metric separable space and $\boldsymbol{I}$ is a σ-ideal of subsets of $\mathfrak{X}$, the field $\mathfrak{S}(\mathfrak{X})$ of all subsets of $\mathfrak{X}$, the field $\mathfrak{B}(\mathfrak{X})$ of all Borel subsets of $\mathfrak{X}$, and the quotient closure algebras $\mathfrak{S}(\mathfrak{X})/\boldsymbol{I}$ and $\mathfrak{B}(\mathfrak{X})/\boldsymbol{I}$ are examples of C-algebras. In general, $\mathfrak{S}(\mathfrak{X})/\boldsymbol{I}$ and $\mathfrak{B}(\mathfrak{X})/\boldsymbol{I}$ are not isomorphic to a σ-field of sets; thus there exist C-algebras which are essentially different from metric spaces, i. e. which are not weakly homeomorphic to a metric space. On the other hand, one can prove that every C-algebra is weakly homeomorphic to a C-algebra $\mathfrak{S}(\mathfrak{X})/\boldsymbol{I}$ where $\mathfrak{X}$ is a metric separable space. The representation problem for C-algebras constitutes the subject of the second part of this paper [3]. The final theorem is a generalization of Urysohn's theorem on the imbedding of topological spaces in the Hilbert cube.

The study of homomorphisms defined on a C-algebra constitutes the subject of the third part. I shall give a definition of the convergence of a sequence of homomorphisms. This definition is a generalization of the notion of convergence of mappings in metric spaces. The representation theorems for C-algebras given in the second part and a general theorem on the inducing of homomorphisms by mappings [4]) enable us to explain more exactly the structure of homomorphisms and of the convergence of homomorphisms. Theorems given in the third part show that it is always possible to reduce the study of properties of homomorphisms to the study of properties of certain mappings between metric spaces.

Terminology and notation.

Boolean algebras. Boolean algebras will be denoted by letters $\boldsymbol{A}, \boldsymbol{B}, \ldots$, their elements by $A, B, \ldots$; $A+B$, AB, $A-B$, and A' will denote the Boolean operations which are analogous to the well-known operations of addition, multiplication, subtraction, and complementation of sets in the general theory of sets. If $A+B=B$, we write $A \subset B$ and we say: A is *contained* in B. $|\boldsymbol{A}|$ and 0 will denote respectively the greatest element and the least element of a Boolean algebra $\boldsymbol{A}$ i. e. $0 \subset A \subset |\boldsymbol{A}|$ for every $A \in \boldsymbol{A}$.

Let S be a set of elements of a Boolean algebra $\boldsymbol{A}$. An element $A_s \in \boldsymbol{A}$ ($A_p \in \boldsymbol{A}$) is called the *sum (product)* of all elements $A \in S$ if it is the least (greatest) element containing (contained in) all elements $A \in S$ [5]). We write then $A_s = \sum_{A \in S} A$ $(A_p = \prod_{A \in S} A)$. If for every set $S \subset \boldsymbol{A}$ of potency $\leqslant \mathfrak{m}$ (or: of arbitrary potency) there exists the sum of all $A \in S$, $\boldsymbol{A}$ is called $\mathfrak{m}$-*complete* (or: *complete*) [6]). $\aleph_0$-complete Boolean algebras will be called also *σ-complete* Boolean algebras. The meaning of the symbols $\sum_{n=1}^{\infty} A_n$, $\prod_{n=1}^{\infty} A_n$ in case of a σ-complete Boolean algebra is clear.

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[3]) The main results of the second part were announced in my paper [3].

[4]) Sikorski [2], p. 19, th. 5.1.

[5]) I. e. 1^0 $A \subset A_s$ for every $A \in S$, and 2^0 if $A \subset A_0$ for every $A \in S$, then $A_s \subset A_0$. Analogously for A_p.

[6]) By the well-known formulas of de Morgan, if $\boldsymbol{A}$ is $\mathfrak{m}$-complete, the product of all $A \in S$ exists also for every set $S \subset \boldsymbol{A}$ of potency $\leqslant \mathfrak{m}$. $\mathfrak{m}$ denotes always a cardinal.

Ideals. Quotient algebras. A class $\boldsymbol{I}$ of elements of a Boolean algebra $\boldsymbol{A}$ is said to be an *ideal* of $\boldsymbol{A}$ provided·

(i) if $A \in \boldsymbol{I}$, $A_1 \in \boldsymbol{A}$, $A_1 \subset A$, then $A_1 \in \boldsymbol{I}$;

(ii) if $A_1 \in \boldsymbol{I}$ and $A_2 \in \boldsymbol{I}$, then $A_1 + A_2 \in \boldsymbol{I}$.

An ideal $\boldsymbol{I}$ (of an $\mathfrak{m}$-complete Boolean algebra $\boldsymbol{A}$) is $\mathfrak{m}$-*additive* if the sum of elements belonging to an arbitrary set $S \subset \boldsymbol{I}$ of potency $\leqslant \mathfrak{m}$ belongs to $\boldsymbol{I}$ also. $\aleph_0$-additive ideals will be called likewise σ-*ideals*.

Let $\boldsymbol{I}$ be an ideal of a Boolean algebra $\boldsymbol{A}$. The symbol $\boldsymbol{A}/\boldsymbol{I}$ denotes the Boolean algebra defined in the following way:

Elements of $\boldsymbol{A}/\boldsymbol{I}$ are disjoint classes of elements of $\boldsymbol{A}$ such that two elements A_1, A_2 belong to the same class if and only if $A_1'A_2 + A_1A_2' \in \boldsymbol{I}$. The class $C \in \boldsymbol{A}/\boldsymbol{I}$ containing an element $A \in \boldsymbol{A}$ will be denoted by $[A]$. We say then that A is a *representative* of the class $C = [A]$ or that A *determines* the class C. The Boolean operations on elements of $\boldsymbol{A}/\boldsymbol{I}$ are defined by the formulas:

(i) $$[A_1]+[A_2]=[A_1+A_2], \quad [A_1][A_2]=[A_1A_2],$$

(ii) $$[A_1]-[A_2]=[A_1-A_2], \quad [A]'=[A'].$$

If $\boldsymbol{A}$ is $\mathfrak{m}$-complete and $\boldsymbol{I}$ is $\mathfrak{m}$-additive, $\boldsymbol{A}/\boldsymbol{I}$ is $\mathfrak{m}$-complete. The formulas (i) hold then also for the infinite sum and product, e.i. if A_s (A_p) is the sum (product) of all elements $A \in S$ (where the set $S \subset \boldsymbol{A}$ is of potency $\leqslant \mathfrak{m}$), then $[A_s]$ ($[A_p]$) is the sum (product) of all elements $[A] \in \boldsymbol{A}/\boldsymbol{I}$ where $A \in \boldsymbol{S}$.

Subalgebras and algebras $\boldsymbol{EA}$. If S is a set of elements of a Boolean algebra $\boldsymbol{A}$ and $E \in \boldsymbol{A}$, the symbol $\boldsymbol{E}S$ denotes the set of all elements EA where $A \in \boldsymbol{S}$.

In particular, $\boldsymbol{EA}$ is the set of all elements $A \subset E$ ($A \in \boldsymbol{A}$). $\boldsymbol{EA}$ is a Boolean algebra [7]): the definition of addition, multiplication, and subtraction in $\boldsymbol{EA}$ is the same as in $\boldsymbol{A}$; the complement of an element $A \in \boldsymbol{EA}$ in the Boolean algebra $\boldsymbol{EA}$ is the element EA'. If $\boldsymbol{A}$ is $\mathfrak{m}$-complete, $\boldsymbol{EA}$ is $\mathfrak{m}$-complete also.

A set $\boldsymbol{S}$ of elements of a σ-complete Boolean algebra $\boldsymbol{A}$ is said to be a σ-*subalgebra* of $\boldsymbol{A}$ provided

(i) if $A \in \boldsymbol{S}$, then $A' \in \boldsymbol{S}$;

(ii) if $A_n \in \boldsymbol{A}$ $(n=1,2,\ldots)$, then $\sum_{n=1}^{\infty} A_n \in \boldsymbol{S}$.

A σ-subalgebra of a σ-complete Boolean algebra is also a σ-complete Boolean algebra.

If $\boldsymbol{I}$ is an ideal of a Boolean algebra $\boldsymbol{A}$ and $\boldsymbol{S}$ is a set of elements of $\boldsymbol{A}$, then $[S]$ denotes the set of all alements $[A] \in \boldsymbol{A}/\boldsymbol{I}$ where $A \in \boldsymbol{S}$. If $\boldsymbol{A}$ is σ-complete, $\boldsymbol{I}$ is a σ-ideal, and $\boldsymbol{S}$ is a σ-subalgebra of $\boldsymbol{A}$, then $[S]$ is a σ-subalgebra of $\boldsymbol{A}/\boldsymbol{I}$.

Fields of sets. For every abstract set $\mathfrak{X}$ the symbol $\mathfrak{S}(\mathfrak{X})$ will denote the class of all subsets of $\mathfrak{X}$. $\mathfrak{S}(\mathfrak{X})$ is a complete Boolean algebra.

σ-subalgebras of $\mathfrak{S}(\mathfrak{X})$ will be called σ-*fields* (of subsets of $\mathfrak{X}$).

If $\boldsymbol{X}$ is a σ-field of subsets of $\mathfrak{X}$ and $\boldsymbol{I}$ is σ-ideal of $\boldsymbol{X}$, the Boolean algebra $\boldsymbol{X}/\boldsymbol{I}$ is called a σ-*quotient algebra* (of $\mathfrak{X}$).

[7]) See Sikorski [5], p. 141.

Homomorphisms and isomorphisms. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two σ-complete Boolean algebras.

A one-one mapping h of $\boldsymbol{A}$ on $\boldsymbol{B}$ is said to be an *isomorphism* provided that $A_1 \subset A_2$ if and only if $h(A_1) \subset h(A_2)$. If there exists an isomorphism of $\boldsymbol{A}$ on $\boldsymbol{B}$, we say that $\boldsymbol{A}$ and $\boldsymbol{B}$ are *isomorphic*.

A mapping f of A in B is called a σ-*homomorphism* [8]) if

$$f(A')=f(A)' \quad \text{and} \quad f\Big(\sum_{n=1}^{\infty} A_n\Big)=\sum_{n=1}^{\infty} f(A_n).$$

Every isomorphism of $\boldsymbol{A}$ on $\boldsymbol{B}$ is a σ-homomorphism.

Let f be a σ-homomorphism of $\boldsymbol{A}$ in $\boldsymbol{B}$ and let $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$.

The symbol Bf will denote the σ-homomorphism g of $\boldsymbol{A}$ in $B\boldsymbol{B}$ defined by the formula

$$g(A)=B\cdot f(A) \quad \text{for } A \in \boldsymbol{A}.$$

If $S \subset \boldsymbol{A}$, the symbol $f|S$ will denote the mapping f restricted to elements of S [9]). In particular, $f|A\boldsymbol{A}$ is a σ-homomorphism of $A\boldsymbol{A}$ in $f(A)\cdot\boldsymbol{B}$. If $\boldsymbol{S}$ is a σ-subalgebra of $\boldsymbol{A}$, then $f|\boldsymbol{S}$ is a σ-homomorphism of $\boldsymbol{S}$ in $f(\boldsymbol{S}) \subset \boldsymbol{B}$.

Examples of homomorphism and isomorphisms. Let $\boldsymbol{A}$ be a σ-complete Boolean algebra and let $\boldsymbol{I}$ be a σ-ideal of $\boldsymbol{A}$.

1) The mapping

$$f(A)=[A] \in \boldsymbol{A}/\boldsymbol{I} \quad \text{for } A \in \boldsymbol{A}$$

is a σ-homomorphism of $\boldsymbol{A}$ on $\boldsymbol{A}/\boldsymbol{I}$. It is called the *natural homomorphism of* $\boldsymbol{A}$ *on* $\boldsymbol{A}/\boldsymbol{I}$.

2) Let $E \in \boldsymbol{A}$ and let us consider the two following Boolean algebras $\boldsymbol{EA}/\boldsymbol{EI}$ and $[E]\boldsymbol{A}/\boldsymbol{I}$. Let $A \in \boldsymbol{EA}$ be a representative of an element $C \in \boldsymbol{EA}/\boldsymbol{EI}$ and let $h(C)$ denote the element of $\boldsymbol{A}/\boldsymbol{I}$ which is determined by A. $h(C)$ does not depend on the choice of the representative A and $h(C) \in [E]\boldsymbol{A}/\boldsymbol{I}$. It is easy to see that h is an isomorphism of $\boldsymbol{EA}/\boldsymbol{EI}$ on $[E]\boldsymbol{A}/\boldsymbol{I}$. h will be called the *natural isomorphism of* $\boldsymbol{EA}/\boldsymbol{EI}$ *on* $[E]\boldsymbol{A}/\boldsymbol{I}$.

3) Let $\boldsymbol{S}$ be a σ-subalgebra of $\boldsymbol{A}$. $[\boldsymbol{S}]$ is a σ-subalgebra of $\boldsymbol{A}/\boldsymbol{I}$. Consider the Boolean algebra $\boldsymbol{S}/\boldsymbol{IS}$. Let $C \in \boldsymbol{S}/\boldsymbol{IS}$, let $A \in \boldsymbol{S}$ be a representative of C and let $h(C)$ denote the element of $\boldsymbol{A}/\boldsymbol{I}$ which is determined by A. $h(C)$ does not depend on the choice of the representative A and $h(C) \in [\boldsymbol{S}]$. It is easy to see that h is an isomorphism of $\boldsymbol{S}/\boldsymbol{IS}$ on $[\boldsymbol{S}]$. h will be called the *natural isomorphism of* $\boldsymbol{S}/\boldsymbol{IS}$ *on* $[\boldsymbol{S}]$.

[8]) A *homomorphism* of $\boldsymbol{A}$ in $\boldsymbol{B}$ is a mapping f such that $f(A')=f(A)'$ and $f(A_1+A_2)=f(A_1)+(A_2)$. All homomorphisms considered in this paper are σ-homomorphisms.

[9]) More generally, if φ is an arbitrary mapping of a set $\mathfrak{X}$ in a set $\mathfrak{Y}$ and $X \subset \mathfrak{X}$, $\varphi|\mathfrak{X}$ will denote the mapping φ restricted to $x \in X$. The letters $f, g, h, \ldots$ will denote always homomorphisms and isomorphisms. Mappings between abstract sets or topological spaces will be denoted by $\varphi, \psi, \ldots$

The inducing of homomorphism by mappings[10]). Let $\mathbf{X}/\mathbf{I}$ and $\mathbf{Y}/\mathbf{J}$ be σ-quotient algebras of sets $\mathfrak{X}$ and $\mathfrak{Y}$ respectively. We say that a σ-homomorphism f of $\mathbf{Y}/\mathbf{J}$ in $\mathbf{X}/\mathbf{I}$ is *induced* by a mapping φ of $\mathfrak{X}$ in $\mathfrak{Y}$ if

(i) $\varphi^{-1}(Y)\in\mathbf{X}$ and $f([Y])=[\varphi^{-1}(Y)]$ for every $Y\in\mathbf{Y}$[11]).

In particular, a σ-homomorphism f of $\mathbf{Y}$ in $\mathbf{X}/\mathbf{I}$ is induced by a mapping φ of $\mathfrak{X}$ in $\mathfrak{Y}$ if

(ii) $\varphi^{-1}(Y)\in\mathbf{X}$ and $f(Y)=[\varphi^{-1}(Y)]$ for every $Y\in\mathbf{Y}$.

A σ-homomorphism f of $\mathbf{Y}$ in $\mathbf{X}$ is induced by a mapping φ of $\mathfrak{X}$ in $\mathfrak{Y}$ if

(iii) $\varphi^{-1}(Y)\in\mathbf{X}$ and $f(Y)=\varphi^{-1}(Y)$ for every $Y\in\mathbf{Y}$.

I. General closure algebras.

1. *Definitions.* A *closure algebra* is by definition a σ-complete Boolean algebra $\mathbf{A}$ in which with every element $A\in\mathbf{A}$ there is associated an element $\bar{A}\in\mathbf{A}$ in such a way that the following axioms of Kuratowski[12]) are satisfied:

I. $\overline{A_1+A_2}=\bar{A}_1+\bar{A}_2$;

II. $\bar{0}=0$;

III. $A\subset\bar{A}$;

IV. $\overline{(\bar{A})}=\bar{A}$.

The element $\bar{A}$ is called the *closure* of A, the element $\mathrm{Int}\,(A)=\overline{(A')}'$ is called the *interior* of A, the element $\mathrm{Fr}\,(A)=\bar{A}\cdot\bar{A}'$ is called the *frontier* of A. An element $A\in\mathbf{A}$ is called respectively

closed, if $A=\bar{A}$,

open, if $A=\mathrm{Int}\,(A)$,

nowhere dense, if $A\subset\mathrm{Fr}\,(\bar{A})$,

of first category, if $A=\sum_n A_n$, where A_n is nowhere dense $(n=1,2,\ldots)$,

dense, if $\bar{A}=|\mathbf{A}|$,

a $\mathbf{G}_\delta$, if $A=\prod_n A_n$ where A_n is open $(n=1,2,\ldots)$

an $\mathbf{F}_\sigma$, if $A=\sum_n A_n$ where A_n is closed $(n=1,2,\ldots)$.

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[10]) Sikorski [2], p. 7 and p. 13.

[11]) Obviously, $[Y]$ denotes here an element of $\mathbf{Y}/\mathbf{J}$ and $[\varphi^{-1}(Y)]$ denotes an element of $\mathbf{X}/\mathbf{I}$.

[12]) Kuratowski [3], p. 181.

The class of all open elements of $\mathbf{A}$ will be denoted by $\mathfrak{G}(\mathbf{A})$, the class of all closed elements of $\mathbf{A}$ will be denoted by $\mathfrak{F}(\mathbf{A})$. $\mathfrak{B}(\mathbf{A})$ will denote the least σ-subalgebra of $\mathbf{A}$ which contains $\mathfrak{F}(\mathbf{A})$. If $A\in\mathfrak{B}(\mathbf{A})$, we say that A is a *Borel element* of $\mathbf{A}$.

If $E\in\mathbf{A}$, then the formula

$$\bar{A}_E=E\bar{A}$$

defines a closure of an element $A\in E\mathbf{A}$ relative to $E\mathbf{A}$.

This closure satisfies also Kuratowski's axioms I-IV. $E\mathbf{A}$ is also a closure algebra and

1.1. $\mathfrak{F}(E\mathbf{A})=E\mathfrak{F}(\mathbf{A})$, $\mathfrak{G}(E\mathbf{A})=E\mathfrak{G}(\mathbf{A})$, $\mathfrak{B}(E\mathbf{A})=E\mathfrak{B}(\mathbf{A})$.

It is easy to show that all theorems given in Kuratowski [1] Chapitre I, § 4, § 5, § 6 I-III, § 8 I-VI, § 8 VIII-IX, § 10 I-II, § 10 IV, § 11 I-III, § 11 V[13]) are also true for arbitrary closure algebras.

If $\mathfrak{X}$ is a topological space[14]), then the field $\mathfrak{S}(\mathfrak{X})$ is a closure algebra. Instead of $\mathfrak{B}(\mathfrak{S}(\mathfrak{X}))$, we shall write simply $\mathfrak{B}(\mathfrak{X})$. $\mathfrak{B}(\mathfrak{X})$ is also a closure algebra.

An example of a closure algebra which is not isomorphic to a σ-field of sets is the following: let $\mathcal{R}$ denote the set of all real numbers and let $\mathbf{L}_0$ be the ideal of all sets of Lebesgue measure zero. In the quotient algebra $\mathfrak{S}(\mathcal{R})/\mathbf{L}_0$ we define the closure of an element $[X]$ as the element $[Y]$ where Y is a G_δ such that $X\subset Y$ and the Lebesgue exterior measures of X and Y are equal.

A general method for the construction of non-trivial closure algebras which are not isomorphic to a σ-field of sets will be given in § 9.

2. *Basis of a closure algebra.* A class $\mathbf{R}$ of open elements of a closure algebra $\mathbf{A}$ is called a *basis* of $\mathbf{A}$ if

(i) $\mathbf{A}$ is $\overline{\overline{\mathbf{R}}}$-complete,

(ii) every open element $G\in\mathbf{A}$ is the sum of elements belonging to a subclass of $\mathbf{R}$.

By this definition every complete closure algebra $\mathbf{A}$ possesses a basis. Namely, the class $\mathfrak{G}(\mathbf{A})$ is a basis of $\mathbf{A}$. In particular, every field $\mathfrak{S}(\mathfrak{X})$, where $\mathfrak{X}$ is a topological space, possesses a basis in the above-mentioned sense. If $\mathfrak{X}$ is a metric separable space, $\mathfrak{S}(\mathfrak{X})$ possesses an enumerable basis. Obviously

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[13]) Or: Kuratowski [2], the same paragraphs and sections.

[14]) A space $\mathfrak{X}$ is called *topological* if the closure operation in $\mathfrak{X}$ satisfies axioms I-IV and if $\overline{(x)}=(x)$ for every $x\in\mathfrak{X}$.

2.1. *If* $\boldsymbol{R}$ *is a basis of a closure algebra* $\boldsymbol{A}$ *and* $E \in \boldsymbol{A}$, *then* $E\boldsymbol{R}$ *is a basis* $E\boldsymbol{A}$ *and* $\overline{\overline{E\boldsymbol{R}}} \leqslant \overline{\overline{\boldsymbol{R}}}$.

The following theorem gives a method for the construction of closure algebras:

2.2. *Let* $\boldsymbol{R}$ *be a class of elements of a* σ*-complete Boolean algebra* $\boldsymbol{A}$ *such that*

(i) $\boldsymbol{A}$ *is* $\overline{\overline{\boldsymbol{R}}}$*-complete,*

(ii) $0 \in \boldsymbol{R}$,

(iii) *the element* $|\boldsymbol{A}|$ *and each product* $R_1 \cdot R_2$, *where* $R_i \in \boldsymbol{R}$ $(i=1,2)$, *are the sum of elements belonging to a subclass of* $\boldsymbol{R}$.

For every $A \in \boldsymbol{A}$ *let* $\operatorname{Int}(A)$ *denote the sum of all* $R \in \boldsymbol{R}$ *which are contained in* A *and let*

(iv) $\bar{A} = (\operatorname{Int}(A'))'$.

Then the closure $\bar{A}$ *satisfies the axioms* I-IV *and* $\boldsymbol{R}$ *is a basis of the closure algebra which we obtain from* $\boldsymbol{A}$ *by definition* (iv)[15].

We say in this case that the closure operation (iv) is *induced by the basis* $\boldsymbol{R}$.

In order to prove theorem 2.2 it is sufficient to remark that (i) and (ii) imply that $\operatorname{Int}(A)$ and $\bar{A}$ are defined for every $A \in \boldsymbol{R}$. By definition $\operatorname{Int}(A) \subset A$ and $\operatorname{Int}(\operatorname{Int}(A)) = \operatorname{Int}(A)$. By (iii) $\operatorname{Int}(|\boldsymbol{A}|) = |\boldsymbol{A}|$ and $\operatorname{Int}(A_1 A_2) = \operatorname{Int}(A_1)\operatorname{Int}(A_2)$. These properties imply immediately the axioms I-IV.

3. *Properties of closure algebras with a basis.* Some theorems about topological spaces cannot be generalized to the case of closure algebras because some infinite operations of addition and multiplication on elements of a closure algebra $\boldsymbol{A}$ are not always feasible. For instance, the well-known theorem on the sum (product) of open (closed) sets can be expressed only in the form:

3.1. *If there exists the sum (product) of a set* $\boldsymbol{S} \subset \boldsymbol{A}$ *of open (closed) elements, it is open (closed) also.*

The above-mentioned difficulties disappear often in the case of closure algebras with a basis. For instance:

3.2. *If a closure algebra* $\boldsymbol{A}$ *possesses a basis, then the sum (product) of an arbitrary set of open (closed) elements exists and is open (closed).*

[15]) Obviously, the above-defined operation „Int" coincides with the operation „Int" defined in par. 1.

Let $\boldsymbol{G}$ be a set of open elements of $\boldsymbol{A}$ and let $\boldsymbol{R}_0$ denote the set of all elements R of the basis of $\boldsymbol{A}$ such that there exists a $G \in \boldsymbol{G}$ which contains R. It is easy to show that the sum G_0 of all $R \in \boldsymbol{R}_0$ is also the sum of all $G \in \boldsymbol{G}$. By 3.1, G_0 is open.

The second part of 3.2 follows from the first.

3.3. *If a closure algebra* $\boldsymbol{A}$ *possesses a basis* $\boldsymbol{R}$, *then every decreasing*[16]) *transfinite sequence* $\{A_\xi\}$ $(0 \leqslant \xi < \alpha)$ *of closed (open) elements is of potency* $\leqslant \overline{\overline{\boldsymbol{R}}}$ *(i. e.* $\bar{\alpha} \leqslant \overline{\overline{\boldsymbol{R}}}$*).*

The proof is the same as the proof of an analogous theorem on topological spaces with an enumerable basis.

By theorem 3.3 we obtain easily that all theorems on developable sets in Kuratowski [1] Chapitre I § 12 I-VIII[17]) can be generalized to the case of closure algebras with a basis[18]). Obviously an element $A \in \boldsymbol{A}$ is called *developable* if there exists a transfinite decreasing sequence $\{A_\xi\}$ of closed elements such that

$$A = \sum_{\xi} (A_{2\xi} - A_{2\xi+1}).$$

In particular we obtain:

3.4. *If the equation* $A = \overline{AA_1} \cdot \overline{AA_2}$ *implies* $A = 0$, *there exists a developable element* E *such that* $EA_2 = 0$ *and* $A_1 \subset E$.

4. *Closure subalgebras and extensions.* Let $\boldsymbol{A}_0$ be a σ-subalgebra of a closure algebra $\boldsymbol{A}$. We say that $\boldsymbol{A}_0$ is a *closure subalgebra* of $\boldsymbol{A}$ if $A \in \boldsymbol{A}_0$ implies $\bar{A} \in \boldsymbol{A}_0$.

4.1. *A closure subalgebra* $\boldsymbol{A}_0$ *of a closure algebra* $\boldsymbol{A}$ *is also a closure algebra (with the same operation of closure) and*

$$\mathfrak{G}(\boldsymbol{A}_0) = \boldsymbol{A}_0 \cdot \mathfrak{G}(\boldsymbol{A}), \quad \mathfrak{F}(\boldsymbol{A}_0) = \boldsymbol{A}_0 \cdot \mathfrak{F}(\boldsymbol{A}), \quad \mathfrak{B}(\boldsymbol{A}_0) = \boldsymbol{A}_0 \cdot \mathfrak{B}(\boldsymbol{A}).$$

4.2. $\mathfrak{B}(\boldsymbol{A})$ *is a closure subalgebra of* $\boldsymbol{A}$ *and*

$$\mathfrak{G}(\mathfrak{B}(\boldsymbol{A})) = \mathfrak{G}(\boldsymbol{A}), \quad \mathfrak{F}(\mathfrak{B}(\boldsymbol{A})) = \mathfrak{F}(\boldsymbol{A}), \quad \mathfrak{B}(\mathfrak{B}(\boldsymbol{A})) = \mathfrak{B}(\boldsymbol{A}).$$

[16]) I. e. $A_\eta \subset A_\xi$ and $A_\eta \neq A_\xi$ for $\xi > \eta$.

[17]) Or: Kuratowski [2], the same sections.

[18]) This follows from the fact that the following lemma holds for any $\mathfrak{m}$-complete Boolean algebra $\boldsymbol{B}$:

If $\{B_\xi\}$ $(0 \leqslant \xi \leqslant \alpha, \bar{\alpha} \leqslant \mathfrak{m})$ *is a decreasing sequence of elements of* $\boldsymbol{B}$ *such that* $\prod_{0 \leqslant \xi < \lambda} B_\xi = B_\lambda$ *for every limit number* $\lambda \leqslant \alpha$, *then*

$$B_0 = \sum_{0 \leqslant \xi < \alpha} (B_\xi - B_{\xi+1}) + B_\alpha.$$

4.3. *If a closure algebra* $\boldsymbol{A}$ *possesses an enumerable basis* $\boldsymbol{R}$, *then each closure subalgebra* $\boldsymbol{A}_0 \subset \boldsymbol{A}$ *possesses also an enumerable basis* $\boldsymbol{R}_0$ [19]).

Let $R \in \boldsymbol{R}$ and let $\{G_\xi\}$ $(0 \leqslant \xi \leqslant \alpha)$ be a transfinite sequence of all open elements of $\boldsymbol{A}_0$ which contains R. Let $H_0 = G_0$ and by induction $H_\xi = \mathrm{Int}\,(\prod_{\eta<\xi} H_\eta)\cdot G_\xi$ (by theorem 3.3, the sequence $\{H_\eta\}$ $(0 \leqslant \eta \leqslant \xi)$ contains only an enumerable number of different sets, since $\{H_\eta\}$ is non-increasing; the multiplication is thus feasible). The element $R^0 = H_\alpha$ possesses the following properties:

(i) $R^0 \in \mathfrak{G}(\boldsymbol{A}_0)$,

(ii) $R \subset R^0$,

(iii) if $G \in \mathfrak{G}(\boldsymbol{A}_0)$ and $R \subset G$, then $R^0 \subset G$.

An immediate consequence of (i), (ii), and (iii) is that the class $\boldsymbol{R}_0$ of all elements R^0 (where $R \in \boldsymbol{R}$) is an enumerable basis of $\boldsymbol{A}_0$.

Let $\boldsymbol{A}$ be an arbitrary closure algebra. Consider the minimal extension [20]) $\widetilde{\boldsymbol{A}}$ of $\boldsymbol{A}$. $\widetilde{\boldsymbol{A}}$ being complete, the class $\mathfrak{G}(\boldsymbol{A})$ fulfils the assumptions of theorem 2.2 and induces a closure operation in $\widetilde{\boldsymbol{A}}$. The closure algebra, which we obtain from $\boldsymbol{A}$ in this way will be called a *minimal closure extension of* $\boldsymbol{A}$ and will be denoted by the same symbol $\widetilde{\boldsymbol{A}}$. By theorem 3.1 $\boldsymbol{A}$ is a closure subalgebra of $\widetilde{\boldsymbol{A}}$ and by theorem 4.1

$$\mathfrak{G}(\boldsymbol{A}) = \mathfrak{G}(\widetilde{\boldsymbol{A}})\cdot\boldsymbol{A}, \quad \mathfrak{F}(\boldsymbol{A}) = \mathfrak{F}(\widetilde{\boldsymbol{A}})\cdot\boldsymbol{A}, \quad \mathfrak{B}(\boldsymbol{A}) = \mathfrak{B}(\widetilde{\boldsymbol{A}})\cdot\boldsymbol{A}.$$

By theorem 3.2 we have

4.4 *If a closure algebra* $\boldsymbol{A}$ *possesses a basis, then*

$$\mathfrak{G}(\boldsymbol{A}) = \mathfrak{G}(\widetilde{\boldsymbol{A}}), \quad \mathfrak{F}(\boldsymbol{A}) = \mathfrak{F}(\widetilde{\boldsymbol{A}}), \quad \mathfrak{B}(\boldsymbol{A}) = \mathfrak{B}(\widetilde{\boldsymbol{A}}).$$

In § 1 we assumed that a closure algebra is σ-complete only. In § 3 we pointed out that this assumption can cause some difficulties since some infinite operations of addition and multiplication are not feasible. These difficulties can be avoided since instead of a closure algebra $\boldsymbol{A}$ we may consider a totally additive minimal closure algebra $\widetilde{\boldsymbol{A}}$. On the other hand, the study of topological

[19]) Theorem 4.3 can be generalized to the case where $\boldsymbol{A}$ and $\boldsymbol{A}_0$ are $\mathfrak{m}$-complete and $\boldsymbol{A}$ possesses a basis of potency $\leqslant \mathfrak{m}$.

[20]) MacNeille [1], p. 437. We assume always that $\boldsymbol{A} \subset \widetilde{\boldsymbol{A}}$, i. e. $\boldsymbol{A}$ is a σ-subalgebra of $\widetilde{\boldsymbol{A}}$.

properties of a closure algebra $\boldsymbol{A}$ reduces often to the study of properties of $\mathfrak{B}(\boldsymbol{A})$. If $\boldsymbol{A}$ possesses a basis, then $\mathfrak{B}(\boldsymbol{A}) = \mathfrak{B}(\widetilde{\boldsymbol{A}})$ by 4.4, and the introduction of the minimal closure algebra $\widetilde{\boldsymbol{A}}$ is in practice unnecessary.

5. *Continuous homomorphisms. Homeomorphisms.* A mapping φ of a topological space $\mathcal{X}$ in a topological space $\mathcal{Y}$ is continuous if and only if the homomorphism f induced by the mapping φ (i. e. $f(Y) = \varphi^{-1}(Y)$ for $Y \subset \mathcal{Y}$) possesses the property:

$$\overline{f(Y)} \subset f(\overline{Y}) \quad \text{for every} \quad Y \in \mathfrak{S}(\mathcal{Y}).$$

This fact permits to introduce the following general definition:

A σ-homomorphism f of a closure algebra $\boldsymbol{A}$ in a closure algebra $\boldsymbol{B}$ is called *continuous* [21]) if $\overline{f(A)} \subset f(\overline{A})$ for every $A \in \boldsymbol{A}$.

This definition implies immediately:

5.1. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be two topological spaces and let* f *be a* σ*-homomorphism of* $\mathfrak{S}(\mathcal{Y})$ *in* $\mathfrak{S}(\mathcal{X})$ *induced by a mapping* φ *of* $\mathcal{X}$ *in* $\mathcal{Y}$ [22]). f *is continuous if and only if* φ *is continuous.*

An analogous theorem holds for homomorphisms of $\mathfrak{B}(\mathcal{Y})$ in $\mathfrak{S}(\mathcal{X})$ (or: in $\mathfrak{B}(\mathcal{X})$).

The following theorems are obvious [23]):

5.2. *In order that a* σ*-homomorphism* f *of* $\boldsymbol{A}$ *in* $\boldsymbol{B}$ *be continuous, it is necessary and sufficient that* $f(A)$ *be open (closed) in* $\boldsymbol{B}$ *for every open (closed) element* $A \in \boldsymbol{A}$.

5.3. *A* σ*-homomorphism* f *of* $\boldsymbol{A}$ *in* $\boldsymbol{B}$ *is continuous if and only if the homomorphism* $f|\mathfrak{B}(\boldsymbol{A})$ *(of* $\mathfrak{B}(\boldsymbol{A})$ *in* $\boldsymbol{B}$*) is continuous.*

5.4. *If* f *is a continuous homomorphism of* $\boldsymbol{A}$ *in* $\boldsymbol{B}$ *and* $A \in \boldsymbol{A}$, $B \in \boldsymbol{B}$, *then the homomorphisms* Bf *(of* $\boldsymbol{A}$ *in* $B\boldsymbol{B}$*) and* $f|A\boldsymbol{A}$ *(of* $A\boldsymbol{A}$ *in* $f(A)\boldsymbol{B}$*) are continuous also.*

Two closure algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ are *homeomorphic* if there exists an isomorphism h of $\boldsymbol{A}$ on $\boldsymbol{B}$ such that both h and h^{-1} are continuous. The isomorphism h is then called a *homeomorphism* (of $\boldsymbol{A}$ on $\boldsymbol{B}$).

[21]) Instead of „a continuous σ-homomorphism" we shall say simply „a continuous homomorphism".

[22]) If $\overline{\overline{\mathcal{Y}}}$ is less than the first inaccessible (in the strict sense) aleph, such a mapping φ exists always for any σ-homomorphism f of $\mathfrak{S}(\mathcal{Y})$ in $\mathfrak{S}(\mathcal{X})$. See Sikorski [2], p. 12.

[23]) In this section $\boldsymbol{A}$ and $\boldsymbol{B}$ denote always two closure algebras.

5.5. *An isomorphism h of $\boldsymbol{A}$ on $\boldsymbol{B}$ is a homeomorphism if and only if $h(\bar{A})=\overline{h(A)}$ for every $A\in\boldsymbol{A}$*[24].

5.6. *An isomorphism h of $\boldsymbol{A}$ on $\boldsymbol{B}$ is a homeomorphism of $\boldsymbol{A}$ on $\boldsymbol{B}$ if and only if $h|\mathfrak{B}(\boldsymbol{A})$ is a homeomorphism of $\mathfrak{B}(\boldsymbol{A})$ on $\mathfrak{B}(\boldsymbol{B})$.*

The following theorem says that the notion of the minimal closure extension of a closure algebra has a topological character.

5.7. *If closure algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ are homeomorphic, then $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{B}}$ are homeomorphic too.*

Let h be a homeomorphism of $\boldsymbol{A}$ on $\boldsymbol{B}$. The isomorphism h can be extended to an isomorphism h_0 of $\tilde{\boldsymbol{A}}$ on $\tilde{\boldsymbol{B}}$[25]). Since h_0 and h_0^{-1} are totally additive[26]), we infer easily that h_0 and h_0^{-1} fulfil the sufficient condition of theorem 5.2 (for open sets). Therefore h_0 is a homeomorphism of $\tilde{\boldsymbol{A}}$ on $\tilde{\boldsymbol{B}}$, q. e. d.

5.8. *Let h be an isomorphism of a closure algebra $\boldsymbol{A}$ in a σ-complete Boolean algebra $\boldsymbol{B}$ and let*

$$\text{(i)}\qquad \bar{B}=h(\overline{h^{-1}(B)})\quad \textit{for every}\quad B\in\boldsymbol{B}.$$

The closure operation (i) *fulfils the axioms* I-IV. *h is a homeomorphism of $\boldsymbol{A}$ on the closure algebra $\boldsymbol{B}$ with the closure operation* (i).

In fact, the isomorphism h and h^{-1} fulfil the sufficient condition of theorem 5.5. We shall say that the closure operation (i) is *induced* in $\boldsymbol{B}$ by the isomorphism h.

We shall say that two closure algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ are *weakly homeomorphic* if the closure algebras $\mathfrak{B}(\boldsymbol{A})$ and $\mathfrak{B}(\boldsymbol{B})$ are homeomorphic. By theorem 5.6, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are homeomorphic, they are also weakly homeomorphic.

5.9. *The following three conditions are equivalent for arbitrary topological spaces $\mathcal{X}$ and $\mathcal{Y}$:*

(i) *the spaces $\mathcal{X}$ and $\mathcal{Y}$ are homeomorphic;*

(ii) *the closure algebras $\mathfrak{S}(\mathcal{X})$ and $\mathfrak{S}(\mathcal{Y})$ are homeomorphic;*

(iii) *the closure algebras $\mathfrak{S}(\mathcal{X})$ and $\mathfrak{S}(\mathcal{Y})$ are weakly homeomorphic (i. e. $\mathfrak{B}(\mathcal{X})$ and $\mathfrak{B}(\mathcal{Y})$ are homeomorphic).*

[24]) See Kuratowski [1], p. 71 or Kuratowski [2], p. 78.

[25]) This follows immediately from the definition of minimal extensions given by Mac Neille in paper [1]. Another proof can be obtained with the help of a theorem on extension of homomorphisms. See Sikorski [6], p. 335 (th. (i) and (ii)).

[26]) I. e. $h_0(\sum_{A\in S}A)=\sum_{A\in S}h_0(A)$ for every set $S\subset\tilde{\boldsymbol{A}}$. This property of isomorphisms (in the case of fields of sets) was observed by E. Marczewski in paper [1], p. 135.

The implication (i)$\to$(ii) follows from 5.1. The implication (ii)$\to$(iii) holds for arbitrary closure algebras. If h is a homeomorphism of $\mathfrak{B}(\mathcal{Y})$ on $\mathfrak{B}(\mathcal{X})$, there exists a one-one mapping φ of $\mathcal{X}$ on $\mathcal{Y}$ such that φ induces h and φ^{-1} induces h^{-1}[27]). By 5.1 φ is a homeomorphism of $\mathcal{X}$ on $\mathcal{Y}$, thus the implication (iii)$\to$(i) is also true.

Theorem 5.9 makes it clear that the above-defined notions of a homeomorphism and of a weak homeomorphism are generalizations of the notion of a homeomorphism between two topological spaces.

6. *Non-continuous homomorphisms.* In this section we consider a closure algebra $\boldsymbol{A}$ with an enumerable basis $(R_1,R_2,...)$ and a σ-homomorphism f of $\boldsymbol{A}$ in a closure algebra $\boldsymbol{B}$. The symbol $D(f)$ will denote the element

$$D(f)=\sum_{n=1}^{\infty}(\overline{f(R_n)}-\mathrm{Int}\,(f(R_n))).$$

6.1. *For every open element $G\in\boldsymbol{A}$*

$$\overline{f(G)}-\mathrm{Int}\,(f(G))\subset D(f).$$

Let $G=\sum_{n=1}^{\infty}R_{m_n}$. We have

$$\overline{f(G)}-\mathrm{Int}\,(f(G))=\sum_{n=1}^{\infty}(\overline{f(R_{m_n})}-\mathrm{Int}\,(f(G)))\subset\sum_{n=1}^{\infty}(\overline{f(R_{m_n})}-\mathrm{Int}\,(f(R_{m_n})))\subset D(f),$$

q. e. d.

By 6.1 we obtain immediately

6.2. *$D(f)$ does not depend on the choice of a basis of $\boldsymbol{A}$. $D(f)$ is the sum of all elements $\overline{f(G)}-\mathrm{Int}\,(f(G))$ where $G\in\mathfrak{G}(\boldsymbol{A})$. $D(f)$ is also the sum of all elements $\overline{f(F)}-f(F)$ where $F\in\mathfrak{F}(\boldsymbol{A})$.*

The final remark follows from the fact that

$$\overline{f(F)}-f(F)=\overline{f(F')}-\mathrm{Int}\,(f(F')).$$

6.3. *f is continuous if and only if $D(f)=0$.*

It is easy to see that if $\boldsymbol{A}=\mathfrak{S}(\mathcal{Y})$ and $\boldsymbol{B}=\mathfrak{S}(\mathcal{X})$ (where $\mathcal{X}$ and $\mathcal{Y}$ are topological spaces) and if f is induced by a mapping φ, then $D(f)$ is the set of all points of the discontinuity of φ[28]). The element $D(f)$ is thus a generalization of the set of all points of the discontinuity of a mapping.

[27]) See Marczewski [1], p. 138.

[28]) See Kuratowski [1], p. 67 or Kuratowski [2], p. 73.

7. *The localization of topological properties.* In this section $\boldsymbol{A}$ denotes always a closure algebra with a basis $\boldsymbol{R}$. $\boldsymbol{I}$ is an ideal of $\boldsymbol{A}$.

Let A be an arbitrary element of $\boldsymbol{A}$. By theorem 3.2 there exists an open element G_0 which is the sum of all open elements G such that $GA \in \boldsymbol{I}$. The closed element G_0' will be denoted by A^*. In symbols:

$$A^* = (\sum G)' = (\sum R)', \quad \text{where } G \in \mathfrak{G}(\boldsymbol{A}),\ GA \in \boldsymbol{I},\ R \in \boldsymbol{R},\ RA \in \boldsymbol{I}.$$

It is easy to show that in the case $\boldsymbol{A} = \mathfrak{S}(\mathfrak{X})$ where $\mathfrak{X}$ is a topological space, the set A^* is the set of all points $x \in \mathfrak{X}$ at which the set $A \subset \mathfrak{X}$ does not possess the property $(\boldsymbol{I})$ in the sense defined by Kuratowski[29]). By an easy modification of the proofs given in Kuratowski [1][30]) we obtain the following formulas (for an arbitrary closure algebra $\boldsymbol{A}$ with a basis):

7.1. (i) $A^* \subset \bar{A}$; (ii) $A^{**} \subset A^*$;

(iii) $(A_1 + A_2)^* = A_1^* + A_2^*$; (iv) $A_1 \subset A_2$ *implies* $A_1^* \subset A_2^*$;

(v) $GA^* = G(GA)^*$ *for every* $G \in G(\boldsymbol{A})$.

All theorems given in Kuratowski [1] Chapter I, § 8 VII, § 9 III, § 10 V-VI, § 11 IV[31]) hold also for an arbitrary closure algebra with a basis.

7.2. *If an* $\mathfrak{m}$*-complete closure algebra* $\boldsymbol{A}$ *possesses a basis* $\boldsymbol{R}$ *of potency* $\leqslant \mathfrak{m}$ *and if* $\boldsymbol{I}$ *is an* $\mathfrak{m}$*-additive ideal of* $\boldsymbol{A}$, *then*

$$A - A^* \in \boldsymbol{I} \quad \textit{for every} \quad A \in \boldsymbol{A}.$$

The conditions: $A^* = 0$ *and* $A \in \boldsymbol{I}$ *are thus equivalent.*

Let $\boldsymbol{R}_0$ be the class of all $R \in \boldsymbol{R}$ such that $RA \in \boldsymbol{I}$. We have

$$(A^*)' = \sum_{R \in \boldsymbol{R}_0} R.$$

$\boldsymbol{I}$ being $\mathfrak{m}$-additive, we obtain

$$A - A^* = \sum_{R \in \boldsymbol{R}_0} RA \in \boldsymbol{I}, \qquad \text{q. e. d.}$$

[29]) See Kuratowski [1], p. 29 or Kuratowski [2], p. 34.
[30]) P. 30 (see also Kuratowski [2], p. 35).
[31]) Or: Kuratowski [2], the same paragraphs and sections.

8. *Boundary ideals.* An ideal $\boldsymbol{I}$ of a closure algebra $\boldsymbol{A}$ is called a *boundary ideal* if $\mathrm{Int}(A) = 0$ for every $A \in \boldsymbol{I}$, i. e. if $0 \in \boldsymbol{A}$ is the sole open element which belongs to $\boldsymbol{I}$.

We suppose in this section that $\boldsymbol{A}$ and $\boldsymbol{I}$ satisfy the assumptions of theorem 7.2.

8.1. *The following conditions are equivalent:*

(i) $\boldsymbol{I}$ *is a boundary ideal;*

(ii) $G \subset G^*$ *for every open element* G;

(iii) $G^* = \bar{G}$ *for every element* G;

(iv) $|\boldsymbol{A}|^* = |\boldsymbol{A}|$.

Proof. (i)→(ii). By 7.2 the open element $G - G^* \in \boldsymbol{I}$. Consequently $G - G^* = 0$, i. e. $G \subset G^*$.

(ii)→(iii). By (ii) and theorem 7.1 (i), we have $G \subset G^* \subset \bar{G}$. Hence $\bar{G} \subset G^* \subset \bar{G}$ since G^* is closed. Thus $\bar{G} = G^*$.

(iii)→(iv). Since $|\boldsymbol{A}|$ is both open and closed, $|\boldsymbol{A}|^* = |\boldsymbol{A}|$ on account of (iii).

(iv)→(i). If G is open and $G \in \boldsymbol{I}$, then $G \cdot |\boldsymbol{A}|^* = 0$ by the definition of $|\boldsymbol{A}|^*$. Since $|\boldsymbol{A}|^* = |\boldsymbol{A}|$, $G = 0$, q. e. d.

8.2. *Let* $E = |\boldsymbol{A}|^* - I$ *where* $I \in \boldsymbol{I}$. *The ideal* $E\boldsymbol{I}$ *is a boundary ideal of the closure algebra* $E\boldsymbol{A}$. *For every* $H \in \mathfrak{G}(\boldsymbol{A})$

$$EH^* = E\overline{EH}.$$

Since $E' \in \boldsymbol{I}$ on account of 7.2,

(i) $$A \in \boldsymbol{I} \text{ if and only if } EA \in E\boldsymbol{I}.$$

Suppose that an element $B \in E\boldsymbol{I}$ is open in $E\boldsymbol{A}$, i. e. $B = EG$ where $G \in \mathfrak{G}(A)$. By (i) $G \in \boldsymbol{I}$, hence $G \subset (|A|^*)' \subset E'$. Consequently $B = 0$. The first part of theorem 8.2 is proved.

It follows from (i) that

$$EH^* = E(\sum G)' = E(\sum EG)', \text{ where } G \in \mathfrak{G}(\boldsymbol{A}) \text{ and } GH \in \boldsymbol{I}$$
$$= E(\sum G)', \text{ where } G \in \mathfrak{G}(E\boldsymbol{A}) \text{ and } GEH \in E\boldsymbol{I}.$$

Thus the element $EH^* \in E\boldsymbol{A}$ is the result of the operation „*" on the element $EH \in E\boldsymbol{A}$ in the closure algebra $E\boldsymbol{A}$ with respect to the ideal $E\boldsymbol{I}$. Since EH is open in $E\boldsymbol{A}$ and $E \cdot \overline{EH}$ is the closure of EH in $E\boldsymbol{A}$ we obtain from 8.1 (iii) that $EH^* = E\overline{EH}$, q. e. d.

9. *The division of closure algebras by ideals.* In this section we suppose also that the considered closure algebra $\mathbf{A}$ and the ideal $\mathbf{I}$ fulfil the assumptions of theorem 7.2, i. e. that $\mathbf{A}$ is $\mathfrak{m}$-complete, $\mathbf{A}$ possesses a basis $\mathbf{R}$ of potency $\leqslant \mathfrak{m}$, and $\mathbf{I}$ is $\mathfrak{m}$-additive.

The following simple theorem gives a method for the construction of closure algebras:

9.1. *The class* $[\mathbf{R}] \subset \mathbf{A}/\mathbf{I}$ *fulfils the assumption of theorem* 2.2 *and induces in the Boolean algebra* $\mathbf{A}/\mathbf{I}$ *a closure operation which satisfies axioms* I-IV.

This follows from the fact that the operations „Σ" and „$\cdot$" are commutative with the operation „[ ]".

The closure algebra which we obtain from $\mathbf{A}/\mathbf{I}$ in the above-mentioned way will be denoted also by $\mathbf{A}/\mathbf{I}$[32]).

9.2. (i) $[\mathbf{R}]$ *is a basis of* $\mathbf{A}/\mathbf{I}$ *of potency* $\leqslant \mathfrak{m}$;

(ii) $\mathfrak{G}(\mathbf{A}/\mathbf{I}) = [\mathfrak{G}(\mathbf{A})]$; (iii) $\mathfrak{F}(\mathbf{A}/\mathbf{I}) = [\mathfrak{F}(\mathbf{A})]$;

(iv) $\mathfrak{B}(\mathbf{A}/\mathbf{I}) = [\mathfrak{B}(\mathbf{A})]$;

i. e. an element of $\mathbf{A}/\mathbf{I}$ *is respectively an open, closed, or Borel element of* $\mathbf{A}/\mathbf{I}$ *if and only if it possesses a representative which is respectively an open, closed, or Borel element of* $\mathbf{A}$.

In order to prove 9.2 it is sufficient to remark that the operations „Σ", „Π", and „$'$" are commutative with the operation „[ ]".

It follows from 9.2 (ii) and (iii) that the closure operation induced by $[\mathbf{R}]$ in $\mathbf{A}/\mathbf{I}$ does not depend on the choice of the basis $\mathbf{R}$ of $\mathbf{A}$.

9.3. *For every* $A \in \mathbf{A}$:

(i) $\overline{[A]} = [A^*]$; (ii) $\overline{[A]} \subset [\bar{A}]$;

(iii) $[\mathrm{Int}\,(A)] \subset \mathrm{Int}\,([A])$; (iv) $\mathrm{Fr}\,([A]) \subset [\mathrm{Fr}\,(A)]$.

The open element $(\overline{[A]})' \in \mathbf{A}/\mathbf{I}$ is the sum[33]) of all elements $[R]$ where $R \in \mathbf{R}$ and $[RA] = [R]\cdot[A] = 0$, i. e. $RA \in \mathbf{I}$. Hence

$$(\overline{[A]})' = \sum [R] = [\sum R] = [(A^*)'] = [A^*]'$$

where $R \in \mathbf{R}$ and $RA \in \mathbf{I}$.

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[32]) Let $\mathcal{R}$ denote the set of all real numbers and let $\mathbf{L}_0$ be the ideal of all sets of measure zero. The above-defined closure algebra $\mathfrak{S}(\mathcal{R})/\mathbf{L}_0$ is different from the closure algebra defined in par. 1.

[33]) See theorem 2.2.

This proves the property (i). The property (ii) follows from (i) and theorem 7.1 (i). The properties (iii) and (iv) are consequences of (ii).

9.4. *Let h be a homeomorphism of* $\mathbf{A}$ *on a closure algebra* $\mathbf{B}$ *and let* $\mathbf{J} = h(\mathbf{I})$. *Then* $\mathbf{A}/\mathbf{I}$ *and* $\mathbf{B}/\mathbf{J}$ *are homeomorphic too.*

Namely the isomorphism h_0 defined by the formula

$$h_0([A]) = [h(A)] \quad \text{for } A \in \mathbf{A}$$

is a homeomorphism of $\mathbf{A}/\mathbf{I}$ on $\mathbf{B}/\mathbf{J}$ on account of 9.2 (ii)-(iii) and 5.2.

Theorem 9.4 shows that the division of a closure algebra by an ideal is a topological operation.

9.5. (i) *The natural homomorphism of* $\mathbf{A}$ *on* $\mathbf{A}/\mathbf{I}$ *is continuous.*

(ii) *The natural isomorphism of* $E\mathbf{A}/E\mathbf{I}$ *on* $[E](\mathbf{A}/\mathbf{I})$ *(where* $E \in \mathbf{A}$*) is a homeomorphism.*

(iii) *The natural isomorphism of* $\mathfrak{B}(\mathbf{A})/(\mathbf{I}\cdot\mathfrak{B}(\mathbf{A}))$ *on* $\mathfrak{B}(\mathbf{A}/\mathbf{I}) = [\mathfrak{B}(\mathbf{A})]$ *is a homeomorphism.*

Theorem 9.5 is an easy consequence of theorems 9.2 (ii)-(iii) and 5.2.

9.6. *If* $\mathbf{I}$ *is a principal ideal of* $\mathbf{A}$, *i. e. if* $\mathbf{I} = E'\mathbf{A}$ *where* $E \in \mathbf{A}$, *then* $\mathbf{A}/\mathbf{I}$ *is homeomorphic to* $E\mathbf{A}$.

In fact, the isomorphism h defined by the formula

$$h([A]) = EA \quad \text{for every} \quad A \in \mathbf{A}$$

is a homeomorphism of $\mathbf{A}/\mathbf{I}$ on $E\mathbf{A}$ on account of 9.2 (ii)-(iii) and 5.2.

Theorem 9.6 shows that all closure algebras $E\mathbf{A}$ are particular cases of closure algebras of the form $\mathbf{A}/\mathbf{I}$.

9.7. *Let* $\mathbf{I}_1$ *be an* $\mathfrak{m}$*-additive ideal of* $\mathbf{A}/\mathbf{I}$ *and let* $\mathbf{I}_2$ *be the* $\mathfrak{m}$*-additive ideal of all* $A \in \mathbf{A}$ *such that* $[A] \in \mathbf{I}_1$. *Then the closure algebras* $(\mathbf{A}/\mathbf{I})/\mathbf{I}_1$ *and* $\mathbf{A}/\mathbf{I}_2$ *are homeomorphic.*

Let A be a representative of an element $C_2 \in \mathbf{A}/\mathbf{I}_2$, let C be the element of $\mathbf{A}/\mathbf{I}$ determined by A and let C_1 be the element of $(\mathbf{A}/\mathbf{I})/\mathbf{I}_1$ determined by C. It is easy to see that C_1 does not depend on the choice of the representative A and that the mapping

$$C_1 = h(C_2)$$

is an isomorphism of A/I_2 on $(A/I)/I_1$. If $C_2 \in \mathfrak{F}(A/I_2)$, we may suppose that $A \in \mathfrak{F}(A)$ (see theorem 9.2 (iii)). Consequently $C \in \mathfrak{F}(A/I)$ and $h(C_2) = C_1 \in \mathfrak{F}((A/I)/I_1)$. The isomorphism h is thus continuous by 5.2. If $C_1 \in \mathfrak{F}((A/I)/I_1)$, there exists an element $C \in \mathfrak{F}(A/I)$ which is a representative of C_1. Element C possesses a representative $A \in \mathfrak{F}(A)$. The element $C_2 \in A/I_2$ determined by A is thus closed in A/I_2. Since $C_2 = h^{-1}(C_1)$, the converse isomorphism h^{-1} is also continuous by 5.2, q. e. d.

Theorem 9.7 shows that the result of a double division of a closure algebra can be obtained by a single division.

II. *C*-algebras. Representation theorems.

10. ***The definition of C-algebras.*** A closure algebra A is called a *C-algebra* if it satisfies the following axiom

V. *There exists an enumerable sequence $\{R_n\}$ of open elements of A with the property:*

(*) *every open element $G \in A$ is the sum of all elements R_n such that $\bar{R}_n \subset G$.*

Every sequence $R_n \in \mathfrak{G}(A)$ possessing the property (*) is called a *C-basis* of A. Obviously *C*-basis of A is a basis of A in the sense defined [34]) in § 2.

C-algebras possess many properties of separable metric spaces and constitute a natural generalization of these spaces. In fact

10.1 *A topological space $\mathfrak{X}$ is separable and metrizable if and only if $\mathfrak{S}(\mathfrak{X})$ is a C-algebra.*

Indeed, every enumerable basis of a separable metric space possesses the property (*). Conversely, every topological space possessing a *C*-basis is separable and regular and therefore it is metrizable.

On the other hand, there exist *C*-algebras which are not weakly homeomorphic to a topological space. The construction of the *C*-algebras is given by the following theorems:

10.2. *If A is a C-algebra and if I is a σ-ideal of A, then A/I is also a C-algebra. If R is a C-basis of A, then $[R]$ is a C-basis of A/I.*

[34]) If a closure algebra A possesses a *C*-basis, every enumerable basis of A is a *C*-basis.

By theorem 9.2 (ii) every open element of A/I is of the form $[G]$ where $G \in \mathfrak{G}(A)$. By (*) there exists a subsequence $R_{m_n} \in R$ such that

$$G = \sum_{n=1}^{\infty} R_{m_n} \quad \text{and} \quad \bar{R}_{m_n} \subset G \qquad (n=1,2,\ldots).$$

Consequently

$$[G] = \sum_{n=1}^{\infty} [R_{m_n}]$$

and by theorem 9.3 (ii)

$$\overline{[R_{m_n}]} \subset [\bar{R}_{m_n}] \subset [G], \qquad \text{q. e. d.}$$

10.3. *Let I be a σ-ideal of subsets of a separable metric space $\mathfrak{X}$. Then $\mathfrak{S}(\mathfrak{X})/I$ is a C-algebra. $\mathfrak{S}(\mathfrak{X})/I$ is homeomorphic to a topological space if and only if I is principal. $\mathfrak{S}(\mathfrak{X})/I$ is weakly homeomorphic to a topological space if and only if $I \cdot \mathfrak{B}(\mathfrak{X})$ is a semi-principal* [35]) *ideal of $\mathfrak{B}(\mathfrak{X})$.*

The first remark follows from 10.1 and 10.2. If I is principal, i. e. $I = E'\mathfrak{S}(\mathfrak{X}) = \mathfrak{S}(E')$ (where $E \subset \mathfrak{X}$), then $\mathfrak{S}(\mathfrak{X})/I$ is homeomorphic to $E\mathfrak{S}(\mathfrak{X}) = \mathfrak{S}(E)$ by theorem 9.6. If I is not principal, then $\mathfrak{S}(\mathfrak{X})/I$ is no isomorph of a σ-field of sets [36]). Consequently $\mathfrak{S}(\mathfrak{X})/I$ is not homeomorphic to a topological space.

The final part of theorem 10.3 follows from theorem 10.5, which we shall prove below (see also theorem 9.5 (iii)).

10.4. *A closure algebra A is a C-algebra if and only if $\mathfrak{B}(A)$ is a C-algebra. Every C-basis of A is a C-basis of $\mathfrak{B}(A)$ and conversely.*

This follows from the fact $\mathfrak{G}(A) = \mathfrak{G}(\mathfrak{B}(A))$ for any closure algebra A.

10.5. *Let I be a σ-ideal of Borel subsets of a separable metric space $\mathfrak{X}$. Then $\mathfrak{B}(\mathfrak{X})/I$ is a C-algebra. $\mathfrak{B}(\mathfrak{X})/I$ is homeomorphic to a C-algebra of all Borel subsets of a topological space if and only if I is a semi-principal* [35]) *ideal of $\mathfrak{B}(\mathfrak{X})$.*

The first remark follows from theorems 10.1, 10.2, and 10.4. If I is semi-principal, i. e. I is formed of all Borel subsets of a set $E' \subset \mathfrak{X}$, then $\mathfrak{B}(\mathfrak{X})/I$ is homeomorphic to $\mathfrak{B}(E)$. If I is not semi-principal, then $\mathfrak{B}(\mathfrak{X})$ is not isomorphic to a σ-field of sets [36]). Therefore $\mathfrak{B}(\mathfrak{X})/I$ is not homeomorphic to a field of all Borel subsets of a topological space.

[35]) An ideal I of a field X (of subsets of $\mathfrak{X}$) is *semi-principal* if I is formed of all subsets of a set $X \subset \mathfrak{X}$ (X may belong to X or not).

[36]) Sikorski [1], p. 253.

10.6. *If $\boldsymbol{A}$ is a C-algebra and $E \in \boldsymbol{A}$, then $E\boldsymbol{A}$ is also a C-algebra. If $\boldsymbol{R}$ is a C-basis of $\boldsymbol{A}$, then $E\boldsymbol{R}$ is a C-basis of $E\boldsymbol{A}$* [37]).

Theorem 10.6 follows directly from the definition of C-algebras.

By 10.4 and 4.4 we obtain

10.7. *The minimal closure extension $\tilde{\boldsymbol{A}}$ of a C-algebra $\boldsymbol{A}$ is also a C-algebra.*

11. *Properties of C-algebras.* A closure algebra $\boldsymbol{A}$ is said to be *normal* provided that for any two closed elements F_1, F_2, $F_1F_2=0$, there exists an open element G such that $F_1 \subset G$ and $\bar{G}F_2=0$.

11.1. *Every C-algebra is normal* [38]).

This theorem is a particular case of the following theorem:

11.2 [39]). *Let A and B be two elements of a C-algebra $\boldsymbol{A}$ such that $\bar{A}B+A\bar{B}=0$. Then there exist two open elements G and H such that $A\subset G$, $B\subset H$, and $GH=0$.*

$\boldsymbol{A}$ being a C-algebra, we have

$$(\bar{B})'=\sum_{n=1}^{\infty} U_n \quad \text{and} \quad (\bar{A})'=\sum_{n=1}^{\infty} V_n$$

where U_n and V_n are open and

$$\bar{U}_n\bar{B}=0 \quad \text{and} \quad \bar{V}_n\bar{A}=0 \qquad (n=1,2,...).$$

Let $G_1=U_1$ and $H_1=V_1-\bar{G}_1$ and by induction:

$$G_n=U_n-\sum_{i=1}^{n-1}\bar{H}_i, \quad H_n=V_n-\sum_{i=1}^{n}\bar{G}_n.$$

The elements $G=\sum_{n=1}^{\infty} G_n$ and $H=\sum_{n=1}^{\infty} H_n$ fulfil the condition of theorem 11.1. In fact

$$A\subset\sum_{n=1}^{\infty} U_n-\sum_{n=1}^{\infty}\bar{H}_n\subset G$$

[37]) In contrast to theorem 4.3, a closure subalgebra of a C-algebra is not, in general, a C-algebra. For example, the sets: $\mathcal{R}$, $\mathcal{R}-(1)$, (1), and the empty set constitute a closure subalgebra of $\mathfrak{S}(\mathcal{R})$. This subalgebra is not a C-algebra ($\mathcal{R}$=the set of all real numbers).

[38]) On the other hand, there exist normal closure algebras with an enumerable basis which are not C-algebras. An example of such a closure algebra is given in footnote [37]).

[39]) The proof of this theorem is an easy modification of the proof of a theorem on regular spaces. See Kuratowski [1], p. 102 or Kuratowski [2], p. 133.

since $A\subset\sum_{n=1}^{\infty} U_n$ and $\bar{H}_n\bar{A}\subset\bar{V}_n\bar{A}=0$. Analogously $B\subset H$. For $n\leqslant m$ we have $H_mG_n\subset(V_n-\bar{G}_n)G_n=0$ and for $n>m$ we have

$$H_mG_n\subset H_m(U_n-\bar{H}_m)=0.$$

Therefore $GH=0$, q. e. d.

Consequently all theorems given in Kuratowski [1] Chapter II § 16 II, § 16 V [40]) hold for an arbitrary C-algebra too.

By axiom V every open element of a C-algebra $\boldsymbol{A}$ is an F_σ. In fact, if $G=\sum_{n=1}^{\infty} R_{m_n}$ where $\bar{R}_{m_n}\subset G$, then $G=\sum_{n=1}^{\infty}\bar{R}_{m_n}$ too. Consequently every closed element of $\boldsymbol{A}$ is G_δ. Therefore all theorems on Borel sets given in Kuratowski [1], Chapter II § 26 I-IX [41]), are also true for arbitrary C-algebras.

In particular we obtain

11.3. *Each developable element of a C-algebra is both an F_σ and a G_δ.*

Since the operations „$\sum_{n=1}^{\infty}$" and „$\prod_{n=1}^{\infty}$" are commutative with the operation „[ ]", it follows from 9.2 (ii)-(iii):

11.4. *Let $\boldsymbol{I}$ be a σ-ideal of a C-algebra $\boldsymbol{A}$. In order that an element $B\in\boldsymbol{A}/\boldsymbol{I}$ be a Borel element of an additive (multiplicative) class α, it is necessary and sufficient that B possess a representative $A\in\boldsymbol{A}$ which is a Borel element of an additive (multiplicative) class α in $\boldsymbol{A}$.*

12. *C-bases.* In this section we shall prove several simple lemmas which will be useful later.

The following lemma is obvious:

12.1. *If a closure algebra $\boldsymbol{A}$ possesses an enumerable basis $R_1, R_2, ...$ such that*

(i) *for every positive integer n there exists a finite or infinite subsequence $\{R_{k_i}\}$ such that $R_n=\sum_i R_{k_i}$ and $\bar{R}_{k_i}\subset R_n$;*

then $\boldsymbol{A}$ is a C-algebra and $R_1, R_2, ...$ is a C-basis of $\boldsymbol{A}$.

12.2. *If there exists a decomposition $|\boldsymbol{A}|=A_1+A_2$ of a closure algebra $\boldsymbol{A}$ such that*

(i) *A_1 and A_2 are both open and closed in $\boldsymbol{A}$;*

(ii) *$A_1\boldsymbol{A}$ and $A_2\boldsymbol{A}$ are C-algebras;*

then $\boldsymbol{A}$ is also a C-algebra.

[40]) Or: Kuratowski [2], § 16 II and § 16 V (theorems 1-6).

[41]) See also Kuratowski [2], § 26 I-VIII and IX (theorem 1).

For if $\boldsymbol{R}_i$ is a C-basis of $A_i\boldsymbol{A}$ $(i=1,2)$, then $\boldsymbol{R}_1+\boldsymbol{R}_2$ is a C-basis of $\boldsymbol{A}$.

12.3. *Let a sequence $S_1, S_2, \ldots$ of elements of a σ-complete Boolean algebra $\boldsymbol{A}$ possess the properties:*

(i) *the element $0 \in \boldsymbol{A}$ is a term of this sequence;*

(ii) $|\boldsymbol{A}|=\sum_{n=1}^{\infty} S_n$;

(iii) *for every pair (i,j) of positive integers there exists a set $N_{i,j}$ of positive integers such that $S_iS_j=\sum_{k \in N_{i,j}} S_k$;*

(iv) *if N_i^0 denotes the set of all integers j such that $S_iS_j=0$ and if N_k denotes the set of all integers i such that*

$$\Big(\sum_{j \in N_i^0} S_j\Big)' \subset S_k,$$

then

$$S_k=\sum_{i \in N_k} S_i.$$

Then the class $(S_1, S_2, \ldots)$ fulfils the assumptions of theorem 2.2 and induces in $\boldsymbol{A}$ a closure operation which satisfies axioms I-V. $(S_1, S_2, \ldots)$ is a C-basis of the C-algebra which we obtain from $\boldsymbol{A}$ in this way.

It is sufficient to prove that axiom V is satisfied. By the definition of closure operation (see theorem 2.2)

$$\sum_{j \in N_i^0} S_j = \mathrm{Int}\,(S_i').$$

Thus N_k is the set of all integers i such that

$$\bar{S}_i = (\mathrm{Int}\,(S_i'))' \subset S_k.$$

By (iv)

$$S_k=\sum_{i \in N_k} S_i.$$

The sequence $S_1, S_2, \ldots$ fulfilling the assumption of theorem 12.1, $\boldsymbol{A}$ is a C-algebra, q. e. d.

It is easy to see that every C-basis of a C-algebra possesses the properties (i)-(iv). Thus theorem 12.3 gives a necessary and sufficient condition for a sequence $S_1, S_2, \ldots$ to be a C-basis of a C-algebra.

13. *C-fields.* If a C-algebra is a σ-field of sets, it is called a *C-field*. For instance, if $\mathcal{Y}$ is a metric separable space, $\mathfrak{S}(\mathcal{Y})$ and $\mathfrak{B}(\mathcal{Y})$ are C-fields on account of 10.1 and 10.4. Conversely

13.1. *Every C-field is weakly homeomorphic to a separable metric space.*

More exactly:

For every C-field $\boldsymbol{X}$ (of subsets of a set $\mathfrak{X}$) there exist a metrizable separable space $\mathcal{Y}$ and a mapping φ of $\mathfrak{X}$ on $\mathcal{Y}$ such that the equation

(i) $$h(X)=\varphi(X) \qquad (\textit{for } X \in \mathfrak{B}(\boldsymbol{X}))$$

defines a homeomorphism h of $\mathfrak{B}(\boldsymbol{X})$ on $\mathfrak{B}(\mathcal{Y})$.

Let $\{R_n\}$ be a C-basis of $\boldsymbol{X}$ and for every $x \in \mathfrak{X}$ let $C(x)$ denote the common part of all sets $\bar{R}_n$ which contain x. Let $\mathcal{Y}$ be the class of all sets $C(x)$ and let $\varphi(x)=C(x)$ for every $x \in \mathfrak{X}$.

The closed sets $C(x)$ are atoms of $\mathfrak{B}(\boldsymbol{X})$ and for every $X \in \mathfrak{B}(\boldsymbol{X})$ $\varphi(X)$ is the class of all atoms contained in X. Therefore the formula (i) defines an isomorphism h of $\mathfrak{B}(\boldsymbol{X})$ on a σ-field $\boldsymbol{Y}$ of subsets of $\mathcal{Y}$ [42]. The field $\boldsymbol{Y}$ with the closure operation induced by the isomorphism h [43] is a closure algebra homeomorphic to $\mathfrak{B}(\boldsymbol{X})$. Since $\mathfrak{B}(\boldsymbol{X})$ is a C-algebra by 10.4, $\boldsymbol{Y}$ is also a C-algebra. Since $(y) \in \boldsymbol{Y}$ for every $y \in \mathcal{Y}$, $\mathfrak{S}(\boldsymbol{Y})$ is the minimal closure extension of $\boldsymbol{Y}$ [44] and $\mathfrak{B}(\mathfrak{S}(\mathcal{Y}))=\mathfrak{B}(\boldsymbol{Y})=\boldsymbol{Y}$ by theorem 4.4. Since every set $C(x)$ is closed in $\boldsymbol{X}$, $\mathfrak{S}(\mathcal{Y})$ satisfies Kuratowski's axiom: $\overline{(y)}=(y)$ for every $y \in \mathcal{Y}$. Thus the set $\mathcal{Y}$ with the above-defined closure operation is a topological space. $\mathfrak{B}(\mathcal{Y})=\boldsymbol{Y}$ being a C-algebra, $\mathcal{Y}$ is a metrizable separable space on account of theorems 10.1 and 10.4. By 5.8 h is a homeomorphism of $\mathfrak{B}(\boldsymbol{X})$ on $\mathfrak{B}(\mathcal{Y})=\boldsymbol{Y}$, q. e. d.

Theorem 13.1 explains the structure of C-fields. The general method for the construction of C-fields is the following:

Let $\mathcal{Y}$ be a metric separable space and let g be an isomorphism of $\mathfrak{B}(\mathcal{Y})$ on a σ-field $\boldsymbol{X}_0$ of subsets of a set $\mathfrak{X}$. Let $\boldsymbol{X}$ be now an arbitrary σ-field of subsets of $\mathfrak{X}$ such that $\boldsymbol{X}_0$ is a subalgebra of $\boldsymbol{X}$. For every $X \in \boldsymbol{X}$ let $\bar{X}$ be the product of all sets $g(Y)$ such that $X \subset g(Y)$ and $Y \in \mathfrak{F}(\mathcal{Y})$. $\boldsymbol{X}$ is a C-field.

Conversely, every C-field $\boldsymbol{X}$ can be obtained by the above construction on account of 13.1.

[42]) See Sikorski [1], p. 251, th. 1.8.

[43]) See theorem 5.8.

[44]) The closure $\bar{Y}$ of a set $Y \subset \mathfrak{X}$ is the product of all sets $\varphi(X)$ where $X \in \mathfrak{F}(\boldsymbol{X})$ and $Y \subset \varphi(X)$.

14. *A representation theorem on $\dot{C}$-algebras.* It is well known that every σ-complete Boolean algebra $\boldsymbol{A}$ is isomorphic to a quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ where $\boldsymbol{X}$ is a σ-field and $\boldsymbol{I}$ is a σ ideal of $\boldsymbol{X}$. The question arises whether every C-algebra is homeomorphic to a closure algebra $\boldsymbol{X}/\boldsymbol{I}$ where $\boldsymbol{X}$ is a C-field and $\boldsymbol{I}$ is a σ-ideal of $\boldsymbol{X}$. The answer is affirmative (theorem 14.2).

14.1. *If a C-algebra $\boldsymbol{A}$ is of the form*

$$\boldsymbol{A}=\boldsymbol{B}/\boldsymbol{I}$$

where $\boldsymbol{B}$ is a σ-complete Boolean algebra and $\boldsymbol{I}$ is a σ-ideal of $\boldsymbol{B}$, then it is possible to define in $\boldsymbol{B}$ a closure operation in such a way that

(i) *$\boldsymbol{B}$ with this closure operation is a C-algebra;*

(ii) *the C-algebra $\boldsymbol{A}$ is identical with the closure algebra which we obtain by the division of the C-algebra $\boldsymbol{B}$ by the ideal $\boldsymbol{I}$* (see theorem 9.1).

Let $\{R_n\}$ be a C-basis of $\boldsymbol{A}$. We may assume that

$$R_1=0,\quad R_2=|A|,\quad \text{and}\quad R_n\neq 0\quad \text{for}\quad n\geqslant 2. \tag{1}$$

Let $N_{i,j}$ denote the set of all positive integers k such that $R_k\subset R_iR_j$, let N_i^0 denote the set of all j such that

$$R_iR_j=0, \tag{2}$$

and let N_k denote the set of all integers i such that

$$\Big(\sum_{j\in N_i^0} R_j\Big)'\subset R_k, \tag{3}$$

this means:

$$\bar{R}_i\subset R_k. \tag{4}$$

Obviously

$$R_iR_j=\sum_{k\in N_{i,j}} R_k \qquad (i,j=1,2,\ldots); \tag{5}$$

$$R_k=\sum_{i\in N_k} R_i \qquad (k=1,2,\ldots). \tag{6}$$

Let $\{B_n\}$ be a sequence of elements of $\boldsymbol{B}$ such that

$$R_n=[B_n] \qquad (n=1,2,\ldots), \tag{7}$$

and in particular

$$B_1=0\quad B_2=|\boldsymbol{B}|. \tag{8}$$

Let

$$B=\sum_{i,j=1}^{\infty}\Big(\Big(B_iB_j-\sum_{k\in N_{i,j}} B_k\Big)+\Big(\sum_{k\in N_{i,j}} B_k-B_iB_j\Big)\Big)+ \\ +\sum_{k=1}^{\infty}\sum_{i\in N_k}\Big(\Big(\sum_{j\in N_i^0} B_j\Big)'-B_k\Big)+\sum_{k=1}^{\infty}\Big(B_k-\sum_{i\in N_k} B_i\Big). \tag{9}$$

On account of (3), (5), (6), and (7) we have

$$[B_i\cdot B_j]=\Big[\sum_{k\in N_{i,j}} B_k\Big];$$

$$\Big[\Big(\sum_{j\in N_i^0} B_j\Big)'\Big]\subset[B_k]\qquad \text{for } i\in N_k;$$

$$[B_k]=\Big[\sum_{i\in N_k} B_i\Big].$$

Consequently

$$B\in \boldsymbol{I}. \tag{10}$$

Let

$$S_n=B'B_n \qquad (n=1,2,\ldots). \tag{11}$$

By (7) and (10)

$$R_n=[S_n] \qquad (n=1,2,\ldots). \tag{12}$$

Consider the Boolean algebra $B'\boldsymbol{B}$. By (8) we have

$$S_1=0, \tag{13}$$

$$S_2=B'=|B'\boldsymbol{B}|. \tag{14}$$

On account of (9) and (11)

$$S_iS_j=\sum_{k\in N_{i,j}} S_k \qquad (i,j=1,2,\ldots). \tag{15}$$

If $S_iS_j=0$, then $R_iR_j=0$ too by (12); therefore $j\in N_i^0$. Conversely, if $j\in N_i^0$, then $R_iR_j=0$, i. e. $N_{i,j}$ contains only one number 1 (see (1)). On account of (15) $S_iS_j=S_1=0$. Consequently

(16) *N_i^0 is the set of all positive integers j such that $S_iS_j=0$.*

If

$$B'\Big(\sum_{j\in N_i^0} S_j\Big)'\subset S_k,$$

then by (10) and (12)

$$\Big(\sum_{j\in N_i^0} R_j\Big)'\subset R_k.$$

Hence $i \in N_k$. Conversely, if $i \in N_k$, then by (9) and (11)

$$B'(\sum_{j \in N_i^0} S_j)' - S_k = B'((\sum_{\in N_i^0} B_j)' - B_k) \subset B'B = 0$$

i. e.

$$B'(\sum_{j \in N_i^0} S_j)' \subset S_k.$$

Consequently

(17) *N_k is the set of all positive integers i such that*

$$B'(\sum_{j \in N_i^0} S_j)' \subset S_k.$$

By (16) and (11)

$$S_i \subset B'(\sum_{j \in N_i^0} S_j)'.$$

Hence if $i \in N_k$, then $S_i \subset S_k$ by (17). Therefore

$$\sum_{i \in N_k} S_i \subset S_k.$$

By (9) and (11)

$$S_k - \sum_{i \in N_k} S_i = B'(B_k - \sum_{i \in N_k} B_i) \subset B'B = 0.$$

Consequently

(18) $$S_k = \sum_{i \in N_k} S_i.$$

We infer from (11), (13), (14), and (15) that the class $\boldsymbol{S} = (B, S_1 S_2, S_3, \ldots)$ fulfils the assumptions of theorem 2.2. Let us consider $\boldsymbol{B}$ as a closure algebra with the closure operation induced by the basis $\boldsymbol{S}$.

By 2.1 the class $B'\boldsymbol{S} = (0, S_1, S_2, S_3, \ldots) = (S_1, S_2, S_3, \ldots)$ is a basis of a closure algebra $B'\boldsymbol{B}$. Since $B'(\sum_{j \in N_i^0} S_j)'$ is the complement of the element $\sum_{j \in N_i^0} S_j$ in the Boolean algebra $B'\boldsymbol{B}$, the sequence $S_1, S_2, S_3, \ldots$, the algebra $B'\boldsymbol{B}$, and the sets $N_{i,j}$, N_i^0, N_k fulfil the assumptions of theorem 12.3 on account of (13), (14), (15), (16), (17), and (18). Thus $B'\boldsymbol{B}$ is a C-algebra.

The basis of $B\boldsymbol{B}$ is the class $(0, B)$, thus $B\boldsymbol{B}$ is also a C-algebra. Since the elements B and B' are both open and closed in $\boldsymbol{B}$ by (14), $\boldsymbol{B}$ is a C-algebra on account of theorem 12.2.

By (12) and theorem 10.2, $\{R_n\}$ is a basis of a C-algebra $\boldsymbol{B}/\boldsymbol{I}$ which obtain by the division of $\boldsymbol{B}$ by $\boldsymbol{I}$. Thus the C-algebra $\boldsymbol{A}$ is identical with the C-algebra $\boldsymbol{B}/\boldsymbol{I}$, q. e. d.

14.2. *Every C-algebra $\boldsymbol{A}$ is homeomorphic to a C-algebra $\boldsymbol{X}/\boldsymbol{I}$ where $\boldsymbol{X}$ is a C-field and $\boldsymbol{I}$ is a σ-ideal*[45]).

In fact, the Boolean algebra $\boldsymbol{A}$ is isomorphic to a σ-quotient algebra $\boldsymbol{B} = \boldsymbol{X}/\boldsymbol{I}$. The isomorphism h of $\boldsymbol{A}$ on $\boldsymbol{B}$ induces in $\boldsymbol{B}$ a closure operation in such a way that $\boldsymbol{B}$ is isomorphic to $\boldsymbol{A}$ (see theorem 5.8). By 14.1 we can define on $\boldsymbol{X}$ a closure operation in such a way that the closure algebra $\boldsymbol{B}$ is identical with the C-algebra $\boldsymbol{X}/\boldsymbol{I}$ which we obtain by the division of the C-field $\boldsymbol{X}$ by $\boldsymbol{I}$. Consequently $\boldsymbol{A}$ is homeomorphic to $\boldsymbol{X}/\boldsymbol{I}$, q. e. d.

15. *A generalization of Urysohn's theorem. Compact C-algebras.*

If we replace the term „homeomorphic" by the term „weakly homeomorphic", we can formulate the representation theorem for C-algebras in the following way:

15.1. *For every C-algebra $\boldsymbol{A}$ there exist a metric separable space $\mathcal{Y}$ and a σ-ideal $\boldsymbol{J}$ of $\mathfrak{S}(\mathcal{Y})$ such that $\boldsymbol{A}$ is weakly homeomorphic to the C-algebra $\mathfrak{S}(\mathcal{Y})/\boldsymbol{J}$ (i. e. $\mathfrak{B}(\boldsymbol{A})$ is homeomorphic to $\mathfrak{B}(\mathcal{Y})/\boldsymbol{J}_0$ where $\boldsymbol{J}_0 = \boldsymbol{J} \cdot \mathfrak{B}(\mathcal{Y})$ is a σ-ideal of $\mathfrak{B}(\mathcal{Y})$*[46]).

By 14.2 $\boldsymbol{A}$ is homeomorphic to $\boldsymbol{X}/\boldsymbol{I}$ where $\boldsymbol{X}$ is a C-field and $\boldsymbol{I}$ is a σ-ideal of $\boldsymbol{X}$. By 13.1 $\boldsymbol{X}$ is weakly homeomorphic to a metric separable space $\mathcal{Y}$. Let h be a homeomorphism of $\mathfrak{B}(\boldsymbol{X})$ on $\mathfrak{B}(\mathcal{Y})$ and let $\boldsymbol{J}_0 = h(\boldsymbol{I} \cdot \mathfrak{B}(\boldsymbol{X}))$. By 9.4 the C-algebra $\mathfrak{B}(\boldsymbol{X}/\boldsymbol{I} \cdot \mathfrak{B}(\boldsymbol{X})$ is homeomorphic to $\mathfrak{B}(\mathcal{Y})/\boldsymbol{J}_0$. Hence by 9.5 (iii) $\mathfrak{B}(\boldsymbol{X}/\boldsymbol{I})$ is homeomorphic to $\mathfrak{B}(\mathfrak{S}(\mathcal{Y})/\boldsymbol{J})$ where $\boldsymbol{J}$ denotes an ideal of $\mathfrak{S}(\mathcal{Y})$ such that $\boldsymbol{J}_0 = \boldsymbol{J} \cdot \mathfrak{B}(\mathcal{Y})$[47]). Consequently $\boldsymbol{A}$ is weakly homeomorphic to $\mathfrak{S}(\mathcal{Y})/\boldsymbol{J}$, q. e. d.

15.2. *For every C-algebra $\boldsymbol{A}$ there exist a σ-ideal $\boldsymbol{I}$ of subsets of the Hilbert cube $\mathcal{H}$ such that $\boldsymbol{A}$ is weakly homeomorphic to $\mathfrak{S}(\mathcal{H})/\boldsymbol{I}$ (i. e. $\mathfrak{B}(\boldsymbol{A})$ is homeomorphic to $\mathfrak{B}(\mathcal{H})/\boldsymbol{I}_0$ where $\boldsymbol{I}_0 = \boldsymbol{I} \cdot \mathfrak{B}(\mathcal{H})$ is a σ-ideal of $\mathfrak{B}(\mathcal{H})$)*[46]).

Let $\mathcal{Y}$ and $\boldsymbol{J}$ have the same meaning as in theorem 15.1. We may assume that $\mathcal{Y} \subset \mathcal{H}$. Let $\boldsymbol{I}$ denote the σ-ideal of all subsets Y

[45]) An analogous theorem holds for closure algebras with an enumerable basis.

[46]) See theorem 9.5 (iii).

[47]) E. g. $\boldsymbol{J}$ is the ideal of all subsets of sets $Y \in \boldsymbol{J}_0$.

of $\mathcal{H}$ such that $Y\mathcal{Y} \in \boldsymbol{J}$. We have: $\mathcal{Y}\boldsymbol{I}=\boldsymbol{J}$. By 9.2 (ii) the C-algebra $\mathfrak{S}(\mathcal{Y})\boldsymbol{J}=\mathcal{Y}\mathfrak{S}(\mathcal{H})/\mathcal{Y}\boldsymbol{I}$ is homeomorphic to $[\mathcal{Y}]\mathfrak{S}(\mathcal{H})/\boldsymbol{I}=\mathfrak{S}(\mathcal{H})/\boldsymbol{I}$ since $\mathcal{H}-\mathcal{Y} \in \boldsymbol{I}$, i. e. $[\mathcal{Y}]=[\mathcal{H}]$. Therefore $\boldsymbol{A}$ is weakly homeomorphic to $\mathfrak{S}(\mathcal{H})/\boldsymbol{I}$, q. e. d.

We have deduced here theorem 15.2 from Urysohn's metrization theorem. Conversely, Urysohn's theorem can be deduced from theorem 15.2.

In fact, let $\mathcal{X}$ be a regular topological space with an enumerable basis $\boldsymbol{R}$. It is easy to see that $\boldsymbol{R}$ is a C-basis. Consequently $\mathfrak{S}(\mathcal{X})$ is a C-algebra. By 15.2 $\mathfrak{B}(\boldsymbol{X})$ is homeomorphic to $\mathfrak{B}(\mathcal{H})/\boldsymbol{I}_0$ where $\boldsymbol{I}_0$ is a σ-ideal of $\mathfrak{B}(\mathcal{H})$ Since $\mathfrak{B}(\boldsymbol{X})$ is a σ-field of sets, the ideal $\boldsymbol{I}_0$ is semi-principal, i. e. $\boldsymbol{I}_0$ is formed of all sets $Y \in \mathfrak{B}(\mathcal{H})$ which are contained in a set $H\subset\mathcal{H}$. Thus $\mathfrak{B}(\mathcal{X})$ is homeomorphic to $\mathfrak{B}(\mathcal{H}-H)$, i. e. $\mathcal{X}$ is homeomorphic to $\mathcal{H}-H\subset\mathcal{H}$ on account of 5.9.

A C-algebra $\boldsymbol{A}$ is said to be *compact*, if $\prod_{n=1}^{\infty} F_n \neq 0$ for every sequence $F_n \in \mathfrak{F}(\boldsymbol{A})$ such that $F_{n+1}\subset F_n\neq 0$ $(n=1,2,3,\ldots)$.

15.3. *Every compact C-algebra $\boldsymbol{A}$ is weakly homeomorphic to a compact metric space.*

By 15.2, $\mathfrak{B}(\boldsymbol{A})$ is homeomorphic to $\mathfrak{B}(\mathcal{H})/\boldsymbol{I}_0$ where $\boldsymbol{I}_0$ is a σ-ideal of $\mathfrak{B}(\mathcal{H})$ Let (x_0) be a one-element set belonging to $\boldsymbol{I}_0$, and let K_n be the set of all $x \in \mathcal{H}$ such that $\varrho(x,x_0)\leqslant(1/n)$. We have:

$$[K_{n+1}]\subset[K_n] \in \mathfrak{F}(\mathfrak{B}(\mathcal{H})/\boldsymbol{I}_0) \quad \text{and} \quad \prod_{n=1}^{\infty}[K_n]=[(x_0)]=0.$$

$\mathfrak{B}(\mathcal{H})/\boldsymbol{I}_0$ being compact, we infer that $[K_n]=0$ for an integer n, i. e. $K_n \in \boldsymbol{I}_0$.

Thus the ideal $\boldsymbol{I}_0$ is formed of all Borel subsets of an open set $H_0\subset\mathcal{H}$. Consequently $\mathfrak{B}(\mathcal{H})/\boldsymbol{I}_0$ is homeomorphic to $\mathfrak{B}(\mathcal{H}-H_0)$, i. e. $\boldsymbol{A}$ is weakly homeomorphic to the compact metric space $\mathcal{H}-H_0$.

III. Homomorphisms.

16. *The convergence in metric spaces.* Not all definitions of topological notions can be generalized to the case of C-algebras in such a simple way as the definitions considered in the preceding paragraphs. For instance, in order to obtain a definition of a convergent sequence of homomorphisms we must first formulate the definitions of a convergent sequence of mappings in a special way (see 16.2 and 16.4). This is the subject of this paragraph.

A sequence $\{G_n\}$ of open subsets of a metric space $\mathcal{Y}$ is said to be *strictly decreasing* if $\overline{G}_{n+1}\subset G_n$ for $n=1,2,\ldots$

16.1. *In order that a sequence $\{y_n\}$ of points of a metric space $\mathcal{Y}$ converge to a point $y \in \mathcal{Y}$ it is necessary and sufficient that for every strictly decreasing sequence $\{G_n\}$ of open sets the two following sentences be equivalent:*

(i) $y \in \prod_{n=1}^{\infty} G_n$;

(ii) *for every positive integer n there exists a positive integer k such that $y_{n+k} \in G_n$.*

Necessity. Let $y=\lim y_n$. If $y \in \prod_{n=1}^{\infty} G_n$, then $y \in G_n$. G_n being a neighbourhood of y, almost all points y_n belong to G_n. Therefore there exists a k such that $y_{n+k} \in G_n$.

If y non $\in \prod_{n=1}^{\infty} G_n$, then there exists an integer m_0 such that y non $\in \overline{G}_{m_0}$. Consequently there exists an integer $n_0\geqslant m_0$ such that

$$y_{n_0+k} \text{ non } \in \overline{G}_{m_0} \quad \text{for } k=1,2,\ldots$$

Since $G_{n_0}\subset\overline{G}_{m_0}$ we obtain

$$y_{n_0+k} \text{ non } \in G_{n_0} \quad \text{for } k=1,2,\ldots,$$

q. e. d.

Sufficiency. Suppose that (i) is equivalent to (ii). We shall prove first that

a) *Every subsequence $\{y_{m_n}\}$ contains a convergent subsequence $\{y_{m_{k_n}}\}$.*

Suppose that a subsequence $\{y_{m_n}\}$ does not possess the property a). Let

$$G_n=\sum_{k=1}^{\infty} K\left(y_{m_{n+k}}, \frac{1}{(k+n)\cdot n}\right)$$

where $K(a,r)$ denotes an open sphere with the center a and the radius r. $\{G_n\}$ is a strictly decreasing sequence of open sets and $\prod_{n=1}^{\infty} G_n=0$. (i) is false and (ii) is true since for every n and $k=m_{n+1}-n>0$ we have $y_{n+k} \in G_n$.

The property a) is proved. Now we shall prove that

b) *Every convergent subsequence $\{y_{m_n}\}$ converges to y.*

Let $y_0 = \lim_{n=\infty} y_{m_n}$ and $G_n = K(y_0, 1/n)$. Then (ii) is true; hence by (i) $y \in (y_0)$, i. e. $y = y_0$, q. e. d.

It follows directly from a) and b) that $y = \lim_{n=\infty} y_n$.

Thus theorem 16.1 is established. As an immediate consequence of 16.1 we obtain

16.2. *Let φ and φ_n $(n=1,2,\ldots)$ be mappings of an abstract set $\mathfrak{X}$ in a metric space $\mathcal{Y}$. In order that $\varphi = \lim_{n=\infty} \varphi_n$ it is necessary and sufficient that*

$$\prod_{n=1}^{\infty} \varphi^{-1}(G_n) = \prod_{n=1}^{\infty} \sum_{k=1}^{\infty} \varphi_{n+k}^{-1}(G_n)$$

for every strictly decreasing sequence $\{G_n\}$ of open sets [48]).

In the case where $\mathcal{Y}$ is compact the necessary and sufficient conditions of 16.1 and 16.2 can be replaced by weaker conditions.

If $\{R_i\}$ is a basis of $\mathcal{Y}$, then $R_{i,n}$ denotes the set of all $y \in \mathcal{Y}$ whose distance from R_i is less than $1/n$.

16.3. *Let $\{R_i\}$ be a basis of a compact metric space $\mathcal{Y}$. In order that a sequence $\{y_n\}$ converge to a point $y \in \mathcal{Y}$ it is necessary and sufficient that for every positive integer i the two following sentences are equivalent:*

(i) $y \in \bar{R}_i$;

(ii) *for every positive integer n there exists a positive integer k such that $y_{n+k} \in R_{i,n}$.*

The necessity follows from 16.1 since the sequence $R_{i,n}$ $(n=1,2,\ldots)$ is a strictly decreasing sequence of open sets and $\bar{R}_i = \prod_{n=1}^{\infty} R_{i,n}$.

Suppose now that y is not the limit of $\{y_n\}$. $\mathcal{Y}$ being compact, there exists a convergent subsequence $\{y_{m_n}\}$ such that

$$\lim_{n=\infty} y_{m_n} = y_0 \neq y.$$

Let i_0 be an integer such that

$$y_0 \in \bar{R}_{i_0} \quad \text{and} \quad y \text{ non} \in \bar{R}_{i_0}.$$

Then (ii) is true (for the integer i_0) and (i) is false. The sufficiency is proved.

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[48]) See Kuratowski [1], p. 185, Kuratowski [2], p. 308 and Hausdorff [1], p. 267.

It follows from 16.3 that

16.4 *Let φ and φ_n $(n=1,2,\ldots)$ be mappings of an abstract set $\mathfrak{X}$ in a compact metric space $\mathcal{Y}$ with a basis $\{R_i\}$. In order that $\varphi = \lim_{n=\infty} \varphi_n$ it is necessary and sufficient that*

$$\varphi^{-1}(\bar{R}_i) = \prod_{n=1}^{\infty} \sum_{k=1}^{\infty} \varphi_{n+k}^{-1}(R_{i,n})$$

for every positive integer i.

17. *The definition of a convergent sequence of homomorphisms.* Let $\boldsymbol{A}$ be a C-algebra and let $\boldsymbol{B}$ be a σ-complete Boolean algebra. On account of theorem 16.2 we admit the following definition:

A sequence $\{f_n\}$ of σ-homomorphisms of $\mathfrak{B}(\boldsymbol{A})$ in $\boldsymbol{B}$ *converges* to a σ-homomorphism f of $\mathfrak{B}(\boldsymbol{A})$ in $\boldsymbol{B}$ (in symbols: $f = \lim_{n=\infty} f_n$) if

$$(*) \qquad \prod_{n=1}^{\infty} f(G_n) = \prod_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{n+k}(G_n)$$

for every strictly decreasing sequence $\{G_n\}$ of open elements of $\boldsymbol{A}$. The homomorphism f is then called the *limit* of the sequence $\{f_n\}$.

Obviously, a sequence $G_n \in \mathfrak{G}(\boldsymbol{A})$ is said to be *strictly decreasing* provided $\bar{G}_{n+1} \subset G_n$ $(n=1,2,\ldots)$.

17.1. *A convergent sequence of σ-homomorphisms possesses only one limit* [49]).

Let $F \in \mathfrak{F}(\boldsymbol{A})$. On account of theorem 11.1 and axiom V it is easy to define a strictly decreasing sequence $G_n \in \mathfrak{G}(\boldsymbol{A})$ such that $F = \prod_{n=1}^{\infty} G_n$. By (*)

$$f(F) = \prod_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{n+k}(G_n).$$

The formula (*) defines the value of the limit f for every $F \in \mathfrak{F}(\boldsymbol{A})$. On the other hand, if g_1 and g_2 are two σ-homomorphisms of $\mathfrak{B}(\boldsymbol{A})$ in $\boldsymbol{B}$ and if $g_1(F) = g_2(F)$ for every $F \in \mathfrak{F}(\boldsymbol{A})$, then $g_1 = g_2$. Thus theorem 16.1 is proved.

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[49]) It may seem singular that we define the convergence of a sequence of homomorphisms which are defined not on $\boldsymbol{A}$ but only on $\mathfrak{B}(\boldsymbol{A})$. We must admit this form of the definition if we want theorem 16.1 to be true. In fact, the formula (*) defines values of the limit f for $A \in \mathfrak{B}(\boldsymbol{A})$ only and, in general, it is possible that two different homomorphisms of $\boldsymbol{A}$ in $\boldsymbol{B}$ coincide on $\mathfrak{B}(\boldsymbol{A})$.

17.2. *Let h be a continuous homomorphism of $\mathfrak{B}(A_1)$ (where A_1 is a C-algebra) in a C-algebra A and let g be a σ-homomorphism of a σ-complete Boolean algebra B in a σ-complete Boolean algebra B_1. If a sequence $\{f_n\}$ of σ-homomorphisms of $\mathfrak{B}(A)$ in B converges to a σ-homomorphism f, then also*

$$\lim_{n=\infty} gf_n h = gfh.$$

For if $\{G_n\}$ is a strictly decreasing sequence of open elements of A_1, then $\{h(G_n)\}$ is a strictly decreasing sequence of open elements of A. Therefore

$$\prod_{n=1}^{\infty} f(h(G_n)) = \prod_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{n+k}(h(G_n))$$

and consequently

$$\prod_{n=1}^{\infty} gfh(G_n) = g\Big(\prod_{n=1}^{\infty} f(h(G_n))\Big) = g\Big(\prod_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{n+k}(h(G_n))\Big) = \prod_{n=1}^{\infty} \sum_{k=1}^{\infty} gf_{n+k}h(G_n)$$

q. e. d.

It follows from 17.2 that

17.3. *Let h be a homeomorphism of $\mathfrak{B}(A_1)$ on $\mathfrak{B}(A)$ (where A and A_1 are C-algebras) and let g be an isomorphism of a σ-complete Boolean algebra B on a σ-complete Boolean algebra B_1. A sequence $\{f_n\}$ of σ-homomorphisms of $\mathfrak{B}(A)$ in B converges to a σ-homomorphism f if and only if*

$$gfh = \lim_{n=\infty} gf_n h.$$

18. *The relation between the convergence of sequences of homomorphisms and the convergence of sequences of mappings.* Our definition of the convergence of a sequence of homomorphisms is a generalization of the notion of the convergence of a sequence of mappings.

In fact, let $\mathfrak{X}$ be an abstract set and let $\mathcal{Y}$ be a metric separable space. By theorem 16.2 a sequence φ_n of mappings of $\mathfrak{X}$ in $\mathcal{Y}$ converges to a mapping φ of $\mathfrak{X}$ in $\mathcal{Y}$ if and only if the sequence $f_n = \varphi_n^{-1}$ of σ-homomorphisms (of $\mathfrak{B}(\mathcal{Y})$ in $\mathfrak{S}(\mathfrak{X})$) induced by the mappings φ_n converges to the σ-homomorphism $f = \varphi^{-1}$.

The question arises now what relation holds between the convergence of a sequence of σ-homomorphisms $\{f_n\}$ and the convergence of mappings φ_n inducing the σ-homomorphisms f_n in the general case where the σ-homomorphisms f_n map a σ-quotient algebra in a σ-quotient algebra.

The answer is given by the following theorem:

18.1. Premises:

(i) *$\mathcal{Y}$ is a separable metric space and J is a σ-ideal of $\mathfrak{B}(\mathcal{Y})$;*

(ii) *X/I is a σ-quotient algebra of a set $\mathfrak{X}$;*

(iii) *$f_0, f_1, f_2, \ldots$ is a sequence of σ-homomorphisms of $\mathfrak{B}(\mathcal{Y})/J$ in X/I induced respectively by mappings $\varphi_0, \varphi_1, \varphi_2, \ldots$ of $\mathfrak{X}$ in $\mathcal{Y}$.*

Thesis: *In order that $f_0 = \lim_{n=\infty} f_n$ it is necessary and sufficient that the sequence $\{\varphi_n\}$ converge to φ_0 almost everywhere (I), i. e. that $\lim_{n=\infty} \varphi_n(x) = \varphi_0(x)$ for all points $x \in \mathfrak{X}$ except at most points belonging to a set $X \in I$.*

Necessity. Suppose that $f_0 = \lim_{n=\infty} f_n$. Let $\varkappa$ be a homeomorphism which maps $\mathcal{Y}$ on a subset of the Hilbert cube $\mathcal{H}$. Since the σ-homomorphism h of $\mathfrak{B}(\mathcal{H})$ in $\mathfrak{B}(\mathcal{Y})/J$ defined by the formula

$$h(H) = [\varkappa^{-1}(H)] \in \mathfrak{B}(\mathcal{Y})/J \quad \text{for} \quad H \in \mathfrak{B}(\mathcal{H})$$

is continuous (see theorem 9.5 (i)) we obtain by 17.2

(iv) $$f_0 h = \lim_{n=\infty} f_n h.$$

Let $\psi_n = \varkappa\varphi_n$ ($n = 0, 1, 2, \ldots$). Obviously

(v) $f_n h(H) = [\psi_n^{-1}(H)] \in X/I$ for every $H \in \mathfrak{B}(\mathcal{H})$ and $n = 0, 1, 2, \ldots$

Let $\{R_i\}$ ($i = 1, 2, \ldots$) be a basis of $\mathcal{H}$. Since the sequence[50] $\{R_{i,n}\}$ ($n = 1, 2, \ldots$) is strictly decreasing and $\bar{R}_i = \prod_{n=1}^{\infty} R_{i,n}$, we obtain by (iv)

$$f_0 h(\bar{R}_i) = \prod_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{n+k} h(R_{i,n}).$$

Hence on account of (v)

(vi) $$[\psi_0^{-1}(\bar{R}_i)] = \Big[\prod_{n=1}^{\infty} \sum_{k=1}^{\infty} \psi_{n+k}^{-1}(R_{i,n})\Big].$$

Let

(vii) $$X = \sum_{i=1}^{\infty} \Big(\psi_0^{-1}(\bar{R}_i) - \prod_{n=1}^{\infty} \sum_{k=1}^{\infty} \psi_{n+k}^{-1}(R_{i,n}) + \sum_{i=1}^{\infty} \Big(\prod_{n=1}^{\infty} \sum_{k=1}^{\infty} \psi_{n+k}^{-1}(R_{i,n}) - \psi_0^{-1}(\bar{R}_i)\Big).$$

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[50]) See the definition on p. 194.

It follows from (vi) and (vii) that $X \in \boldsymbol{I}$ and

(viii) $$\mathfrak{X}_0\varphi_0^{-1}(R_i)=\prod_{n=1}^{\infty}\sum_{k=1}^{\infty}\mathfrak{X}_0\varphi_{n+k}^{-1}(R_{i,n})$$

where $\mathfrak{X}_0=\mathfrak{X}-X$.

Let [51]) $\overline{\varphi}_n=\varphi_n|\mathfrak{X}_0$ $(n=0,1,2,\ldots)$. By (viii)

$$\overline{\varphi}_0^{-1}(\overline{R}_i)=\prod_{n=1}^{\infty}\sum_{k=1}^{\infty}\overline{\varphi}_{n+k}^{-1}(R_{i,n}),$$

i. e. $\overline{\varphi}_0=\lim_{n=\infty}\overline{\varphi}_n$ on account of 16.4. Therefore $\psi_0(x)=\lim_{n=\infty}\psi_n(x)$ for every $x \in \mathfrak{X}-X$. $\varkappa$ being a homeomorphism, we obtain

$$\varphi_0(x)=\lim_{n=\infty}\varphi_n(x) \quad \text{for} \quad x \in \mathfrak{X}-X, \qquad \text{q. e. d.}$$

Sufficiency. Suppose now that $\varphi(x)=\lim_{n=\infty}\varphi_n(x)$ for $x \in \mathfrak{X}-X$ where $X \in \boldsymbol{I}$. Let $\{[G_n]\}$ be a strictly decreasing sequence of open elements of $\mathfrak{B}(\mathcal{Y})/\boldsymbol{I}$. By theorem 9.2 (ii) we may suppose that G_n is an open subset of $\mathcal{Y}$. By theorem 9.3 (i) we have

(ix) $$[G_{n+1}^*]=[\overline{G_{n+1}}]\subset G_n \qquad (n=1,2,\ldots),$$

hence

(x) $$\sum_{n=1}^{\infty}(G_{n+1}^*-G_n) \in \boldsymbol{I}.$$

Let

(xi) $$\mathcal{Y}_0=\mathcal{Y}^*-\sum_{n=1}^{\infty}(G_{n+1}^*-G_n)$$

and $H_n=\mathcal{Y}_0 G_n$. By (x) and (xi)

(xii) $$[H_n]=[G_n] \in \mathfrak{B}(\mathcal{Y})/\boldsymbol{I}.$$

By (x) and theorem 8.2

$$\mathcal{Y}_0\overline{H_{n+1}}=\mathcal{Y}_0\overline{\mathcal{Y}_0 G_{n+1}}=\mathcal{Y}_0 G_{n+1}^*\subset\mathcal{Y}_0 G_n=H_n,$$

i. e. $\{H_n\}$ is a strictly decreasing sequence of open subsets of the space $\mathcal{Y}_0$. Let

$$\mathfrak{X}_0=\prod_{n=0}^{\infty}\varphi_n^{-1}(\mathcal{Y}_0)-X$$

and

$$\psi_n=\varphi_n|\mathfrak{X}_0 \qquad (n=0,1,2,\ldots).$$

[51]) See footnote [9]).

The mappings ψ_n map $\mathfrak{X}_0$ in $\mathcal{Y}_0$ and $\lim_{n=\infty}\psi_n=\psi_0$. By theorem 16.2

$$\prod_{n=1}^{\infty}\psi_0^{-1}(H_n)=\prod_{n=1}^{\infty}\sum_{k=1}^{\infty}\psi_{n+k}^{-1}(H_n).$$

Consequently

$$\prod_{n=1}^{\infty}[\psi_0^{-1}(H_n)]=\prod_{n=1}^{\infty}\sum_{k=1}^{\infty}[\psi_{n+k}^{-1}(H_n)] \in \boldsymbol{X}/\boldsymbol{I}.$$

Since $\mathcal{Y}-\mathcal{Y}_0 \in \boldsymbol{J}$, we have

$$[\varphi_n^{-1}(\mathcal{Y}-\mathcal{Y}_0)]=f_n([\mathcal{Y}-\mathcal{Y}_0])=f_n(0)=0,$$

i. e. $\varphi_n^{-1}(\mathcal{Y}-\mathcal{Y}_0) \in \boldsymbol{I}$. Hence

$$\mathfrak{X}-\mathfrak{X}_0=X+\sum_{n=0}^{\infty}\varphi_n^{-1}(\mathcal{Y}-\mathcal{Y}_0) \in \boldsymbol{I}$$

and

$$[\psi_i^{-1}(H_n)]=[\varphi_i^{-1}(H_n)] \quad (i=0,1,2,\ldots;\ n=1,2,\ldots).$$

Therefore by (xii) and (iii)

$$\prod_{n=1}^{\infty}f_0([G_n])=\prod_{n=1}^{\infty}\sum_{k=1}^{\infty}f_{n+k}([G_n]).$$

$\{[G_n]\}$ being an arbitrary strictly decreasing sequence of open elements of $\mathfrak{B}(\mathcal{Y})/\boldsymbol{J}$, we infer that $f_0=\lim f_n$, q. e. d.

The relation established by theorem 18.1 is important for the following reason: for every C-algebra $\boldsymbol{A}$, for every σ-complete Boolean algebra $\boldsymbol{B}$, and for every σ-homomorphism f of $\mathfrak{B}(\boldsymbol{A})$ in $\boldsymbol{B}$ we may always assume without restricting the generality of our consideration that

a) $\mathfrak{B}(\boldsymbol{A})$ is of the form $\mathfrak{B}(\mathcal{Y})/\boldsymbol{J}$ where $\mathcal{Y}$ is a metric separable space, e. g. $\mathcal{Y}$ is the Hilbert cube (see theorems 15.1 and 15.2);

b) $\boldsymbol{B}$ is a σ-quotient algebra $\boldsymbol{X}/\boldsymbol{I}$ of a set $\mathfrak{X}$ [52]);

c) f is induced by a mapping φ of $\mathfrak{X}$ in $\mathcal{Y}$ [53]).

Theorem 18.1 shows that we can always reduce the study of the convergence of sequences of homomorphisms to the study of the convergence „almost everywhere $(\boldsymbol{I})$" of mappings.

[52]) See Loomis [1], p. 757, and Sikorski [1], p. 256.

[53]) See Sikorski [2], p. 19.

As such an application of theorem 18.1 we shall prove the following theorem:

18.2. *Let* A *be a C-algebra and let* $f, f_1, f_2, \ldots$ *be σ-homomorphisms of* $\mathfrak{B}(A)$ *in a σ-complete Boolean algebra* B. *If*

(i) $$f = \lim_{n=\infty} f_n,$$

then also

(ii) $$f = \lim_{n=\infty} f_{m_n}$$

for every subsequence $\{f_{m_n}\}$ *of the sequence* $\{f_n\}$.

By theorem 14.2 there exist a σ-ideal J of Borel subsets of the Hilbert cube $\mathcal{H}$ and homeomorphism h of $\mathfrak{B}(\mathcal{H})/J$ on $\mathfrak{B}(A)$. Similarly there exist a σ-quotient algebra X/I of a set $\mathcal{X}$ and an isomorphism g of A on X/I. By (i) and 17.3

$$gfh = \lim_{n=\infty} gf_n h.$$

Let φ, φ_n be mappings of $\mathcal{X}$ in $\mathcal{H}$ inducing respectively the homeomorphisms f and f_n $(n=1,2,\ldots)$. By 18.1

$$\varphi(x) = \lim_{n=\infty} \varphi_n(x)$$

almost everywhere (I). Consequently

$$\varphi(x) = \lim \varphi_{m_n}(x)$$

almost everywhere (I), i. e.

$$gfh = \lim_{n=\infty} gf_n h$$

on account of 18.1. By 17.3 the equality (ii) is true, q. e. d.

19. *The* (*)-*convergence*. It follows immediately from 17.1 and 18.2 that [54])

19.1 *The set of all σ-homomorphisms of* $\mathfrak{B}(A)$ *in* B *together with the definition of limit given in* § 17 *is a Fréchet space* $\mathcal{L}$ [55]).

In general, the set of all σ-homomorphisms of $\mathfrak{B}(A)$ in B is not [56]) Kuratowski's space $\mathcal{L}^*$ [55]).

[54]) In this section A and B denote always a C-algebra and a σ-complete Boolean algebra respectively.

[55]) See Kuratowski [1], pp. 76-77 or Kuratowski [2], pp. 83-84.

[56]) Theorem I in my paper [4] was falsely formulated.

For example, let $\mathcal{R}$ denote the set of all real numbers and let L and L_0 be respectively the field of all measurable subsets of the interval $0 \leqslant x \leqslant 1$ and the ideal of all sets $X \in L$ of measure zero. Let $\{\varphi_n(x)\}$ $(0 \leqslant x \leqslant 1)$ be a sequence of measurable functions such that

(i) φ_n converges asymptotically to a function φ;

(ii) φ_n does not converge almost everywhere (L_0) to φ [57]).

Let f and f_n be σ-homomorphisms of $\mathfrak{B}(\mathcal{R})$ in L/L_0 induced respectively by φ and φ_n $(n=1,2,\ldots)$. By (ii) and theorem [58]) 18.1 the sequence $\{f_n\}$ does not converge to f. Let $\{f_{m_n}\}$ be any subsequence of $\{f_n\}$. Since $\{\varphi_{m_n}\}$ converges asymptotically to φ by (i), there exists a subsequence $\{\varphi_{m_{k_n}}\}$ which converges to φ almost everywhere (L_0). Consequently $f = \lim_{n=\infty} f_{m_{k_n}}$ by 18.1. Thus the axiom [55]) 3° is not satisfied, q. e. d.

It is possible, however, to introduce another definition of the convergence of a sequence of homomorphisms in such a way that the set of all homomorphisms of $\mathfrak{B}(A)$ in B will be a space $\mathcal{L}^*$. This follows from the following known remark [58a]) on spaces $\mathcal{L}$:

Let $\mathcal{P}$ be a space $\mathcal{L}$ with a primitive term $p = \lim_{n=\infty} p_n$. A sequence $\{p_n\}$ is said to be (*)-*convergent* to an element $p \in \mathcal{P}$ (in symbols: $p = \lim_{n=\infty} p_n\,(*)$) if every subsequence $\{p_{m_n}\}$ of $\{p_n\}$ contains a subsequence $\{p_{m_{k_n}}\}$ such that $p = \lim_{n=\infty} p_{m_{k_n}}$. The space $\mathcal{P}^*$ which we obtain from $\mathcal{P}$ by admitting the above-defined notion $p = \lim_{n=\infty} p_n\,(*)$ as the primitive term is Kuratowski's space $\mathcal{L}^*$. Obviously $\lim_{n=\infty} p_n = p$ implies $\lim_{n=\infty} p_n = p\,(*)$. Therefore the fundamental topological notions (e. g. closure of a set, open set, closed set, etc.) are the same in the space $\mathcal{P}^*$ as in the space $\mathcal{P}$.

In accordance with the general definition we shall say that a sequence $\{f_n\}$ of σ-homomorphisms of $\mathfrak{B}(A)$ in B (*)-*converges* to a σ-homomorphism f of $\mathfrak{B}(A)$ in B, in symbols:

$$f = \lim_{n=\infty} f_n\,(*)$$

if every subsequence $\{f_{m_n}\}$ contains a subsequence $\{f_{m_{k_n}}\}$ convergent to f.

Analogously, if X/I is a σ-quotient algebra of a set $\mathcal{X}$ and if $\varphi_0, \varphi_1, \varphi_2, \ldots$ are mappings of $\mathcal{X}$ in a metric space $\mathcal{Y}$ such that

$$\varphi^{-1}(Y) \in X \quad \text{for every} \quad Y \in \mathfrak{B}(\mathcal{Y}) \qquad (n=0,1,2,\ldots),$$

[57]) An example of such a sequence $\{\varphi_n\}$ is given in Saks [1], p. 61.

[58]) Where J is the „null" ideal, i. e. J has only one element: the empty set.

[58a]) See Kantorovitch [1], p. 143.

we shall say that $\{\varphi_n\}$ (*)-*converges almost everywhere* ($\boldsymbol{I}$) to φ_0 if every subsequence $\{\varphi_{m_n}\}$ contains a subsequence $\{\varphi_{m_{k_n}}\}$ which converges to φ_0 almost everywhere ($\boldsymbol{I}$).

It follows immediately from the above definitions and theorem 18.1 that

19.2. *Let $\mathfrak{X}$, $\mathcal{Y}$, $\boldsymbol{X}$, $\boldsymbol{I}$, $\boldsymbol{J}$, f_n, φ_n have the same meaning as in the theorem* 18.1. *In order that $f_0=\lim\limits_{n=\infty} f_n(*)$ it is necessary and sufficient that the sequence $\{\varphi_n\}$ (*)-converge almost everywhere ($\boldsymbol{I}$) to φ_0.*

Obviously

19.3. *The set of all σ-homomorphisms of $\mathfrak{B}(\boldsymbol{A})$ in $\boldsymbol{B}$ with the (*)-limit as the primitive term is a space $\mathcal{L}^*$.*

20. *Baire homomorphisms.* Let f be a σ-homomorphism of a C-algebra $\boldsymbol{A}$ in a C-algebra $\boldsymbol{B}$. We say that f is of class α if $f(F)$ is a Borel element of the multiplicative class α for every $F\in\mathfrak{F}(\boldsymbol{A})$. Obviously

20.1. *f is of class α if and only if $f|\mathfrak{B}(\boldsymbol{A})$ is of class α.*

Axiom V implies that every closed element is the product of a strictly decreasing sequence of open elements. It follows immediately from this fact and from the definition of convergence that

20.2 [59]). *Let f, f_n $(n=1,2,\ldots)$ be σ-homomorphisms of $\boldsymbol{A}$ in $\boldsymbol{B}$. If all homomorphisms f_n are of class α and*

$$f|\mathfrak{B}(\boldsymbol{A})=\lim_{n=\infty} f_n|\mathfrak{B}(\boldsymbol{A})$$

then f is of class $\alpha+1$.

An analogous theorem holds also for (*)-convergence.

Baire's theorems on mappings of first class can be generalized in the following way:

20.3. *If f is a σ-homomorphism of $\boldsymbol{A}$ in $\boldsymbol{B}$ of first class, then $D(f)$ is of first category in $\boldsymbol{B}$.*

The proof is the same as the proof of an analogous theorem on mappings [60]).

20.4. *Let f be a σ-homomorphism of $\boldsymbol{A}$ in $\boldsymbol{B}$. If* [61]) $B-D(Bf)\neq 0$ *for every $B\in\mathfrak{F}(\boldsymbol{B})$, $B\neq 0$, then f is of first class* [62]).

Let $F\in\mathfrak{F}(\boldsymbol{A})$. By axiom V, $F=\sum\limits_{n=1}^{\infty}F_n$ where $F_n\in\mathfrak{F}(\boldsymbol{A})$.

[59]) See footnote [49]).

[60]) See Kuratowski [1], p. 189 or Kuratowski [2], p. 301.

[61]) The symbol $D(f)$ was defined in § 6, p. 177.

[62]) The proof of this theorem is an easy modification of an analogous theorem given in Kuratowski [1], p. 190. See also Kuratowski [2], pp. 301-302.

We shall prove first that the equality

$$B=\overline{Bf(F)}\cdot\overline{Bf(F_n)} \tag{i}$$

implies: $B=0$. In fact, if B satisfies (i), B is closed and

$$B=(\overline{Bf(F)}-Bf(F)+Bf(F))\cdot(\overline{Bf(F_n)}-Bf(F_n)+Bf(F_n))\subset$$
$$\subset(\overline{Bf(F)}-Bf(F))+(\overline{Bf(F_n)}-Bf(F_n))\subset D(Bf).$$

By the assumption, this inclusion is possible only in the case $B=0$.

On account of 3.4 there exists a developable element E_n such that $f(F)\subset E_n$ and $f(F_n)=0$. Consequently $f(F)=\prod\limits_{n=1}^{\infty}E_n$. By 11.3 $f(F)$ is a $\boldsymbol{G}_\delta$, q. e. d.

21. *The inducing of homomorphisms.* In this section $\mathfrak{X}$ and $\mathcal{Y}$ are two separable metric spaces and $\boldsymbol{I}$ and $\boldsymbol{J}$ are two σ-ideals of $\mathfrak{B}(\mathfrak{X})$ and $\mathfrak{B}(\mathcal{Y})$ respectively.

The following theorem explains the structure of σ-homomorphisms of class α (in particular, in the case $\alpha=0$: of continuous homomorphisms) [63]):

21.1. *Let f be a σ-homomorphism of $\mathfrak{B}(\mathcal{Y})/J$ in $\mathfrak{B}(\mathfrak{X})/I$. If f is of class α, every mapping φ inducing f* [64]) *has the property:*

(i) *there exists a set $X\in\boldsymbol{I}$ such that the mapping $\varphi|\mathfrak{X}-X$ is of class α.*

Conversely, if a mapping φ inducing the σ-homomorphism f has the property (i), *f is of class α.*

Suppose that φ induces a σ-homomorphism f of class α. Let $\{R_n\}$ be a basis of $\mathcal{Y}$. Since $[R_n]\in\mathfrak{G}(\mathfrak{B}(\mathcal{Y})/\boldsymbol{J})$, $f([R_n])$ is of additive class α in $\mathfrak{B}(\mathfrak{X})/\boldsymbol{I}$. By 11.4 there exists Borel set $X_n\subset\mathfrak{X}$ of additive class α such that $f([R_n])=[X_n]$. Consequently

$$[\varphi^{-1}(R_n)]=[X_n]. \tag{ii}$$

Let

$$X=\sum_{n=1}^{\infty}(\varphi^{-1}(R_n)-X_n)+(X_n-\varphi^{-1}(R_n)). \tag{iii}$$

By (ii) $X\in\boldsymbol{I}$. The mapping $\psi=\varphi|\mathfrak{X}-X$ is of class α since by (iii) the set

$$\psi^{-1}(R_n)=X_n(\mathfrak{X}-X)$$

is of additive class α in $\mathfrak{X}-X$. Thus φ possesses the property (i).

[63]) See the remarks on p. 199.

[64]) If $\mathcal{Y}$ is homeomorphic to a Borel subset of the Hilbert cube, such a mapping φ always exists. See Sikorski [2], p. 19.

Suppose now that φ possesses the property (i) and let $\psi=\varphi|\mathfrak{X}-X$. Since $X\in\boldsymbol{I}$, we have

$$f([F])=[\varphi^{-1}(F)]=[\psi^{-1}(F)] \quad \text{for every } F\in\mathfrak{F}(\mathcal{Y}).$$

$\psi^{-1}(F)$ is of multiplicative class α in $\mathfrak{X}-X$, i. e. $\psi^{-1}(F)=H-X$ where H is of multiplicative class α in $\mathfrak{X}$. Since $f([F])=[H-X]=[H]$, $f([F])$ is of multiplicative class α in $\mathfrak{B}(\mathfrak{X})/\boldsymbol{I}$ by 11.4. $[F]$ being an arbitrary closed element of $\mathfrak{B}(\mathcal{Y})/\boldsymbol{J}$ (see 9.2 (iii)), the σ-homomorphism f is of class α, q. e. d.

The following theorem explains the structure of weak homeomorphisms between two C-algebras [65]):

21.2. *Let h be a homeomorphism of $\mathfrak{B}(\mathfrak{X})/\boldsymbol{I}$ on $\mathfrak{B}(\mathcal{Y})/\boldsymbol{J}$. If mappings ψ and φ induce the isomorphisms h and h^{-1} respectively* [66]), *there exist two sets $X_0\in\boldsymbol{I}$ and $Y_0\in\mathcal{Y}$ such that the mappings $\varphi_0=\varphi|\mathfrak{X}-X_0$ and $\psi_0=\psi|\mathcal{Y}-Y_0$ are one-one and continuous and $\varphi_0=\psi_0^{-1}$. The mapping φ_0 is thus a homeomorphism of $\mathfrak{X}-X_0$ on $\mathcal{Y}-Y_0$ such that*

(i) $h([X])=[\varphi_0(X)]$ *for every* $X\in\mathfrak{B}(\mathfrak{X})$ *and*

(ii) $\varphi_0(X)\in\boldsymbol{J}$ *if and only if* $X\in\boldsymbol{I}$.

Conversely, if there exist two sets $X_0\in\boldsymbol{I}$ and $Y_0\in\boldsymbol{J}$ and a homeomorphism φ_0 of $\mathfrak{X}-X_0$ on $\mathcal{Y}-Y_0$ satisfying the condition (ii), *then the formula* (i) *defines a homeomorphism h of $\mathfrak{B}(\mathfrak{X})/\boldsymbol{I}$ on $\mathfrak{B}(\mathcal{Y})/\boldsymbol{J}$.*

Since ψ and φ induce the homomorphisms h and h^{-1} respectively, there exist two sets $X_1\in\boldsymbol{I}$ and $Y_1\in\boldsymbol{J}$ such that the mappings $\varphi_1=\varphi|\mathfrak{X}-X_1$ and $\psi_1=\psi|\mathcal{Y}-Y_1$ are one-one, $\varphi_1=\psi_1^{-1}$ and [67])

(iii) $h([X])=[\varphi_1(X)]$ for every $X\in\mathfrak{B}(\mathfrak{X})$.

By theorem 21.1 there exist two sets $X_2\in\boldsymbol{I}$ and $Y_2\in\boldsymbol{J}$ such that the mappings $\varphi|\mathfrak{X}-X_2$ and $\psi|\mathcal{Y}-Y_2$ are continuous. Let

$$X_0=X_1+X_2+\varphi_1^{-1}(Y_2) \quad\text{and}\quad Y_0=Y_1+Y_2+\varphi_1(X_2).$$

[65]) See footnote [63]).

[66]) If $\mathfrak{X}$ and $\mathcal{Y}$ are homeomorphic to Borel subsets of the Hilbert cube, such mappings φ and ψ always exists. See Sikorski [2], p. 20.

[67]) The proof of this fact is the same as the proof of theorem 6.1 in my paper [2], pp. 20-21.

By (iii), $\varphi_1^{-1}(Y_2)\in\boldsymbol{I}$ and $\varphi_1(X_2)\in\boldsymbol{J}$; hence $X_0\in\boldsymbol{I}$ and $Y_0\in\boldsymbol{J}$. The mappings

$$\varphi_0=\varphi|\mathfrak{X}-X_0=\varphi|(\mathfrak{X}-X_2)-(X_1+\varphi_1^{-1}(Y_2))$$

and

$$\psi_0=\psi|\mathcal{Y}-Y_0=\psi|(\mathcal{Y}-Y_2)-(Y_1+\varphi_1(X_2))$$

are continuous. They are also one-one and $\varphi_0=\psi_0^{-1}$ since $\varphi_1=\psi_1^{-1}$,

$$\varphi_1(X_2+\varphi_1^{-1}(Y_2))=Y_2+\varphi_1(X_2),$$
$$\varphi_0=\varphi_1|(\mathfrak{X}-X_1)-(X_2+\varphi_1^{-1}(Y_2)),$$
$$\psi_0=\psi_1|(\mathcal{Y}-Y_1)-(Y_2+\varphi_1(X_2)).$$

For any $X\in\mathfrak{B}(\mathfrak{X})$

$$h([X])=[\varphi_1(X)]=[\varphi_0(X)+\varphi_1(XX_0)]=[\varphi_0(X)]+[\varphi_1(XX_0)]=$$
$$=[\varphi_0(X)]+h([XX_0])=[\varphi_0(X)]$$

since $XX_0\in\boldsymbol{I}$. The equality (i) is true. (ii) follows immediately from (i). The first part of theorem 21.2 is proved.

Suppose now that the assumptions of the second part of 21.2 are fulfilled. It is clear that (i) defines then an isomorphism h of $\mathfrak{B}(\mathfrak{X})/\boldsymbol{I}$ on $\mathfrak{B}(\mathcal{Y})/\boldsymbol{J}$. The proof that h and h^{-1} are continuous is analogous to the proof of the second part of theorem 21.1.

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On the boundary values of functions of several complex variables, I.

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1. Introduction. Let $f(\zeta)$ be a function regular for $|\zeta|<1$. Fatou's classical result asserts that if $f(\zeta)$ is bounded there, then for almost every point $e^{i\theta}$ on $|\zeta|=1$ the limit of $f(\zeta)$ exists as ζ approaches $e^{i\theta}$ along any non-tangential path. A more general result of Nevanlinna and Ostrowski asserts that the conclusion of Fatou's theorem holds if the boundedness of the function f is replaced by the condition

$$\int_0^{2\pi} \log^+ |f(\varrho e^{i\theta})|\, d\theta = O(1). \tag{1}$$

In a sense, this is the best possible result since, if $\omega(u)$ is any positive and increasing function tending to $+\infty$ with u but less rapidly than $\log u$, then there is a function $f(\zeta)$ regular for $|\zeta|<1$, satisfying the condition $\int_0^{2\pi} \omega(|f(\varrho e^{i\theta})|)\, d\theta = O(1)$, and deprived even of radial limit for almost every θ [1]).

The main problem of this paper is that of boundary values of regular functions of several complex variables. Let us begin by the simplest case of two variables, and let $f(z,\zeta)$ be regular in the bicylinder $|z|<1$, $|\zeta|<1$. Thus

$$f(z,\zeta)=\sum_{0,0}^{\infty} c_{mn} z^m \zeta^n.$$

The first question that naturally occurs is whether an analogue of Fatou's theorem holds here. The answer is affirmative: if $f(z,\zeta)$ is bounded for $|z|<1$, $|\zeta|<1$, then

$$\lim_{z\to e^{i\chi},\, \zeta\to e^{i\theta}} f(z,\zeta)$$

[1]) See Paley and Zygmund [1].