Considérons effectivement un K_2 et les t.u.c. $T: K_2 \to H_1 \times H_2$, et soit $y \in K_2$. On aura $Ty = x_1 \times x_2$, $x_n \in H_n$. Par suite, $T_n: y \to x_n$ est une t.u.c. $K_2 \to H_n$, dite *projection* de T sur H_n. Les deux projections déterminent T, et T_1 est normale (la normalité ne se pose pas pour T_2).

Soit $A = A_1 \times A_2$, où A_n est un point fixe de H_n. Lorsque T_n varie dans une classe déterminée, il en est de même de T. Profitons en pour ramener T_1, T_2 à des réduites: $T_n K_{n-1} = A_n$. La T résultante est dite *réduite* également.

Soient σ_n^i les n-simplexes de K_2 et supposons T réduite. $T_n \sigma_n^i$ aura alors un degré déterminé m_n^i sur H_n. Les m sont les *caractères* de T. Toute réduite aux mêmes caractères que T est de même classe. C'est une conséquence immédiate de la même propriété pour les T_n.

Comme les m_1^i sont les caractères de T_1, ils doivent satisfaire à la condition de normalité pour cette dernière. D'ailleurs, à part cela, les m sont arbitraires. En effet, supposons qu'on se les donne avec les m_1^i satisfaisant simplement à cette condition. On peut alors se donner $T_1 K_1$ sur H_1. Puisque toute H_n, où $n > 1$, est inessentielle sur H_1, on pourra étendre T_1 à K_2 tout entier.

Il s'agit maintenant de construire T_2. A cet effet, on prend d'abord $T_2 \sigma_n^i = A_2$, où $n = 0,1$. Puis on prendra $T_2 \sigma_2^i$ telle qu'elle recouvre H_2 avec le degré m_2^i. Ceci définit T_2, donc T pour K_2 tout entier.

On pourra alors définir l'addition $T + T'$ par la condition que les projections de la somme soient les sommes des projections, et de même, de façon évidente, pour $-T$. D'ailleurs, T est inessentielle quand, et seulement quand ses projections le sont. On écrira alors $T = 0$. Ceci donne ainsi lieu à un groupe additif G. En s'en rapportant au Théorème 12, on voit donc que l'on a:

Théorème 13. *Les classes de t.u.c. $K_2 \to H_1 \times H_2$ donnent lieu à un groupe additif abélien G, et l'on a $G = G_{21} \oplus G_{22}$ (somme directe), où les facteurs sont les groupes des projections sur les H_n.*

Ce sont d'ailleurs les G_{2n} du Théorème 12, et par conséquent ils sont isomorphes aux groupes d'homologie g^1, g^2 des cocycles.

On locally bicompact spaces.

By

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The object of this paper is to define and study two classes of locally bicompact spaces [2]) which may be considered as generalizations of the class of locally compact, separable, metrisable spaces. Both generalizations are based on the well known fact that a characteristic property of spaces of this class is that of being the sum of a sequence [3]) of compact metrisable spaces, each of which is interior to the next.

Guided by this, *a space will be said to belong to the class S_1* [4]) *if it is the sum of a sequence of bicompact spaces each of which is interior to the next, and will be said to belong to the class S_2 if each of the bicompact spaces in question has the property that each of its closed subsets is a G_δ* [5]). Evidently the spaces which belong to the class S_2 also belong to the class S_1, while the class of metrisable spaces belonging to either class is exactly that of the locally compact, separable, metrisable spaces.

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[1]) Alfred H. Lloyd Fellow, University of Michigan.

[2]) All spaces considered in this paper will be Hausdorff spaces. The word *neighborhood* (of a point) will be used to denote any set to which the point in question is interior. A space is said to be *bicompact* if every covering of the space by open sets is reducible to a finite one, and is said to be *locally bicompact* if each point has a neighborhood which is a bicompact space. For a detailed study of such spaces, see Alexandroff and Urysohn, *Mémoire sur les espaces topologiques compacts*, Verh. Akad. Wet. Amsterdam, **14** (1929), pp. 1-96.

[3]) By a *sequence* is meant a denumerable ordered set which is ordinally similar to the set of positive integers.

[4]) It seems undesirable, at this time, to add another name to the large number already in use in the theory of abstract spaces. Such a step can well be delayed until the definitions have proved their utility.

[5]) See [2]), p. 35, also Čech, *Sur la dimension des espaces parfaitement normaux*, Bull. Intern. Acad. Sc. de Bohème, 1932.

Theorem 1. *In order that a space belong to the class S_1 it is necessary and sufficient that it be locally bicompact and that every covering of the space by open sets be reducible to a denumerable covering.*

Proof. Necessity: Let $M=\sum_{i=1}^{\infty} M_i$ where M_i is bicompact and contained in the interior (with respect to M) of M_{i+1}. If a is a point of M there exists an index i such that a belongs to M_i. Then M_{i+1} is a bicompact neighborhood of a. If (U) is any covering of M by open sets there exists, for each i, a finite subclass of (U) which covers M_i. The sum of all these subclasses is a denumerable covering of M.

Sufficiency: Suppose that M is locally bicompact and that every covering of M by open sets is reducible to a denumerable covering. Then, since each point of M is contained in an open set whose closure[6] is bicompact, there exists a denumerable covering (U_n) of M by sets having this property. Let $M_1=\overline{U}_1$. Supposing that $M_1, M_2, \ldots, M_i$ have been defined so that each is bicompact and contains the preceding in its interior, let $F_i=\overline{M-M_i}\cdot M_i$. This is a closed subset of the bicompact space M_i and hence is bicompact. As a consequence there is a finite subfamily of (U_n) which covers F_i, and the sum of the closures of these sets is a bicompact space N_i. Let $M_{i+1}=M_i+N_i+\overline{U}_{i+1}$. Then M_{i+1} is a bicompact space containing M_i in its interior. Since M_{i+1} contains U_{i+1},

$$M=\sum_{i=1}^{\infty} M_i.$$

Corollary. *If G is an F_σ subset of a space which belongs to the class S_1 every covering of G by open sets is reducible to a denumerable covering.*

Proof. A closed subset of a space which belongs to the class S_1 also belongs to the class S_1 and, by the preceding theorem, has the desired property. From the nature of the property itself it follows that it holds for F_σ sets.

It has been shown by Alexandroff[7] that the class of locally bicompact spaces is identical with the class of open subsets of

[6] If A is a subset of the space M, the *closure* of A is the set obtained from A by adding its limit points and is denoted by $\overline{A}$.

[7] See [2], p. 70.

bicompact spaces. The spaces belonging to the class S_1 allow of a similar characterization, as follows:

Theorem 2. *In order that a space belong to the class S_1 it is necessary and sufficient that it be an open F_σ subset of a bicompact space.*

Proof. Necessity: This follows at once from the theorem of Alexandroff quoted above and from the fact that a bicompact space is closed in every space in which it is topologically imbedded[8].

Sufficiency: Suppose that M is a bicompact space, $F=\prod_{i=1}^{\infty} G_i$ a closed G_δ subset of M. Let $H_0=M$ and $H_1=G_1$. Suppose now, that $H_1, H_2, \ldots, H_i$ have been determined in such a way that H_i is an open set containing F, contained in G_i, and whose closure is contained in H_{i-1}. Since M is normal[9] there exists an open set H_{i+1}^* containing F whose closure is contained in H_i. Then $H_{i+1}=G_{i+1}H_{i+1}^*$ is an open set containing F, contained in G_{i+1}, whose closure is contained in H_i. Evidently $F=\prod_{i=1}^{\infty} H_i$ and, if $M_i=M-H_i$, $M-F=\sum_{i=1}^{\infty} M_i$. Since, for each i, M_i is a closed subset of M, M_i is bicompact and, since the closure of H_i is contained in H_{i-1}, the interior of M_i contains M_{i-1}. Consequently $M-F$ is a space belonging to the class S_1.

As suggested by the proof of the preceding theorem, one of the most important properties of bicompact spaces is that they are normal. The next theorem shows that this property holds also for the spaces which belong to the class S_1.

Theorem 3. *A space which belongs to the class S_1 is normal.*

Proof. Let $M=\sum_{i=1}^{\infty} M_i$ where M_i is bicompact and interior to M_{i+1}. Let E and F be disjoint closed subsets of M and let $E_i=M_i\cdot E$, $F_i=M_i\cdot F$. Let $G_0=H_0=0$. Since M_3 is normal there exist open sets G_1^* and H_1^* containing $\overline{G}_0+E_2$ and $\overline{H}_0+F_2$ respectively, contained in M_3 and having disjoint closures. Let $G_1=G_1^*[M_2-\overline{M-M_2}]$, and $H_1=H_1^*[M_2-\overline{M-M_2}]$. Then G_1 and H_1 are open sets containing $\overline{G}_0+E_1$ and $\overline{H}_0+F_1$ respectively and such that $\overline{G}_1+E_2$ and $\overline{H}_1+F_2$ are disjoint. Suppose now that open sets $G_0, G_1, \ldots, G_i$,

[8] See [2], p. 47.

[9] See [2], p. 26.

$H_0, H_1, \ldots, H_i$ such that G_i and H_i contain $\bar{G}_{i-1}+E_i$ and $\bar{H}_{i-1}+F_i$ respectively, are contained in M_{i+1}, and have the property that $\bar{G}_i+E_{i+1}$ and $\bar{H}_i+F_{i+1}$ are disjoint. Then $\bar{G}_i+E_{i+2}$ and $\bar{H}_i+F_{i+2}$ are disjoint closed subsets of M_{i+2} and there exist open sets G^*_{i+1} and H^*_{i+1} containing these sets respectively, contained in M_{i+3} and having disjoint closures. Let $G_{i+1}=G^*_{i+1}\,[M_{i+2}-\overline{M-M_{i+2}}]$, and $H_{i+1}=H^*_{i+1}\,[M_{i+2}-\overline{M-M_{i+2}}]$. These are open sets containing $\bar{G}_i+E_{i+1}$ and $\bar{H}_i+F_{i+1}$ respectively, contained in M_{i+2}, and such that $\bar{G}_{i+1}+E_{i+2}$ and $\bar{H}_{i+1}+F_{i+2}$ are disjoint. The sets $G=\sum_{i=1}^{\infty}G_i$ and $H=\sum_{i=1}^{\infty}H_i$ are disjoint open sets containing E and F respectively.

As might be expected, the distinction between compact and perfectly compact [10]) sets disappears when considering spaces which belong to the class S_1.

Theorem 4. *Every compact subset of a space which belongs to the class S_1 is perfectly compact.*

Proof. Let M be a space which belongs to the class S_1 and let E be a compact subset of M. By Theorem 3, M is normal and hence [11]) $\bar{E}$ is a closed and compact set. From the corollary to Theorem 1 it follows that every covering of $\bar{E}$ by open sets is reducible to a denumerable covering while, since $\bar{E}$ is a compact space, every denumerable covering can be reduced to a finite covering [12]). Hence $\bar{E}$ is a bicompact space and E, being a subset of $\bar{E}$, is perfectly compact [13]).

It will now be shown that the first three theorems have precise analogues in the case of spaces belonging to the class S_2.

Theorem 5. *In order that a space belong to the class S_2 it is necessary and sufficient that it be locally bicompact and that every covering of any subset of the space by open sets be reducible to a denumerable covering.*

[10]) A set is called *compact* if every infinite subset has a limit point; it is called *perfectly compact* if, to each infinite subset E, there corresponds a point a such that every neighborhood of a contains a subset of E having the same cardinal number as the set E itself.

[11]) See [2]), p. 57.

[12]) See [2]), p. 7.

[13]) See [2]), p. 8.

Proof. Necessity: Let $M=\sum_{i=1}^{\infty}M_i$ where M_i is a bicompact space contained in the interior of M_{i+1} and has the property that each of its closed subsets is a G_δ. As before, M is locally bicompact. Also [14]), if E is any subset of M, $E_i=M_i\cdot E$, and (U) is any covering of E by open sets, a denumerable subfamily of these sets serves to cover E_i and E is, itself, covered by a denumerable subfamily of (U).

Sufficiency: Let M be a locally bicompact space having the property that each covering of any of its subsets by open sets is reducible to a denumerable covering. Then, as in Theorem 1, $M=\sum_{i=1}^{\infty}M_i$ where M_i is bicompact and contained in the interior of M_{i+1}. Evidently M_i also has the any-to-denumerable covering property, from which it follows [14]) that every closed subset of M_i is a G_δ.

As an analogue of Theorem 2 one obtains the following:

Theorem 6. *In order that a space belong to the class S_2 it is necessary and sufficient that it be an open subset of a bicompact space in which every open set is an F_σ.*

Proof. Necessity: Let $M=\sum_{i=1}^{\infty}M_i$ where M_i is a bicompact space contained in the interior of M_{i+1} and has the property that each of its closed subsets is a G_δ. Since M is locally bicompact there exists [7]) a (unique) bicompact space $\tilde{M}$ and a point a_0 such that $M=\tilde{M}-(a_0)$. It is sufficient to show that each closed subset F of $\tilde{M}$ is a G_δ. Since, from the structure of M, the point a_0 of $\tilde{M}$ is of denumerable character [15]), every closed subset of $\tilde{M}$ which does not contain a_0 or which has a_0 as an isolated point is a G_δ. Suppose then that F is a closed subset of $\tilde{M}$ which has a_0 as a limit point, and let $F_i=M_i\cdot F$. Then $F_i=\prod_{j=1}^{\infty}G^*_{ij}$ where G^*_{ij} is open and contains G^*_{ij+1}, and $a_0=\prod_{j=1}^{\infty}G_{0j}$ where G_{0j} is open and contains G_{0j+1}.

[14]) See [2]), p. 38.

[15]) The *character* of a point in a neighborhood space is the least cardinal number χ such that there exists a family of neighborhoods of the point, χ in number, which is equivalent to the family of all neighborhoods of the points. See [3]), p. 2.

Let $G_{1j}=G_{1j}^*$ and, for $i>1$, let $G_{ij}=G_{ij}^*-M_{i-1}$. The set $G_j=\sum_{i=0}^{\infty}G_{ij}$ is an open subset of $\widetilde{M}$ which contains F. Now suppose that a is a point of $\prod_{j=1}^{\infty}G_j$, distinct from a_0. There exists an index i such that a belongs to M_i and hence does not belong to any of the sets G_{kj} for $k>i$. As a consequence, a is a point of the set $\prod_{j=1}^{\infty}\sum_{k=1}^{i}G_{kj}$. Since i is finite there exists a $k'\leqslant i$ such that a belongs to $G_{k'j}$ for an infinite number of indices j and, hence, to $F_{k'}$. Since a_0 is, by hypothesis, a point of F, it follows that $F=\prod_{j=1}^{\infty}G_j$, or, is a G_δ in M.

Sufficiency: It follows, as in Theorem 2, that an open subset of a bicompact space in which every open set is an F_σ belongs to the class S_1. That such a space belongs to the class S_2 is then a consequence of the obvious fact that, if M is a space each of whose closed sets is a G_δ, any subset of M is a space having the same property.

Corollary. *Every closed subset of a space which belongs to the class S_2 is a G_δ.*

For the spaces which belong to the class S_2, Theorem 3 can be replaced by the following stronger theorem:

Theorem 7. *A space which belongs to the class S_2 is completely normal.*

Proof. This follows at once from the preceding corollary and from a theorem due to Urysohn[16]).

A large number of properties may be shown to hold for spaces which belong to the class S_2 largely by noting that they are properties of those bicompact spaces in which every closed set is a G_δ. Some of the more important are gathered into the following theorem:

[16]) P. Urysohn, *Über die Mächtigkeit der zusammenhängenden Mengen*, Math. Ann. **94** (1925), pp. 274-83, note [41]).

Theorem 8. *A space which belongs to the class S_2 has the following properties:*

1. *It is of denumerable character.*
2. *The set of points of the space is denumerable or has the cardinal number of the continuum.*
3. *Every non-denumerable analytic set contains a perfect set and has the cardinal number of the continuum.*
4. *Every clairsemé set is denumerable.*
5. *Every non-denumerable set contains a condensation point.*
6. *Every closed set is the sum of a perfect set and a denumerable set.*
7. *Every well-ordered decreasing sequence of closed sets is at most denumerable.*

Proof. 1. This follows from the fact that the space is locally bicompact and every closed set is a G_δ.

2, 3 and 6: These are properties of bicompact spaces in which every closed set is a G_δ which, of their own nature, extend to spaces which belong to the class S_2.

4, 5 and 7. These properties are equivalent to the property that every closed set is a G_δ [17]).

[17]) See [2]), pp. 35-41.