6. Finally we have

Theorem 6. *The hypotheses of Theorem 2 are satisfied whenever $f(x)$ is a fractional integral of positive non-zero order of a Lebesgue integrable function.*

Proof. Let us take $f(x)$ to be a fractional integral of order $\alpha>0$ of a function $g(x)$. Choose p so that $1<p<1/(1-\alpha)$. Then $p(\alpha-1)>-1$, and we have by (15)

$$f(x+t)-f(x)=\int_0^{2\pi} g(x-u)\{\Psi_\alpha(u+t)-\Psi_\alpha(u)\}\,du.$$

Therefore, if $1/p+1/p'=1$,

$$|f(x+t)-f(x)|^p \leqslant \Big(\int_0^{2\pi}|g(x-u)|\,du\Big)^{p/p'}\int_0^{2\pi}|g(x-u)|\,|\Psi_\alpha(u+t)-\Psi_\alpha(u)|^p\,du$$
$$\leqslant B\int_0^{2\pi}|g(x-u)|\,|\Psi_\alpha(u+t)-\Psi_\alpha(u)|^p\,du,$$

where B depends on g and p only. Hence

$$\int_0^{2\pi}|f(x+t)-f(x)|^p\,dx \leqslant B_1\int_0^{2\pi}|\Psi_\alpha(u+t)-\Psi_\alpha(u)|^p\,du$$
$$\leqslant B_2 t^\beta,$$

where $\beta>0$, by Lemma 3. It is now evident that the hypothesis (6) of Theorem 2 is satisfied and the theorem is proved.

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Complete normality of cartesian products.

By

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All spaces we consider are Hausdorff spaces.

Theorem 1. *Let $\mathfrak{m}$ be an infinite cardinal. Let P and Q be spaces such that $P\times Q$ is completely normal*[1]*). Then either every subset of Q with potency $\leqslant\mathfrak{m}$ is closed or the pseudocharacter*[2]*) of every closed subset of P is $\leqslant\mathfrak{m}$.*

Proof. Suppose there exists an $M\subset Q$ with potency $\leqslant\mathfrak{m}$ and a $b\in\overline{M}-M$. Let $F\subset P$ have pseudocharacter $>\mathfrak{m}$. Let us put

$$A=F\times M,\quad B=(P-F)\times(b).$$

Then $\overline{A}\subset F\times Q$, $\overline{B}\subset P\times(b)$ whence A and B are separated. Hence there exists an open $G\supset A$ such that $\overline{G}B=0$. For each $y\in M$ let G_y denote the set of all $x\in P$ such that $(x,y)\in G$. Clearly $y\in M$ implies G_y open, $G_y\supset F$. The potency of the family $\{G_y\}$ being $\leqslant\mathfrak{m}$ we have $\prod_{y\in M}G_y\neq F$. Choose $c\in\prod G_y-F$. For any $y\in M$ we have then $(c,y)\in G$, whence $(c,b)\in\overline{G}$ implying the contradiction $\overline{G}B\neq0$.

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[1]) A topological space is called *completely normal* if any two separated sets A, B (i. e. such that $A\overline{B}+\overline{A}B=0$) are contained in disjoint open sets.

It is easy to show that a topological space is completely normal if and only if it is hereditarily normal, i. e. every subspace is normal.

[2]) Let S be a space, let $M\subset S$ and let $\mathfrak{A}$ be a family of neighborhoods of the set M. The collection $\mathfrak{A}$ is said to be a *complete family* of neighborhoods of M if there exists, for any neighborhood H of the set M, a set $A\in\mathfrak{A}$ such that $M\subset A\subset H$. The collection $\mathfrak{A}$ is said to be a *pseudocomplete family* of neighborhoods of M if the intersection of all $A\in\mathfrak{A}$ is equal to M.

The minimal potency of a complete (pseudocomplete) family of neighborhoods of a set M in a space S is called the *character (pseudocharacter)* of M in S and is denoted by $\chi(M)$ or more explicitly by $\chi_S(M)$ (respectively, by $\psi(M)$ or $\psi_S(M)$).

Corollary 1. *If $P\times Q$ is completely normal, then either every countable subset of Q is closed or P is perfectly normal*[3]).

Corollary 2. *A compact* (i. e. bicompact) *space P is metrizable if, and only if, the space $P\times P\times P$ is completely normal.*

Proof. The necessity being evident suppose the condition to hold true. If P is infinite, it contains a non-closed countable subset. Hence theorem 1 says that the pseudocharacter of the „diagonal" D of $P\times P$ is countable. The characters in a compact space being equal to pseudocharacters, there exists a countable basic system $\{H_n\}$ of neighborhoods of D. For each n there exist open sets $G_{nk}\subset P$ $(k=1,\dots,p_n)$ such that

$$D\subset\sum_{k=1}^{p_n} G_{nk}\times G_{nk}\subset H_n.$$

It is easy to see that $\{G_{nk}\}$ is an open base of P so that P is separable.

I do not know whether, for compact P, the complete normality of $P\times P$ implies metrizability of P. In theorem 1, the hypothesis of the existence of a non-closed subset of Q is essential, which is shown in the following

Example 1. Let P_1 have potency $\mathfrak{m}>\aleph_0$. Let all points of P_1 be isolated with the exception of a single point ∞ whose neighborhoods are sets $(\infty)\times G$ with P_1-G finite. Then the pseudocharacter of ∞ equals $\mathfrak{m}$ so that P_1 is not perfectly normal. Nevertheless we shall show $P_1\times P_1$ to be completely normal. To this end let us put

$$A_1=(\infty)\times P_1,\quad A_2=P_1\times(\infty),\quad A_3=P_1\times P_2-A_1-A_2.$$

We clearly have: if $M\subset A_i$, $N\subset A_j$ $(i=j$ or $i\neq j)$ and if the sets M, N are separated (in $P_1\times P_2$), they can be separated by open sets. Suppose now M, N to be two separated subsets of $P_1\times P_2$. There exist open sets G_{ij}, H_{ij} $(i,j=1,2,3,)$ such that

$$G_{ij}\supset MA_i,\quad H_{ij}\supset NA_j,\quad G_{ij}H_{ij}=0.$$

[3]) A space is called *perfectly normal* if it is normal and every closed subspace is the intersection of a countable number of open sets.

A perfectly normal space is completely normal (cf. P. Urysohn, *Über die Mächtigkeit der zusammenhängenden Mengen*, Math. Ann. **94** (1925)).

Putting

$$G=\sum_i\prod_j G_{ij},\quad H=\sum_j\prod_i H_{ij}$$

we obtain $G\supset M$, $H\supset N$, $GH=0$.

I do not know whether there exists a space P such that $P\times P$ is completely normal, contains a non-closed countable set, and is not perfectly normal.

Theorem 2. *If all spaces $P_1\times\dots\times P_n$ $(n=1,2,\dots)$ are perfectly normal, then the space $P=\mathfrak{P}_{n=1}^{\infty} P_n$ is perfectly normal as well.*

Proof. Let $A\subset P$ be closed. Let π_n denote the projection of P onto $P_1\times\dots\times P_n$. There exists a continuous function $g_n(y)$ on $P_1\times\dots\times P_n$ such that $0\leqslant g_n(y)\leqslant 1$ for all y and $g_n(y)=0$ if, and only if, $y\in\overline{\pi_n(A)}$. For $x\in P$ let us put

$$f_n(x)=g_n(\pi_n(x)),\quad f(x)=\sum_{n=1}^{\infty}\frac{1}{2^n}f_n(x).$$

Clearly $f(x)$ is a continuous function on P such that $0\leqslant f(x)\leqslant 1$ for all x and $f(x)=0$ for any $x\in A$. If $x\in P-A$, then, for a convenient m, we have $\pi_m(x)\,\text{non}\in\overline{\pi_m(A)}$, whence $f_m(x)>0$ and $f(x)>0$. Hence $f(x)=0$ if, and only if, $x\in A$. This proves the theorem since, by a well known theorem of Urysohn, P is perfectly normal if, and only if, there exists, for every closed $A\subset P$, a continuous function f such that $f(x)=0$ if, and only if, $x\in A$.

Theorem 3. *Let the spaces P_n $(n=1,2,\dots)$ contain more than one point. The space $P=\mathfrak{P}_{n=1}^{\infty} P_n$ is completely normal if, and only if, it is perfectly normal.*

Proof. Urysohn having shown every perfectly normal space to be completely normal, let P be completely normal. We may suppose P infinite so that it contains (the discontinuum of Cantor and, therefore) a countable non-closed set. The same holds true for any $\mathfrak{P}_{n=m}^{\infty} P_n$. Applying now corollary 1 we see that the spaces $P_1\times\dots\times P_n$ are perfectly normal. By theorem 2, the same must hold true for P.

Theorem 4. *The cartesian product of a countable number of countable regular spaces is perfectly normal.*

Proof. Let $P=\mathop{\mathfrak{P}}_{n=1}^{\infty} P_n$, P_n being countable and regular. The spaces $P_1\times\ldots\times P_n$ $(n=1,2,\ldots)$ are countable and regular, hence, as shown by Urysohn, perfectly normal and it suffices to apply theorem 2.

Example 2. If the spaces $P_1\times\ldots\times P_n$ are completely normal, the space $\mathop{\mathfrak{P}}_{n=1}^{\infty} P_n$ need not be completely normal. Choosing $P_n=P_1$ for all n, where P_1 denotes the space of example 1, we may easily show (analogously as for $P_1\times P_1$ in example 1) that $P_1\times\ldots\times P_n$ are completely normal. On the other hand, the space $\mathop{\mathfrak{P}}_{n=1}^{\infty} P_n$, where $P_n=P_1$, is not perfectly normal, for its subspace P_1 is not. Hence $\mathop{\mathfrak{P}}_{n=1}^{\infty} P_n$ is not completely normal by theorem 3.

On Area and Length.

By

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1. This paper is concerned with intrinsic definitions of area and of length. Although the definitions are new, they are obtained by combining ideas which are quite familiar to anyone working in this field: the ideas of Banach [1, 2] which have been the basis of researches on area for twenty years [2, 14, 11, 12] and which consist in effect in introducing our intrinsic definitions in a special case (the case of a surface situated in a plane); and the well-known theory of measure of Carathéodory [5, 8]. Moreover the old definitions, based on simplicial approximations, have long been regarded as unsatisfactory: examples of space-filling curves which constitute surfaces of zero area though of positive volume have been known for forty years; the examples recently produced by Besicovith [3, 4] are even more conclusive.

The value of a particular definition however, depends mainly on its usefulness as a tool, and in this connection the Lebesgue-Fréchet definition of area has rendered great services. It has shown itself quite satisfactory for Lipschitzian surfaces (often misleadingly termed „rectifiable") and has led to important semi-continuity theorems in the Calculus of Variations. Above all, it has had sufficient depth to serve as background to Banach's fundamental methods already referred to.

The greater part of these results and methods remain when we adopt instead the present intrinsic definitions. We show in particular that the definitions agree for Lipschitzian surfaces. Moreover the new definitions are framed for the purpose of developing tools which are needed as a preliminary to the study of „generalized