On the functional equation $f(x+y)=f(x)+f(y)$.

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§ 1. If $f(x)$ is a real function and satisfies

(1) $$f(x+y)=f(x)+f(y) \quad \text{for all real } x \text{ and } y,$$

and $f(x)$ is measurable in some interval, then it is known that

(2) $$f(x)\equiv xf(1).$$

In a recent volume of *Fundamenta* [1]) two new proofs of this were given, together with references to earlier proofs by Fréchet, Sierpiński, and Banach. We give another proof below, which is quite elementary and assumes less than measurability for $f(x)$.

Theorem 1. *Suppose $f(x)$ is a real additive function (i. e.* (1) *holds) and $f(x)$ is bounded on a set E of positive measure. Then* (2) *holds* [2]).

Proof: By a theorem of Steinhaus [3]), there is a positive number δ such that if $|\Theta|<\delta$, then $\Theta=x-y$ for suitable x, y in E, and so, by (1), if M is the upper bound of $|f|$ on E,

$$|f(\Theta)|=|f(x-y)|=|f(x)-f(y)|\leqslant 2M;$$

hence, by (1)

(3) $$|\sigma|<\delta/n \quad \text{implies} \quad |f(\sigma)|\leqslant 2M/n \qquad (n=1,2,\ldots).$$

Let ξ be any real number; if r is rational and $|r-\xi|<\delta/n$, we have by (1) and (3) for $n=1,2,\ldots$

$$|f(\xi)-\xi f(1)|=|f(\xi-r)+(r-\xi)f(1)|\leqslant(2M+|f(1)|\delta)/n,$$

which means $f(\xi)=\xi f(1)$.

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1) A. Alexiewicz et W. Orlicz, Fund. Math. **33** (1945), 314.

2) See *Addendum*.

3) H. Steinhaus, Fund. Math. **1** (1920), 93.

Corollary 1. *If $f(x)$ satisfies* (1) *and is measurable in some set of positive measure, then* (2) *holds because the set of x for which $|f(x)|<N$ has positive measure if N is large enough.*

Corollary 2. *Every discontinuous solution of* (1) *is unbounded on every set of positive interior measure.*

§ 2. It may be appropriate to give an elementary proof of Steinhaus' theorem in the Euclidean space R_n.

Theorem (Steinhaus). *Let C be a closed bounded set in R_n with $|C|>0$. Then a positive number δ exists such that*

(4) *every vector v in R_n satisfying $|v|<\delta$ may be expressed as $x-y$ for suitable x, y in C.*

Proof: Let U be an open set covering C and satisfying $|U-C|<\frac{1}{2}|C|$, let δ be the (positive) distance between the closed sets C and R_n-U, and let v be any vector with $|v|<\delta$. If C_v is the set of all x for which $x+v\in C$, we have to show $CC_v\neq 0$. Since $|v|<\delta$, we have $C_v\subset U$ and so

$$|U-C_v|=|U-C|<\tfrac{1}{2}|C|;$$

hence the set of points of U which are not in C or C_v has measure less than $|C|$, which means $|CC_v|>0$.

It will be noticed that the assumption in Theorem 1 that $|E|>0$ was only made in order to justify the analogue of (4). It is not necessary for the truth of (4) that $|C|>0$. For example, if C is the set of all real numbers in $[0,1]$ which have decimal expansions (in scale 10) which miss the integer 5, it is easy to prove that C is closed and null. If now $x=0,a_1a_2\ldots$ is any number in $(0,1)$, it is the difference between two numbers in C, e. g.

$$x=\sum_{r=1}^{\infty} b_r 10^{-r}-\sum_{r=1}^{\infty} c_r 10^{-r}$$

where $b_r=6$ and $c_r=1$ whenever $a_r=5$, and $b_r=a_r$, $c_r=0$ if $a_r\neq 5$. On the other hand, (4) is not always true if C is null and perfect. The set of all numbers in $[0,1]$ which have decimal expansions (in scale 10) which use only the integers 0 and 1 is perfect, but clearly the difference between two such numbers can never be $5\cdot(10^{-n})$ where n is an integer. It is easy to show (see § 3) that if E is a set in R_n for which a sphere U exists such that $U-EU$ is of first category, then (4) holds with E in place of C; such sets E can be null.

§ **3.** The argument of Theorem 1 can obviously be applied to additive operations defined in normalised vector spaces. Let S be such a space; if $E \subset S$ and there is a positive number δ such that every point v of S with $\|v\|<\delta$ can be expressed as $v=x-y$ where x and y belong to E, we shall write $E \in \mathcal{S}$ (the „set of distances" of E contains a full sphere).

Theorem 2. *Let $U(x)$ be an additive operation with domain and contradomain S_1 and S_2 respectively, both normalised vector spaces. Suppose $U(x)$ is bounded on a set E and $E \in \mathcal{S}$. Then $U(x)$ is homogeneous, i. e. $U(tx)=tU(x)$ for all real t.*

Proof: $U(x)$ being additive, $U(rx)=rU(x)$ whenever r is rational, and it follows easily that r may be replaced by t if $U(x)$ is continuous at the null element of S_1. Let M be the upper bound of $\|U(x)\|$ on E. By hypothesis, there is a positive number δ such that if $\|\Theta\|<\delta$, then $\Theta=x-y$ for suitable x and y in E, and so

$$\|U(\Theta)\|=\|U(x-y)\|=\|U(x)-U(y)\|\leqslant 2M.$$

It now follows from the additive property of $U(x)$ that

$$\|z\|<\delta/n \quad \text{implies} \quad \|U(z)\|\leqslant 2M/n \qquad (n=1,2,\ldots),$$

and this shows that $U(x)$ is continuous at the null element of S_1.

Corollary: *If $f(x)$ is a vector in R_n, defined for all vectors x in R_m, and if each of the coordinate axes of R_m contains a linear $\mathcal{S}$ set (e. g. a set of positive measure) on which $f(x)$ is bounded, then the solution of the functional equation $f(x+y)=f(x)+f(y)$ is*

$$f((t_1,t_2,\ldots,t_m))=\sum_{r=1}^{m} t_r\lambda_r$$

where $\lambda_1,\ldots,\lambda_m$ are arbitrary constant vectors in R_n.

Corollary: *If $U(z)$ is an additive operation defined for all complex numbers $z(=\sigma+i\tau)$ and $\|U(z)\|$ is bounded on two linear $\mathcal{S}$ sets, one on the real axis and one on the imaginary axis, then*

$$U(\sigma+i\tau)=\sigma U(1)+\tau U(i).$$

If the space S is complete, then any set E for which a sphere U exists such that $U-EU$ is of the first category is an $\mathcal{S}$ set. To see this, let ϱ be the radius of U and let V be a concentric sphere of radius $\frac{1}{2}\varrho$. Plainly VE is of the second category (V being complete), whereas the set of all x satisfying $x+v \in U-EU$ is of the first category, v being any chosen vector with $|v|<\frac{1}{2}\varrho$. Hence VE includes points ξ such that $\xi+v \bar{\in} U-EU$; but $\xi+v \in U$, since $|v|<\frac{1}{2}\varrho$ and $\xi \in V$; hence $\xi+v \in E$, and this completes the proof.

Addendum. The result of Theorem 1 is not new; I am grateful to the Editors for pointing this out by referring me to a paper by A. Ostrowski, *Jahresberichte d. Deutscher Mathematiker Vereinigung* (38) 1929, p. 56, in which the linearity of $f(x)$ is shown to follow from (1) and the slightly weaker assumption that $f(x)$ is bounded above on some set of positive measure. Ostrowski's proof is however different from that given above; his slightly better result may be derived in the manner of Theorem 1 as follows.

Suppose $f(x)$ real and additive and that $f(x)<M<\infty$ on a set E of positive measure: we deduce that $f(x)$ is bounded in some interval. Since $mE>0$, there must be a number a and a positive number η such that if $E_1=E(a-\eta,a)$ and $E_2=E(a,a+\eta)$, then mE_1 and mE_2 both exceed $\frac{1}{2}\eta$. Plainly E_2 and the reflection of E_1 in a (i. e. the set of y satisfying $(2a-y) \in E_1$) have in common a set E_3 with $mE_3>0$. By Theorem 2, there is a positive number δ such that if $|\Theta|<\delta$, then $\Theta=x-y$ for suitable x and y in E_3; since $\Theta=x+2a-y-2a$ and $(2a-y) \in E_1$, we have by (1) $f(\Theta)<2M-f(2a)$. Since $-f(\Theta)=f(-\Theta)$, it follows that $f(\Theta)$ is bounded for $|\Theta|<\delta$.