Ajoutons au sujet du th. 2 que dans les conditions (18) et (20) on a

$$\mathscr{P}\{s^* \geqslant x\} \leqslant 32\, e^{-x/4M} \qquad \text{pour} \quad x \geqslant B/M \tag{24}$$

et dans les conditions (18bis) et (20bis) on a

$$\mathscr{P}\{s^* \geqslant x\} \leqslant 32\, e^{-x/16H} \qquad \text{pour} \quad x \geqslant B/4H. \tag{24 bis}$$

L'inégalité (24) s'obtient de (23), en y posant $a=1/M$. L'inégalité (24 bis) se démontre d'une façon analogue. En même temps, si l'on remplace dans (24) et (24 bis) s^* par s, le facteur 32 dans les membres droits peuvent être remplacés par 2 (cf. Kolmogoroff, loc. cit.).

La seconde partie du th. 2 permet aussi de généraliser légèrement le théorème sur le logarithme itéré, en l'étendant aux variables pas nécessairement bornées. Soit $X_1, X_2, \ldots$ une suite infinie de variables aléatoires indépendantes, à valeurs moyennes nulles. Admettons que les X_ν satisfassent aux inégalités (18bis), où $H=H_\nu$ $(\nu=1,2,\ldots)$. Posons $\overline{H}_\nu=\operatorname{Max}(H_1,H_2,\ldots,H_\nu)$ et supposons que

$$\overline{H}_n = o\sqrt{B_n/\log\log B_n} \qquad (B_n \to \infty). \tag{25}$$

Dans ces hypothèses, la probabilité de l'inégalité (3) est égale à 1.

La démonstration de ce théorème ne diffère pas essentiellement de celle de M. Kolmogoroff. Notons seulement que la démonstration de la première partie de (3), à savoir de l'inégalité $\overline{\lim} \ldots \leqslant 1$, est une conséquence facile de (21 bis).

Il résulte du th. 1 que le symbole o ne peut pas être remplacé par O dans la condition (25).

Algebraic Characterizations of Special Boolean Rings[1]).

By

M. H. Stone (Cambridge, Mass. U. S. A.).

In a paper entitled *The Theory of Representations for Boolean Algebras* [2]), we have introduced and discussed a certain classification of the ideals in a Boolean ring (or generalized Boolean algebra). Here we propose to carry out a detailed study of that classification, with the particular purpose of discovering what types of Boolean ring can be characterized by properties of the ideal-structure. In order to make our examination complete, we have to consider many details of a somewhat tedious and uninteresting nature. For the convenience of the reader who prefers to pass over such details, we adopt a synthetic, rather than analytic, form of presentation; and formulate our results in a series of theorems and tables which, we hope, can be easily and rapidly surveyed.

[1]) Parts of this paper (in particular §§ 1, 5, 12 and most of §§ 6, 7, 8, 10, 11) were written in 1933-4 and communicated to the Polish Mathematical Society at a meeting in Warsaw on September 12, 1935. Other parts (in particular §§ 2, 3, 4, 9 and certain aspects of §§ 6, 7, 8, 10, 11) were obtained in 1936-7 while the writer was a Fellow of the John Simon Guggenheim Memorial Foundation in residence at the Institute for Advanced Study (Princeton) as a temporary member.

[2]) M. H. Stone, Trans. Amer. Math. Soc. 40 (1936), pp. 37-111. A knowledge of this paper is assumed here, and references to it made by such citations as „R Th. 24", „R Def. 8", and so on.

Our central problem and the contributions made to its solution in these pages are of interest under two different aspects. In the first place, it is known [3]) that the classification of Boolean rings is equivalent to the classification of the totally-disconnected locally-bicompact Hausdorff spaces. Accordingly, the present investigation may be regarded as a test of the power and effectiveness of a purely algebraic attack upon a problem of topology. The fact that this attack proves to be a relatively feeble one is hardly surprising but is perhaps worthy of detailed consideration. In the second place, it is known that the structural problems of the symbolic (Aristotelian) logic of propositions are mathematically equivalent to the structural problems of the theory of Boolean rings. Speaking more precisely, we may say that the theory of deductive systems developed in recent years by Tarski [4]) is mathematically identical with the theory of ideals in Boolean rings (with unit). A brief digression at this point will permit us to establish this identity. The elements $a, b, c, \dots$ of a Boolean ring A (even one without unit) may be regarded as propositions, $a+b$ and ab may be interpreted as the propositions "a if and only if b" and "a or b" respectively, and the equation $a=0$ may be interpreted as the assertion $\vdash a$ or "a is true" [5]). The propositions "a and b" and "a implies b" may then be introduced by the respective definitions $a \& b = a+b+ab = a \vee b$, $a \rightarrow b = b+ab$. A subclass $\mathfrak{a}$ of A is then called a *deductive system* if it has the three following properties: (1) if $a=0$, then $a \epsilon \mathfrak{a}$; (2) if $a \epsilon \mathfrak{a}$ and $b \epsilon \mathfrak{a}$, then $a \& b \epsilon \mathfrak{a}$; (3) if $a \epsilon \mathfrak{a}$ and $a \rightarrow b \epsilon \mathfrak{a}$, then $b \epsilon \mathfrak{a}$. Obviously, a deductive system $\mathfrak{a}$ is non-void, by (1), and contains $a \vee b$ together with a and b, by (2). Since $a \rightarrow ab = ab + a(ab) = 0$, we see further than $a \rightarrow ab \epsilon \mathfrak{a}$ by (1) and hence that $a \epsilon \mathfrak{a}$ implies $ab \epsilon \mathfrak{a}$ for arbitrary b in accordance with (3). Thus every deductive system $\mathfrak{a}$ is an ideal, by virtue of $\mathfrak{R}$ Th. 16. Conversely, we can show that every ideal $\mathfrak{a}$ is a deductive system. Properties (1) and (2) are obvious from $\mathfrak{R}$ Th. 16. To establish property (3) we first note that $a \epsilon \mathfrak{a}$ implies $ab \epsilon \mathfrak{a}$ for arbitrary b; and we then observe that $a \epsilon \mathfrak{a}$ and $b+ab = a \rightarrow b \epsilon \mathfrak{a}$ imply $b = (b+ab) \& ab = (a \rightarrow b) \& ab \epsilon \mathfrak{a}$. In view of this identity between ideals and deductive systems, the present investigation bears directly on the classification of deductive systems. The fact that the purely algebraic attack proves to be relatively ineffective means, in this connection, that the profounder aspects of the theory of deductive systems must be studied by the general methods of topology. The interesting case for the theory of deductive systems is that where the Boolean ring A of propositions is countable. According to A, Ch. I, the problem of classifying such Boolean rings and their ideals, considered as subrings, is the problem of classifying the closed subsets of the Cantor discontinuum and their (relatively) open subsets — or, equivalently, the problem of classifying all zero-dimensional compact metric spaces and their open subsets. In view of the special significance of the case where A is countable, we shall show (in §§ 9, 10) how our general results appear under it.

As we have already indicated, we find only a few special types of Boolean ring which can be characterized in terms of the ideal-classification of $\mathfrak{R}$. By way of recompense, we find that most of these types can be characterized in many different, equivalent ways. Our special types fall into two main groups. On the one hand we have a series of distinct types which can all be obtained from infinite totally additive Boolean rings by appropriate combinations of the following operations: selection of a non-normal invariant subring or ideal, adjunction of a unit, and direct summation. These types are analysed in § 2. On the other hand, we have two types (not distinct from those in the first group) which have a fairly general structure. They can be obtained from Boolean rings with unit by the following operations: selection of a special type of prime ideal, and direct summation. These types are discussed in § 3. The only countable Boolean rings, other than the finite ones, occurring under these various types belong to three of the most restricted types in the first group.

[3]) M. H. Stone, *Applications of the Theory of Boolean to Rings General Topology*, Trans. Amer. Math. Soc. 41 (1937), pp. 375-481, cited hereafter by the letter A. In the present connection we refer particularly to Chapter I.

[4]) A. Tarski, Fund. Math. 25 (1935), pp. 503-526, and 26 (1936), pp. 283-301. Accordingly many theorems of our paper $\mathfrak{R}$, especially those in Chapter II, duplicate results of Tarski. I regret that due acknowledgement of this connection was not made in $\mathfrak{R}$: my manuscript was prepared and submitted for publication before Tarski's first paper was available; and I did not have the opportunity to recognize its bearing on my own paper while the latter was still under press. While both theories originated several years before publication, Tarski's priority at the points of close contact seems quite clear. See also the bibliographical indications contained in a footnote in the end of this paper.

[5]) These remarks are developed in greater detail by M. H. Stone, Amer. Journ. Math. 59 (1937), pp. 506-514.

The general plan of the paper is as follows: in § 1 we present in tabular form the essential data concerning the behavior of ideals in various specified classes under the algebraic operations and relativisation; in §§ 2, 3 we discuss the special types of Boolean ring described in the preceding paragraph; in § 4 we apply results of our paper A to obtain topological constructions and interpretations related to §§ 2, 3; in §§ 5-9 we obtain ideal-structural characterizations and properties of various special Boolean rings, chiefly those described in §§ 2, 3; in §§ 10, 11 we tabulate (and complete) the earlier work according to the different possible types of ideal-structure under the fundamental classification of R. Finally, we show that the tables of § 1 give "best possible" results except for special types of Boolean ring occurring among those obtained in the earlier sections. The notations of the paper will be taken directly from R; but we shall allow ourselves on occasion to replace the term "Boolean ring" by the term "ring", since no other type of ring is considered here.

§ 1. Algebraic Operations on Ideals. From the ideals $\mathfrak{a}$ and $\mathfrak{b}$ we can form ideals $\mathfrak{a}\vee\mathfrak{b}$, $\mathfrak{a}\mathfrak{b}$, $\mathfrak{a}'$; and, if $\mathfrak{a}$ is contained in $\mathfrak{b}$, we can perform the operation — relativisation — of presenting $\mathfrak{a}$ as an ideal in the ring $\mathfrak{b}$. We shall devote the present section to a study of the behavior of ideals in the various fundamental classes under these algebraic operations. A knowledge of the results is essential in subsequent proofs. In § 12 we shall show that these results are "the best possible".

We begin by a consideration of possible inferences about the classification of $\mathfrak{a}$ and $\mathfrak{b}$ from the assumption that $\mathfrak{a}$ is contained in $\mathfrak{b}$. The only generally valid assertion we can make is the following

Theorem 1.1. *If the ideals $\mathfrak{a}$ and $\mathfrak{b}$ in a ring A have the property that $\mathfrak{a}\subset\mathfrak{b}$, then*

(1) $\mathfrak{a}\,\epsilon\,\mathfrak{S}$ *and* $\mathfrak{b}\,\epsilon\,\mathfrak{P}$ *imply* $\mathfrak{a}\,\epsilon\,\mathfrak{P}$;

(2) $\mathfrak{a}\,\epsilon\,\mathfrak{P}^*\varDelta\mathfrak{P}$ *and* $\mathfrak{b}\,\epsilon\,\mathfrak{S}$ *imply* $\mathfrak{b}\,\epsilon\,\mathfrak{P}^*\varDelta\mathfrak{P}$.

We establish (1) as follows: if $\mathfrak{a}$ is simple and $\mathfrak{b}$ principal, then $\mathfrak{a}=\mathfrak{a}\mathfrak{b}$ is principal by $\mathfrak{R}$ Th. 26. We then obtain (2) as follows: if $\mathfrak{a}$ is semiprincipal but not principal, then $\mathfrak{a}'$ is principal by $\mathfrak{R}$ Th. 32 (4_2); if $\mathfrak{b}$ is simple, then $\mathfrak{b}'$ is simple by $\mathfrak{R}$ Th. 30; hence the relation $\mathfrak{b}'\subset\mathfrak{a}'$ of $\mathfrak{R}$ Th. 20 (1) combined with (1) above shows that $\mathfrak{b}'$ is principal; thus $\mathfrak{b}$ is semiprincipal by $\mathfrak{R}$ Th. 32; and the fact that $\mathfrak{a}$ is not principal shows by (1) that $\mathfrak{b}$ cannot be principal.

We pass now successively to various similar inferences.

Theorem 1.2. *The three accompanying tables exhibit the dependence of the classification of the ideals $\mathfrak{a}'$, $\mathfrak{a}\vee\mathfrak{b}$, and $\mathfrak{a}\mathfrak{b}$, respectively and in that order, upon the classifications of $\mathfrak{a}$ and $\mathfrak{b}$: the class of $\mathfrak{a}$ is shown at the left, that of $\mathfrak{b}$ at the top (in the second and third tables), and that of the resultant ideal in the appropriate row and column.*

$\mathfrak{P}$	$\mathfrak{P}^*$
$\mathfrak{P}^*$	$\mathfrak{P}^*$
$\mathfrak{S}$	$\mathfrak{S}$
$\mathfrak{N}$	$\mathfrak{N}$
$\mathfrak{I}$	$\mathfrak{N}$

	$\mathfrak{P}$	$\mathfrak{P}^*$	$\mathfrak{S}$	$\mathfrak{N}$	$\mathfrak{I}$
$\mathfrak{P}$	$\mathfrak{P}$	$\mathfrak{P}^*$	$\mathfrak{S}$	$\mathfrak{N}$	$\mathfrak{I}$
$\mathfrak{P}^*$	$\mathfrak{P}^*$	$\mathfrak{P}^*$	$\mathfrak{S}$	$\mathfrak{N}$	$\mathfrak{I}$
$\mathfrak{S}$	$\mathfrak{S}$	$\mathfrak{S}$	$\mathfrak{S}$	$\mathfrak{N}$	$\mathfrak{I}$
$\mathfrak{N}$	$\mathfrak{N}$	$\mathfrak{N}$	$\mathfrak{N}$	$\mathfrak{I}$	$\mathfrak{I}$
$\mathfrak{I}$	$\mathfrak{I}$	$\mathfrak{I}$	$\mathfrak{I}$	$\mathfrak{I}$	$\mathfrak{I}$

	$\mathfrak{P}$	$\mathfrak{P}^*$	$\mathfrak{S}$	$\mathfrak{N}$	$\mathfrak{I}$
$\mathfrak{P}$	$\mathfrak{P}$	$\mathfrak{P}$	$\mathfrak{P}$	$\mathfrak{N}$	$\mathfrak{I}$
$\mathfrak{P}^*$	$\mathfrak{P}$	$\mathfrak{P}^*$	$\mathfrak{S}$	$\mathfrak{N}$	$\mathfrak{I}$
$\mathfrak{S}$	$\mathfrak{P}$	$\mathfrak{S}$	$\mathfrak{S}$	$\mathfrak{N}$	$\mathfrak{I}$
$\mathfrak{N}$	$\mathfrak{N}$	$\mathfrak{N}$	$\mathfrak{N}$	$\mathfrak{N}$	$\mathfrak{I}$
$\mathfrak{I}$	$\mathfrak{I}$	$\mathfrak{I}$	$\mathfrak{I}$	$\mathfrak{I}$	$\mathfrak{I}$

With the exception of the result that $\mathfrak{a}\,\epsilon\,\mathfrak{N}$ and $\mathfrak{b}\,\epsilon\,\mathfrak{S}$ imply $\mathfrak{a}\vee\mathfrak{b}\,\epsilon\,\mathfrak{N}$, the three tables can be filled in by reference to $\mathfrak{R}$ Def. 8 and $\mathfrak{R}$ Ths. 23, 24, 26, 27, 30, 31, and 32. We prove the one result still necessary, in the following manner: if a is an arbitrary element in $(\mathfrak{a}\vee\mathfrak{b})''$, then $\mathfrak{a}(a)=\mathfrak{a}(a)(\mathfrak{b}\vee\mathfrak{b}')=\mathfrak{a}(a)\mathfrak{b}\vee\mathfrak{a}(a)\mathfrak{b}'\subset\mathfrak{b}\vee(\mathfrak{a}\vee\mathfrak{b})''\mathfrak{b}'=$ $=\mathfrak{b}\vee[(\mathfrak{a}\vee\mathfrak{b})'\vee\mathfrak{b}]'=\mathfrak{b}\vee(\mathfrak{a}'\mathfrak{b}'\vee\mathfrak{b})'=\mathfrak{b}\vee[\mathfrak{a}'\mathfrak{b}'\vee(\mathfrak{a}'\mathfrak{b}\vee\mathfrak{b})]'=\mathfrak{b}\vee[\mathfrak{a}'\vee\mathfrak{b}]'=$ $=\mathfrak{b}\vee\mathfrak{a}''\mathfrak{b}'=\mathfrak{b}\vee\mathfrak{a}\mathfrak{b}'=\mathfrak{b}\vee\mathfrak{a}=\mathfrak{a}\vee\mathfrak{b}$ by virtue of the relations $\mathfrak{a}''=\mathfrak{a}$, $\mathfrak{b}\vee\mathfrak{b}'=\mathfrak{e}$; and we conclude that $(\mathfrak{a}\vee\mathfrak{b})''\subset\mathfrak{a}\vee\mathfrak{b}\subset(\mathfrak{a}\vee\mathfrak{b})''$, $\mathfrak{a}\vee\mathfrak{b}=(\mathfrak{a}\vee\mathfrak{b})''$, $\mathfrak{a}\vee\mathfrak{b}\,\epsilon\,\mathfrak{N}$, as we wished.

Theorem 1.3. *The three accompanying tables exhibit the dependence of the classification of $\mathfrak{a}$ relative to $\mathfrak{b}$ when $\mathfrak{a}\subset\mathfrak{b}$, of $\mathfrak{a}\mathfrak{b}$ relative to $\mathfrak{b}$, and of $\mathfrak{a}$ relative to $\mathfrak{a}\vee\mathfrak{b}$, respectively and in that order, upon the classification of the ideals $\mathfrak{a}$ and $\mathfrak{b}$ in the ring A: the class of $\mathfrak{a}$ is shown at the left, the class of $\mathfrak{b}$ at the top, and the relative class of $\mathfrak{a}$ in $\mathfrak{b}$, $\mathfrak{a}\mathfrak{b}$ in $\mathfrak{b}$, or of $\mathfrak{a}$ in $\mathfrak{a}\vee\mathfrak{b}$, respectively, in the appropriate row and column.*

	$\mathfrak{P}$	$\mathfrak{P}^*$	$\mathfrak{S}$	$\mathfrak{N}$	$\mathfrak{I}$
$\mathfrak{P}$	$\mathfrak{P}$	$\mathfrak{P}$	$\mathfrak{P}$	$\mathfrak{P}$	$\mathfrak{P}$
$\mathfrak{P}^*$	$\mathfrak{P}$	$\mathfrak{P}^*$	$\mathfrak{P}^*$	$\mathfrak{S}$	$\mathfrak{S}$
$\mathfrak{S}$	$\mathfrak{P}$	$\mathfrak{S}$	$\mathfrak{S}$	$\mathfrak{S}$	$\mathfrak{S}$
$\mathfrak{N}$	$\mathfrak{N}$	$\mathfrak{N}$	$\mathfrak{N}$	$\mathfrak{N}$	$\mathfrak{N}$
$\mathfrak{I}$	$\mathfrak{I}$	$\mathfrak{I}$	$\mathfrak{I}$	$\mathfrak{I}$	$\mathfrak{I}$

	$\mathfrak{P}$	$\mathfrak{P}^*$	$\mathfrak{S}$	$\mathfrak{N}$	$\mathfrak{I}$
$\mathfrak{P}$	$\mathfrak{P}$	$\mathfrak{P}$	$\mathfrak{P}$	$\mathfrak{S}$	$\mathfrak{S}$
$\mathfrak{P}^*$	$\mathfrak{P}$	$\mathfrak{P}^*$	$\mathfrak{P}^*$	$\mathfrak{S}$	$\mathfrak{S}$
$\mathfrak{S}$	$\mathfrak{P}$	$\mathfrak{S}$	$\mathfrak{S}$	$\mathfrak{S}$	$\mathfrak{S}$
$\mathfrak{N}$	$\mathfrak{N}$	$\mathfrak{N}$	$\mathfrak{N}$	$\mathfrak{N}$	$\mathfrak{N}$
$\mathfrak{I}$	$\mathfrak{I}$	$\mathfrak{I}$	$\mathfrak{I}$	$\mathfrak{I}$	$\mathfrak{I}$

	$\mathfrak{P}$	$\mathfrak{P}^*$	$\mathfrak{S}$	$\mathfrak{N}$	$\mathfrak{I}$
$\mathfrak{P}$	$\mathfrak{P}$	$\mathfrak{P}$	$\mathfrak{P}$	$\mathfrak{P}$	$\mathfrak{P}$
$\mathfrak{P}^*$	$\mathfrak{P}^*$	$\mathfrak{P}^*$	$\mathfrak{P}^*$	$\mathfrak{S}$	$\mathfrak{S}$
$\mathfrak{S}$	$\mathfrak{P}^*$	$\mathfrak{S}$	$\mathfrak{S}$	$\mathfrak{S}$	$\mathfrak{S}$
$\mathfrak{N}$	$\mathfrak{N}$	$\mathfrak{N}$	$\mathfrak{N}$	$\mathfrak{N}$	$\mathfrak{N}$
$\mathfrak{I}$	$\mathfrak{I}$	$\mathfrak{I}$	$\mathfrak{I}$	$\mathfrak{I}$	$\mathfrak{I}$

If $\mathfrak{a}$ and $\mathfrak{c}$ are arbitrary ideals, then $\mathfrak{R}$ Th. 22 shows that the ideal $\mathfrak{ac}$ in $\mathfrak{c}$ has orthocomplement $\mathfrak{a}'\mathfrak{c}$ relative to $\mathfrak{c}$; and that the ideal $\mathfrak{a}'\mathfrak{c}$ in $\mathfrak{c}$ similarly has orthocomplement $\mathfrak{a}''\mathfrak{c}$ relative to $\mathfrak{c}$. Thus $\mathfrak{a}\epsilon\mathfrak{N}$ implies $\mathfrak{a}=\mathfrak{a}''$, $\mathfrak{ac}=\mathfrak{a}''\mathfrak{c}$, so that $\mathfrak{ac}$ is normal relative to $\mathfrak{c}$. Similarly $\mathfrak{a}\epsilon\mathfrak{S}$ implies $\mathfrak{a}\vee\mathfrak{a}'=\mathfrak{e}$, $\mathfrak{ac}\vee\mathfrak{a}'\mathfrak{c}=\mathfrak{ec}=\mathfrak{c}$, so that $\mathfrak{ac}$ is simple relative to $\mathfrak{c}$. Accordingly, all entries $\mathfrak{J}$, $\mathfrak{N}$, $\mathfrak{S}$ in the three tables are obviously justified; and, furthermore, we see that the remaining entries (actually $\mathfrak{P}$ and $\mathfrak{P}^*$) cannot be "worse" than $\mathfrak{S}$. Now in order that $\mathfrak{ac}$ be principal relative to $\mathfrak{c}$ it is necessary and sufficient that $\mathfrak{ac}$ be principal in A. In case $\mathfrak{a}\subset\mathfrak{b}$ we take $\mathfrak{c}=\mathfrak{b}$, $\mathfrak{ac}=\mathfrak{ab}=\mathfrak{a}$. With the aid of Th. 1.1 (1) we then see that all entries in the first table are justified. For the second table, we take $\mathfrak{c}=\mathfrak{b}$ and use the table for $\mathfrak{ab}$ in Th. 1.2, obtaining justification for all the entries $\mathfrak{P}$. Similarly for the third table, we take $\mathfrak{c}=\mathfrak{a}\vee\mathfrak{b}$, $\mathfrak{ac}=\mathfrak{a}(\mathfrak{a}\vee\mathfrak{b})=\mathfrak{a}$ and conclude that $\mathfrak{a}$ is principal relative to $\mathfrak{a}\vee\mathfrak{b}$ whenever it is principal in A.

We still have to justify the various entries $\mathfrak{P}^*$ in the three tables. If $\mathfrak{a}$ is semiprincipal and $\mathfrak{c}$ is simple, then at least one of the ideals $\mathfrak{a}$ and $\mathfrak{a}'$ is principal by $\mathfrak{R}$ Def. 8 and $\mathfrak{R}$ Th. 32; and hence the corresponding one of the ideals $\mathfrak{ac}$ and $\mathfrak{a}'\mathfrak{c}$ must be principal by virtue of Th. 1.2. Thus we conclude that in $\mathfrak{c}$ either $\mathfrak{ac}$ or its (relative) orthocomplement $\mathfrak{a}'\mathfrak{c}$ is principal. Since $\mathfrak{ac}$ is simple relative to $\mathfrak{c}$ by our preceding results it must be semiprincipal relative to $\mathfrak{c}$. If we take $\mathfrak{a}\subset\mathfrak{b}=\mathfrak{c}$, $\mathfrak{ac}=\mathfrak{a}$, we obtain justification for the entries $\mathfrak{P}^*$ in the first table. If we take $\mathfrak{c}=\mathfrak{b}$, $\mathfrak{ac}=\mathfrak{ab}$, we similarly obtain justification for the entries $\mathfrak{P}^*$ in the second table. If we take $\mathfrak{c}=\mathfrak{a}\vee\mathfrak{b}$, $\mathfrak{ac}=\mathfrak{a}$, and note that $\mathfrak{a}\epsilon\mathfrak{P}^*$, $\mathfrak{b}\epsilon\mathfrak{S}$ imply $\mathfrak{a}\vee\mathfrak{b}\epsilon\mathfrak{S}$ by Th. 1.2, we obtain justification for all entries $\mathfrak{P}^*$ in the third table, with the exception of that in the third row. In the exceptional case we know that $\mathfrak{a}$ is simple relative to $\mathfrak{a}\vee\mathfrak{b}$ from our preceding results; since the orthocomplement of $\mathfrak{a}$ relative to $\mathfrak{a}\vee\mathfrak{b}$ is $\mathfrak{a}'(\mathfrak{a}\vee\mathfrak{b})=\mathfrak{a}'\mathfrak{b}$ and since $\mathfrak{a}\epsilon\mathfrak{S}$, $\mathfrak{b}\epsilon\mathfrak{P}$ imply $\mathfrak{a}'\mathfrak{b}\epsilon\mathfrak{P}$ by virtue of Th. 1.2, we conclude that $\mathfrak{a}$ is semiprincipal relative to $\mathfrak{a}\vee\mathfrak{b}$. This completes our discussion.

Theorem 1.4. *The accompanying table exhibits the dependence of the classification of the ideal $\mathfrak{a}$ in a ring A upon the classification of $\mathfrak{a}$ relative to an ideal $\mathfrak{b}$ containing it, and the classification of $\mathfrak{b}$ in A: the class of $\mathfrak{a}$ relative to $\mathfrak{b}$ is shown at the left, the class of $\mathfrak{b}$ in A is shown at the top, and the class of $\mathfrak{a}$ in A is shown in the appropriate row and column.*

	$\mathfrak{P}$	$\mathfrak{P}^*$	$\mathfrak{S}$	$\mathfrak{N}$	$\mathfrak{J}$
$\mathfrak{P}$	$\mathfrak{P}$	$\mathfrak{P}$	$\mathfrak{P}$	$\mathfrak{P}$	$\mathfrak{P}$
$\mathfrak{P}^*$	$\mathfrak{P}$	$\mathfrak{P}^*$	$\mathfrak{S}$	$\mathfrak{N}$	$\mathfrak{J}$
$\mathfrak{S}$	$\mathfrak{P}$	$\mathfrak{S}$	$\mathfrak{S}$	$\mathfrak{N}$	$\mathfrak{J}$
$\mathfrak{N}$	$\mathfrak{N}$	$\mathfrak{N}$	$\mathfrak{N}$	$\mathfrak{N}$	$\mathfrak{J}$
$\mathfrak{J}$	$\mathfrak{J}$	$\mathfrak{J}$	$\mathfrak{J}$	$\mathfrak{J}$	$\mathfrak{J}$

If $\mathfrak{a}$ is principal relative to $\mathfrak{b}$, then it is principal also in A, as we have observed previously. If $\mathfrak{b}$ is principal in A, it is a ring with unit; hence if $\mathfrak{a}$ is simple relative to $\mathfrak{b}$, it is principal relative to $\mathfrak{b}$ by virtue of Th. 1.1 (1) and therefore principal in A. Thus all the entries $\mathfrak{P}$ in the table are justified. The orthocomplement of $\mathfrak{a}$ relative to $\mathfrak{b}$ is $\mathfrak{a}'\mathfrak{b}$, of $\mathfrak{a}'\mathfrak{b}$ relative to $\mathfrak{b}$ is $\mathfrak{a}''\mathfrak{b}$, in accordance with $\mathfrak{R}$ Th. 22. Thus if $\mathfrak{a}$ is normal relative to $\mathfrak{b}$ we have $\mathfrak{a}=\mathfrak{a}''\mathfrak{b}$; by Th. 1.2 the ideal $\mathfrak{a}''$ is normal in A; and hence, if $\mathfrak{b}$ is normal in A, the product $\mathfrak{a}''\mathfrak{b}=\mathfrak{a}$ is normal in A by virtue of Th. 1.2. Similarly, if $\mathfrak{a}$ is simple relative to $\mathfrak{b}$ and $\mathfrak{b}$ is simple in A, we have $\mathfrak{a}\vee\mathfrak{a}'\mathfrak{b}=\mathfrak{b}$, $\mathfrak{b}<\mathfrak{b}'=\mathfrak{e}$, $\mathfrak{a}'\supset\mathfrak{b}'$ and hence $\mathfrak{a}\vee\mathfrak{a}'=\mathfrak{a}\vee\mathfrak{a}'(\mathfrak{b}\vee\mathfrak{b}')=(\mathfrak{a}\vee\mathfrak{a}'\mathfrak{b})\vee\mathfrak{a}'\mathfrak{b}'=\mathfrak{b}\vee\mathfrak{a}'\mathfrak{b}'=$ $=\mathfrak{b}\vee\mathfrak{b}'=\mathfrak{e}$, so that $\mathfrak{a}$ is simple in A. Thus the entries $\mathfrak{N}$ and $\mathfrak{S}$ are justified. Finally, if $\mathfrak{a}$ is semiprincipal but not principal relative to $\mathfrak{b}$, there exists an element a in $\mathfrak{b}$ such that $\mathfrak{a}=\mathfrak{a}'(a)\mathfrak{b}$, where $\mathfrak{a}'(a)\mathfrak{b}$ is the orthocomplement relative to $\mathfrak{b}$ of the principal ideal $\mathfrak{a}(a)\subset\mathfrak{b}$; if $\mathfrak{b}$ is semiprincipal but not principal in A, then there exists an element b in A such that $\mathfrak{b}=\mathfrak{a}'(b)$; and hence we find that $\mathfrak{a}=\mathfrak{a}'(a)\mathfrak{b}=$ $=\mathfrak{a}'(a)\mathfrak{a}'(b)$ is semiprincipal but not principal in A by virtue of $\mathfrak{R}$ Th. 32 or Th. 1.2. Taken with the results already established, this justifies the entry $\mathfrak{P}^*$.

We now consider the possible inferences about the classification of $\mathfrak{a}$ relative to an ideal $\mathfrak{b}$ containing it from a knowledge of the class of $\mathfrak{b}$ in A; and also the possible inferences about the classification of $\mathfrak{b}$ in A from a knowledge of the classification of $\mathfrak{a}$ relative to $\mathfrak{b}$. The only generally valid assertion we can make is a repetition of a previous result:

Theorem 1.5. *If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in a ring A such that $\mathfrak{a}\subset\mathfrak{b}$ and if $\mathfrak{b}$ is principal in A, then $\mathfrak{a}$ is simple relative to $\mathfrak{b}$ if and only if it is principal relative to $\mathfrak{b}$ (and hence also in A).*

We still have to study the behavior of prime ideals in a similar way. The remainder of the present section will be devoted to the necessary investigations. We first have:

Theorem 1.6. *If $\mathfrak{a}$ is an arbitrary ideal and $\mathfrak{p}$ a prime ideal in a ring A, then the classification of the ideal $\mathfrak{ap}$ is determined as follows:*

(1) *if $\mathfrak{ap}=\mathfrak{a}$ or if $\mathfrak{p}$ is normal, then $\mathfrak{a}$ and $\mathfrak{ap}$ have the same classification in A;*

(2) *if $\mathfrak{ap}\neq\mathfrak{a}$ and $\mathfrak{p}$ is not normal, then $\mathfrak{ap}$ is not normal.*

According to R Ths. 38, 39, 41, we may make the following preliminary remarks: one and only one of the relations $\mathfrak{a}\mathfrak{p}=\mathfrak{a}$ and $\mathfrak{a}\mathfrak{p}\neq\mathfrak{a}$ is valid; if $\mathfrak{a}\mathfrak{p}=\mathfrak{a}$ and $\mathfrak{p}$ is normal, then $\mathfrak{p}$ is semiprincipal, $\mathfrak{p}'$ is principal, and $\mathfrak{p}\vee\mathfrak{p}'\neq\mathfrak{e}$, $\mathfrak{a}\mathfrak{p}'=\mathfrak{p}'\neq\mathfrak{o}$; if $\mathfrak{a}\mathfrak{p}\neq\mathfrak{a}$ and $\mathfrak{p}$ is not normal, then $\mathfrak{a}'\subset\mathfrak{p}$ and $\mathfrak{p}'=\mathfrak{o}$. The case where $\mathfrak{o}\mathfrak{p}=\mathfrak{a}$ is trivial. We turn therefore to the case where $\mathfrak{a}\mathfrak{p}\neq\mathfrak{a}$. First, let $\mathfrak{p}$ be normal. Then Th. 1.2 shows that $\mathfrak{a}\mathfrak{p}$, where $\mathfrak{p}$ is semiprincipal, belongs to the same class (or a preceding class) as does $\mathfrak{a}$ in the sequence $\mathfrak{P}, \mathfrak{P}^*, \mathfrak{S}, \mathfrak{N}, \mathfrak{J}$. On the other hand, if we write $\mathfrak{a}=\mathfrak{a}(\mathfrak{p}\vee\mathfrak{p}')=\mathfrak{a}\mathfrak{p}\vee\mathfrak{p}'$ where $\mathfrak{p}'$ is principal in accordance with the preceding remarks, we see by Th. 1.2 again that $\mathfrak{a}$ belongs to the same class (or a preceding class) as does $\mathfrak{a}\mathfrak{p}$ in the sequence $\mathfrak{P}, \mathfrak{P}^*, \mathfrak{S}, \mathfrak{N}, \mathfrak{J}$. Hence $\mathfrak{a}$ and $\mathfrak{a}\mathfrak{p}$ have the same classification in A. Finally we treat the case where $\mathfrak{a}\mathfrak{p}\neq\mathfrak{a}$ and $\mathfrak{p}$ is not normal. Since $\mathfrak{a}\mathfrak{p}\subset\mathfrak{a}$, its orthocomplement relative to $\mathfrak{a}$ is given by $(\mathfrak{a}\mathfrak{p})'\mathfrak{a}$ and also by $\mathfrak{p}'\mathfrak{a}=\mathfrak{o}$. Hence we have $(\mathfrak{a}\mathfrak{p})'\mathfrak{a}=\mathfrak{o}$, $\mathfrak{a}\subset(\mathfrak{a}\mathfrak{p})''$. Since $\mathfrak{a}\mathfrak{p}$ is contained in $\mathfrak{a}$ but distinct from $\mathfrak{a}$, we must therefore have $\mathfrak{a}\mathfrak{p}\neq(\mathfrak{a}\mathfrak{p})''$. Thus $\mathfrak{a}\mathfrak{p}$ is not normal in this case.

Theorem 1.7. *If $\mathfrak{a}$ is an arbitrary ideal, $\mathfrak{p}$ a prime ideal in a ring A, then the classification of the ideal $\mathfrak{a}\mathfrak{p}$ relative to $\mathfrak{a}$ is determined as follows:*

(1) *if $\mathfrak{a}\mathfrak{p}=\mathfrak{a}$, then $\mathfrak{a}\mathfrak{p}$ is semiprincipal relative to $\mathfrak{a}$;*

(2) *if $\mathfrak{a}\mathfrak{p}\neq\mathfrak{a}$, then $\mathfrak{a}\mathfrak{p}$ is a prime ideal in $\mathfrak{a}$ which is normal relative to $\mathfrak{a}$ if and only if $\mathfrak{p}$ is normal in A.*

If $\mathfrak{a}\mathfrak{p}=\mathfrak{a}$, then $\mathfrak{a}\mathfrak{p}$ coincides with the orthocomplement $\mathfrak{o}'\mathfrak{a}=\mathfrak{e}\mathfrak{a}=\mathfrak{a}$ of the principal ideal $\mathfrak{o}=\mathfrak{a}(0)$ relative to $\mathfrak{a}$; and $\mathfrak{a}\mathfrak{p}$ is thus semiprincipal relative to $\mathfrak{a}$. If $\mathfrak{a}\mathfrak{p}\neq\mathfrak{a}$, we can show as follows that $\mathfrak{a}\mathfrak{p}$ is prime in $\mathfrak{a}$: if a and b are elements of $\mathfrak{a}$ such that $ab\in\mathfrak{a}\mathfrak{p}$, then $ab\in\mathfrak{p}$ and hence $a\in\mathfrak{p}$ or $b\in\mathfrak{p}$; now $a\in\mathfrak{a}$ and $a\in\mathfrak{p}$ would imply $a\in\mathfrak{a}\mathfrak{p}$ and similarly $b\in\mathfrak{a}$ and $b\in\mathfrak{p}$ would imply $b\in\mathfrak{a}\mathfrak{p}$; and we therefore conclude that $ab\in\mathfrak{a}\mathfrak{p}$, $a\in\mathfrak{a}$, $b\in\mathfrak{a}$ imply $a\in\mathfrak{a}\mathfrak{p}$ or $b\in\mathfrak{a}\mathfrak{p}$. If $\mathfrak{a}\mathfrak{p}$ is normal in $\mathfrak{a}$, then its orthocomplement relative to $\mathfrak{a}$ is $\mathfrak{a}\mathfrak{p}'\neq\mathfrak{o}$ in accordance with R Th. 38; and we conclude that $\mathfrak{p}'\neq\mathfrak{o}$, $\mathfrak{p}\in\mathfrak{N}$. On the other hand, if $\mathfrak{p}$ is normal in A, we have $\mathfrak{p}'\neq\mathfrak{o}$, $\mathfrak{a}\mathfrak{p}'=\mathfrak{p}'\neq\mathfrak{o}$; and, by R Th. 38 again, we conclude that $\mathfrak{a}\mathfrak{p}$ is normal relative to $\mathfrak{a}$.

Theorem 1.8. *If $\mathfrak{a}$ is an arbitrary ideal and $\mathfrak{p}$ a prime ideal in a ring A, then the classification of $\mathfrak{a}\mathfrak{p}$ relative to $\mathfrak{p}$ is connected with the classification of $\mathfrak{a}$ in A in the following manner:*

(1) *if $\mathfrak{p}$ is normal in A, then the class of $\mathfrak{a}\mathfrak{p}$ relative to $\mathfrak{p}$ is the same as the class of $\mathfrak{a}$ in A;*

(2) *if $\mathfrak{p}$ is not normal in A and $\mathfrak{a}\mathfrak{p}\neq\mathfrak{a}$, then the class of $\mathfrak{a}\mathfrak{p}$ relative to $\mathfrak{p}$ is the same as the class of $\mathfrak{a}$ in A, except for the special situations described as follows:*

 (i) *$\mathfrak{a}\mathfrak{p}$ is never principal relative to $\mathfrak{p}$;*

 (ii) *if $\mathfrak{a}$ is principal in A and A has no unit, then $\mathfrak{a}\mathfrak{p}$ is simple but not semiprincipal relative to $\mathfrak{p}$;*

 (iii) *if $\mathfrak{a}$ is simple in A and A has a unit, then $\mathfrak{a}$ is principal in A and $\mathfrak{a}\mathfrak{p}$ is semiprincipal but not principal relative to $\mathfrak{p}$.*

(3) *if $\mathfrak{p}$ is not normal in A and $\mathfrak{a}\mathfrak{p}=\mathfrak{a}$, then the class of $\mathfrak{a}$ in A determines the class of $\mathfrak{a}\mathfrak{p}=\mathfrak{a}$ relative to $\mathfrak{p}$ in accordance with Th. 1.3, first table, second column; if $\mathfrak{a}\mathfrak{p}$ is principal relative to $\mathfrak{p}$, then $\mathfrak{a}=\mathfrak{a}\mathfrak{p}$ is principal in A; but, even in the case where $\mathfrak{a}\mathfrak{p}$ is semiprincipal relative to $\mathfrak{p}$, the ideal $\mathfrak{a}=\mathfrak{a}\mathfrak{p}$ may be non-normal in A (as in the special instance $\mathfrak{a}=\mathfrak{p}$).*

To prove (1), we first note that, by R Th. 38, the normal prime ideal $\mathfrak{p}$ is semiprincipal. If we know the class of $\mathfrak{a}$ in A, we can therefore apply Th. 1.3, second table, second column, to find the class of $\mathfrak{a}\mathfrak{p}$ relative to $\mathfrak{p}$. As a result we see that, if $\mathfrak{a}$ is normal (simple, semiprincipal, principal) in A, then the corresponding statement is valid for $\mathfrak{a}\mathfrak{p}$ in $\mathfrak{p}$. On the other hand, Th. 1.6 (1) shows that the class of $\mathfrak{a}\mathfrak{p}$ in A is the same as the class of $\mathfrak{a}$ in A. Hence the second column of the table in Th. 1.4 may be regarded as yielding the class of $\mathfrak{a}$ in A when the class of $\mathfrak{a}\mathfrak{p}$ relative to $\mathfrak{p}$ is known. As a result we see that, if $\mathfrak{a}\mathfrak{p}$ is normal (simple, semiprincipal, principal) in $\mathfrak{p}$, then the corresponding statement is valid for $\mathfrak{a}$ in A. Combining these results, we find that $\mathfrak{a}\mathfrak{p}$ is normal, simple, semiprincipal, or principal in $\mathfrak{p}$ if and only if the corresponding statement is valid for $\mathfrak{a}$ in A.

We now consider (2). If $\mathfrak{a}$ is normal in A, then $\mathfrak{a}\mathfrak{p}$ is normal relative to $\mathfrak{p}$ by Th. 1.3; and also, if $\mathfrak{a}$ is simple in A, then $\mathfrak{a}\mathfrak{p}$ is simple relative to $\mathfrak{p}$. On the other hand, if $\mathfrak{a}\mathfrak{p}$ is normal relative to $\mathfrak{p}$, the relation $\mathfrak{a}\mathfrak{p}=\mathfrak{a}''\mathfrak{p}$ taken together with the relation $\mathfrak{a}\vee\mathfrak{p}=\mathfrak{e}$

established in R Th. 39 yields $\mathfrak{a}=\mathfrak{a}\vee\mathfrak{a}\mathfrak{p}=\mathfrak{a}''\mathfrak{a}\vee\mathfrak{a}''\mathfrak{p}=\mathfrak{a}''(\mathfrak{a}\vee\mathfrak{p})=\mathfrak{a}''$, so that $\mathfrak{a}$ is normal in A. Likewise, if $\mathfrak{a}\mathfrak{p}$ is simple relative to $\mathfrak{p}$, we have $\mathfrak{a}\vee\mathfrak{a}'\neq\mathfrak{p}$, $\mathfrak{a}\vee\mathfrak{a}'\supset(\mathfrak{a}\vee\mathfrak{a}')\mathfrak{p}=\mathfrak{a}\mathfrak{p}\vee\mathfrak{a}'\mathfrak{p}=\mathfrak{p}$; since $\mathfrak{p}$ is prime, we conclude that $\mathfrak{a}\vee\mathfrak{a}'=\mathfrak{e}$, $\mathfrak{a}\epsilon\mathfrak{S}$. Thus we have shown that $\mathfrak{a}\mathfrak{p}$ is normal (simple) relative to $\mathfrak{p}$ if and only if $\mathfrak{a}$ is normal (simple) in A. If $\mathfrak{a}\mathfrak{p}$ were principal relative to $\mathfrak{p}$, it would be principal, and hence normal in A; but Th. 1.6 (2) shows that $\mathfrak{a}\mathfrak{p}$ is not normal in A under the present conditions. Thus if $\mathfrak{a}\mathfrak{p}$ is semiprincipal relative to $\mathfrak{p}$, it is necessarily non-principal relative to $\mathfrak{p}$; and its orthocomplement $\mathfrak{a}'\mathfrak{p}$ relative to $\mathfrak{p}$ is therefore principal both in $\mathfrak{p}$ and in A. Since $\mathfrak{a}'\subset\mathfrak{p}$ by R Th. 41 and since $\mathfrak{a}$ is simple in A by preceding results, the ideals $\mathfrak{a}',\mathfrak{a}$ are respectively principal and semiprincipal in A. Moreover, $\mathfrak{a}$ is principal in A if and only if A has a unit, as we see by reference to R Th. 25. On the other hand, if $\mathfrak{a}$ is semiprincipal in A, then $\mathfrak{a}\mathfrak{p}$ is simple relative to $\mathfrak{p}$ and has the orthocomplement $\mathfrak{a}'\mathfrak{p}=\mathfrak{a}'$ in $\mathfrak{p}$. Since $\mathfrak{a}\mathfrak{p}$ is not principal relative to $\mathfrak{p}$, it is semiprincipal relative to $\mathfrak{p}$ if and only if $\mathfrak{a}'$ is principal both in $\mathfrak{p}$ and in A. Thus, if A has no unit and $\mathfrak{a}$ is not principal, then $\mathfrak{a}\mathfrak{p}$ is semiprincipal relative to $\mathfrak{p}$; and if A has no unit and $\mathfrak{a}$ is principal, then $\mathfrak{a}\mathfrak{p}$ is simple but not semiprincipal relative to $\mathfrak{p}$. Likewise, if A has a unit and $\mathfrak{a}$ is simple, then $\mathfrak{a}$ and $\mathfrak{a}'$ are both principal and $\mathfrak{a}\mathfrak{p}$ is semiprincipal relative to $\mathfrak{p}$. This completes the discussion of (2).

The statement (3) is obvious.

§ 2. Totally Additive, Totally Multiplicative, and Related Boolean Rings. In this section we shall study those Boolean rings in which it is possible to form unrestricted (logical) sums or products; and shall discuss certain special types of Boolean ring which can be constructed from them by simple algebraic operations. Each of our fundamental types is thus characterized by a certain constructive representation. We shall prove further that each such representation is unique except for isomorphisms and certain internal modifications. We introduce all our fundamental definitions at once.

First we define unrestricted sums and products as follows:

Definition 2.1. *A non-void subclass* $\mathfrak{a}$ *of a Boolean ring* A *is said to have the sum* b *if* b *is an element of* A *with the properties*:

(1) $b>a$ *for every* a *in* $\mathfrak{a}$;

(2) *if* $c>a$ *for every* a *in* $\mathfrak{a}$, *then* $c>b$.

Definition 2.2. *A non-void subclass* $\mathfrak{a}$ *of a Boolean ring* A *is said to have the product* b *if* b *is an element of* A *with the properties*:

(1) $b<a$ *for every* a *in* $\mathfrak{a}$;

(2) *if* $c<a$ *for every* a *in* $\mathfrak{a}$, *then* $c<b$.

It is immediately evident that the sum and product of $\mathfrak{a}$ are unique whenever they exist. It is also evident that, in case $\mathfrak{a}$ is a finite class consisting of elements $a_1,\ldots,a_n$, its sum and product are the elements $a_1\vee\ldots\vee a_n$ and $a_1\ldots a_n$ respectively.

In terms of the definitions just given for sums and products, we next introduce:

Definition 2.3. *A Boolean ring in which every non-void subclass has a sum is said to be totally additive.*

Definition 2.4. *A Boolean ring in which every non-void subclass has a product is said to be totally multiplicative.*

The types of Boolean ring to be considered in the present section are now indicated in the definitions which follow.

Definition 2.5. *An infinite, totally additive Boolean ring is said to be of type* (α).

Definition 2.6. *A Boolean ring which is isomorphic to a non-normal ideal* $\mathfrak{a}$ *with* $\mathfrak{a}'=\mathfrak{o}$ *in a Boolean ring of type* (α) *is said to be of type* (β_1) *if* $\mathfrak{a}$ *is prime, of type* (β_3) *if* $\mathfrak{a}$ *has an atomic basis* [6]*, of type* (β_2) *if* $\mathfrak{a}$ *neither is prime nor has an atomic basis.*

Definition 2.7. *A Boolean ring which is obtained by adjunction of a unit* [7] *to one of type* (β_k) *is said to be of type* (β_k^*), $k=1,2,3$.

Definition 2.8. *A Boolean ring which is the direct sum* [8] *of two Boolean rings of respective types* (*) *and* (**) *is said to be of type* (*, **).

We turn now to a study of totally additive and totally multiplicative rings. We begin with conditions for the existence of sum or product.

---

[6]) See R Def. 3 and 4, where sum means the finite ring-sum. Since $ab=0$ implies $a+b=a\vee b$, R Def. 4 may be interpreted in terms of the finite logical sum.

[7]) See R Th. 1.

[8]) See R p. 86.

Theorem 2.1. *If $\mathfrak{a}$ is an arbitrary non-void subclass of a ring A, the subclasses $\mathfrak{b}_1=\mathfrak{a}''$ and $\mathfrak{b}_2=\mathop{\boldsymbol{P}}_{a\in\mathfrak{a}}\mathfrak{a}(a)$ are normal ideals in A. In order that $\mathfrak{a}$ have a sum, it is necessary and sufficient that $\mathfrak{b}_1$ be principal, the sum of $\mathfrak{a}$ being the generating element of $\mathfrak{b}_1$; and, in order that $\mathfrak{a}$ have a product, it is necessary and sufficient that $\mathfrak{b}_2$ be principal, the product of $\mathfrak{a}$ being the generating element of $\mathfrak{b}_2$.*

From, $\mathfrak{R}$ Ths. 19 and 27, we see that $\mathfrak{b}_1$ and $\mathfrak{b}_2$ are normal ideals. In order that $c>a$ for every a in $\mathfrak{a}$, it is obviously necessary and sufficient that $\mathfrak{a}(c)\supset\mathfrak{a}$. Moreover, the relations $\mathfrak{a}(c)\supset\mathfrak{a}$ and $\mathfrak{a}(c)\supset\mathfrak{b}_1$ are equivalent: for $\mathfrak{a}(c)\supset\mathfrak{a}$ implies $\mathfrak{a}(c)=\mathfrak{a}''(c)\supset\mathfrak{a}''=\mathfrak{b}_1$ and $\mathfrak{a}(c)\supset\mathfrak{b}_1$ implies $\mathfrak{a}(c)\supset\mathfrak{b}_1=\mathfrak{a}''\supset\mathfrak{a}$, by $\mathfrak{R}$ Th. 20. Now if $\mathfrak{a}$ has a sum b_1, we must have $\mathfrak{a}(b_1)\supset\mathfrak{b}_1$ in accordance with the results just obtained. We can also prove the relation $\mathfrak{a}(b_1)\subset\mathfrak{b}_1$ as follows: if $c\in\mathfrak{b}_1'$, we have $\mathfrak{a}(c)\subset\mathfrak{b}_1$, $\mathfrak{a}'(c)\supset\mathfrak{b}_1''=\mathfrak{b}_1$, $\mathfrak{a}(b_1)\mathfrak{a}'(c)\supset\mathfrak{b}_1\supset\mathfrak{a}$; since $\mathfrak{a}(b_1)\mathfrak{a}'(c)$ is a principal ideal, we must have $\mathfrak{a}(b_1)\mathfrak{a}'(c)\supset\mathfrak{a}(b_1)$ by the definition of the sum b_1; hence we have $\mathfrak{a}(b_1)\mathfrak{a}(c)=\mathfrak{o}$; now by virtue of the fact that c may be chosen arbitrarily in $\mathfrak{b}_1'$, we conclude that $\mathfrak{a}(b_1)\mathfrak{b}_1'=\mathfrak{o}$; and it follows finally that $\mathfrak{a}(b_1)\subset\mathfrak{b}_1''=\mathfrak{b}_1$. Combining these results, we find that $\mathfrak{b}_1=\mathfrak{a}(b_1)$, as we wished to prove. On the other hand, the relation $\mathfrak{b}_1=\mathfrak{a}(b_1)$ shows that $\mathfrak{a}(b_1)\supset\mathfrak{a}$ and also that $\mathfrak{a}(c)\supset\mathfrak{a}$ implies $\mathfrak{a}(c)\supset\mathfrak{b}_1=\mathfrak{a}(b_1)$ or, equivalently $c>b_1$. Hence b_1 is the sum of $\mathfrak{a}$ in accordance with Def. 2.1. In order that $c<a$ for every a in $\mathfrak{a}$, it is evidently necessary and sufficient that $\mathfrak{a}(c)\subset\mathop{\boldsymbol{P}}_{a\in\mathfrak{a}}\mathfrak{a}(a)=\mathfrak{b}_2$. Thus, if $\mathfrak{a}$ has a product b_2, the relation $\mathfrak{a}(b_2)\subset\mathfrak{b}_2$ is valid; and, if c is any element in $\mathfrak{b}_2$, the relations $\mathfrak{a}(c)\subset\mathfrak{b}_2$, $\mathfrak{a}(c)\subset\mathfrak{a}(b_2)$ are valid, so that $\mathfrak{b}_2\subset\mathfrak{a}(b_2)$. Hence we find that $\mathfrak{b}_2=\mathfrak{a}(b_2)$, as we wished to prove. On the other hand, if $\mathfrak{b}_2=\mathfrak{a}(b_2)$, the relation $\mathfrak{a}(b_2)\subset\mathfrak{b}_2$ is trivial; and $\mathfrak{a}(c)\subset\mathfrak{b}_2$ implies $\mathfrak{a}(c)\subset\mathfrak{a}(b_2)$ or, equivalently, $c<b_2$. Hence b_2 is the product of $\mathfrak{a}$ in accordance with Def. 2.2.

It is now easy to characterize totally additive and totally multiplicative rings. We have

Theorem 2.2. *The following properties of a Boolean ring A are equivalent*

(1) *A is totally additive;*
(2) *every normal ideal in A is principal;*
(3) *A has a unit and is totally multiplicative.*

In particular, every finite Boolean ring is totally additive.

It is obvious from Th. 2.1 that (2) implies (1). On the other hand, if we assume (1) and take $\mathfrak{a}$ as an arbitrary normal ideal in A, Th. 2.1 shows that $\mathfrak{a}=\mathfrak{a}''$ is principal; hence (1) implies (2). Since A is a normal ideal relative to itself, (2) implies that A has a unit; and, moreover, (2) also implies in accordance with Th. 2.1 that A is totally multiplicative. Thus (2) implies (3). It is also easy to show that (3) implies (2). If A has a unit, every normal ideal $\mathfrak{a}$ is the product of the principal ideals containing it, by virtue of $\mathfrak{R}$ Th. 27. Thus if (3) holds, Th. 2.1 can be applied with the result that $\mathfrak{a}$ is principal, as we wished to show. The equivalence of (1), (2) and (3) is thereby fully established. A finite Boolean ring is obviously totally additive, since it contains only finite subclasses.

Theorem 2.3. *The following properties of a Boolean ring A are equivalent:*

(1) *A is totally multiplicative;*
(2) *every normal ideal in A is simple;*
(3) *every principal ideal in A is totally additive.*

In particular, every Boolean ring with an atomic basis is totally multiplicative.

First let us prove that (1) implies (3). Let $\mathfrak{a}(a)$ be an arbitrary principal ideal in A. Then it is evident that, considered as a ring, $\mathfrak{a}(a)$ is totally multiplicative. Since $\mathfrak{a}(a)$ has a as its unit, Th. 2.2 shows that $\mathfrak{a}(a)$ is totally additive. Next we show that (3) implies (2). If $\mathfrak{a}$ is a normal ideal in A and $\mathfrak{a}(a)$ is an arbitrary principal ideal in A, then $\mathfrak{a}\mathfrak{a}(a)$ is a normal ideal relative to $\mathfrak{a}(a)$ in accordance with Th. 1.3. Thus (3) implies by Th. 2.2 that $\mathfrak{a}\mathfrak{a}(a)$ is principal in $\mathfrak{a}(a)$ and hence also in A. By $\mathfrak{R}$ Th. 26, we find that $\mathfrak{a}$ is a simple ideal. Finally, we show that (2) implies (1). Let $\mathfrak{a}$ be an arbitrary non-void subclass of A, let a_0 be a selected element of $\mathfrak{a}$, and let $\mathfrak{b}_2=\mathop{\boldsymbol{P}}_{a\in\mathfrak{a}}\mathfrak{a}(a)$ be the normal ideal considered in Th. 2.1. By hypothesis, $\mathfrak{b}_2$ must be simple. Moreover, since $\mathfrak{b}_2=\mathfrak{b}_2\mathfrak{a}(a_0)$, we see by $\mathfrak{R}$ Th. 26 that $\mathfrak{b}_2$ must even be principal. Hence A is totally multiplicative in accordance with Th. 2.1. The equivalence of (1), (2), and (3) is thus established. If a Boolean ring has an atomic basis, then every principal ideal is obviously finite. Hence (3) above is satisfied by virtue of Th. 2.2; and it follows that the ring is totally multiplicative.

We next prove two fundamental imbedding theorems.

Theorem 2.4. *Let A be a Boolean ring; $\mathfrak{P}$ the class of all its principal ideals, considered as a Boolean ring in accordance with R Th. 31; and $\mathfrak{N}$ the class of all its normal ideals, considered as a Boolean ring in accordance with R Th. 29.*

Then $\mathfrak{N}$ is a totally additive Boolean ring; and the correspondence $\mathfrak{a}\leftrightarrow\mathfrak{a}(a)$ carries A isomorphically into the subring $\mathfrak{P}$ of $\mathfrak{N}$ in such, a way that the sum (product) of a subclass of A is carried, when it exists, into the sum (product) in $\mathfrak{N}$ of the corresponding subclass of $\mathfrak{P}$.

In particular, A is totally additive if and only if it is isomorphic to $\mathfrak{N}$.

In R Ths. 29 and 31, we have already shown that the indicated correspondance carries A isomorphically into $\mathfrak{P}$, that $\mathfrak{P}$ is a subring of $\mathfrak{N}$, and that $\mathfrak{N}$ has the property that its normal ideals are all principal. By reference to Th. 2.2, we now see that $\mathfrak{N}$ is totally additive. If A were isomorphic to $\mathfrak{N}$, it would obviously be totally additive; on the other hand, if A were totally additive, then the relation $\mathfrak{N}=\mathfrak{P}$ would hold by Th. 2.2 and the isomorphism between A and $\mathfrak{P}$ would reduce to one between A and $\mathfrak{N}$. Now let $\mathfrak{a}$ be an arbitrary non-void subclass of A, and let $\mathfrak{A}$ be its correspondent in $\mathfrak{P}$ under the isomorphism $\mathfrak{A}\leftrightarrow\mathfrak{P}$. Then the normal ideals $\mathfrak{b}_1$ and $\mathfrak{b}_2$ associated with $\mathfrak{a}$ in the manner described in Th. 2.1 are the sum and product respectively of the class $\mathfrak{A}$ in $\mathfrak{N}$. First let us consider $\mathfrak{b}_1$. A member $\mathfrak{a}(a)$ of $\mathfrak{A}$ obviously satisfies the relation $\mathfrak{b}_1\supset\mathfrak{a}(a)$ since $\mathfrak{b}_1\supset\mathfrak{a}$; and, if $\mathfrak{c}$ is a normal ideal with the property that $\mathfrak{c}\supset\mathfrak{a}(a)$ for every $\mathfrak{a}(a)$ in $\mathfrak{A}$, we have $\mathfrak{c}\supset\mathfrak{a}$, $\mathfrak{c}''=\mathfrak{c}\supset\mathfrak{a}''=\mathfrak{b}_1$. Thus $\mathfrak{b}_1$ is identified as the sum of $\mathfrak{A}$ in $\mathfrak{N}$. The discussion of $\mathfrak{b}_2$ is similar. It is obvious that $\mathfrak{b}_2\subset\mathfrak{a}(a)$ for every $\mathfrak{a}(a)$ in $\mathfrak{A}$; and if $\mathfrak{c}$ is any ideal, whether normal or not, the relation $\mathfrak{c}\subset\mathfrak{a}(a)$ holding for every $\mathfrak{a}(a)$ in $\mathfrak{A}$ implies $\mathfrak{c}\subset\underset{a\epsilon\mathfrak{a}}{P}\mathfrak{a}(a)=\mathfrak{b}_2$. Thus $\mathfrak{b}_2$ is the product of $\mathfrak{A}$ in $\mathfrak{N}$. Th. 2.1 now shows that, if $\mathfrak{a}$ has sum b_1, then $\mathfrak{b}_1=\mathfrak{a}(b_1)$ is the correspondent of b_1 under the isomorphism $A\leftrightarrow\mathfrak{P}$; and that, if $\mathfrak{a}$ has product b_2, then $\mathfrak{b}_2=\mathfrak{a}(b_2)$ is the correspondent of b_2 under this isomorphism.

The theorem just established is significant in two senses. In the present context, it is important because it provides us with a construction for totally additive rings and shows that all possible totally additive rings can be obtained by the construction described: one has merely to start with an arbitrary ring and pass

to the ring of its normal ideals. It has an additional interest in that it shows that an arbitrary Boolean ring can be imbedded in a totally additive ring with preservation of all sums and products, even the infinite ones, which are already present [9]). From R Th. 29 we can now read off the algebraic behavior of sums and products without further difficulty.

Theorem 2.5. *Let a Boolean ring A be contained as an ideal $\mathfrak{a}$ in a totally additive Boolean ring B. Then the ideal $\mathfrak{a}''$ in B is a Boolean ring A_0 containing A as the ideal $\mathfrak{a}$; A_0 is totally additive; A is totally multiplicative; and in A_0 the ideal $\mathfrak{a}$ has the property that $\mathfrak{a}'=\mathfrak{o}$. Next, let a Boolean ring A be contained as an ideal $\mathfrak{a}$ with $\mathfrak{a}'=\mathfrak{o}$ in a totally additive Boolean ring B; and let A_0 be a subring of B containing A. Then A_0 contains A as the ideal $\mathfrak{a}$; A is totally multiplicative; and in A_0 the ideal a has the property that $\mathfrak{a}'=\mathfrak{o}$. Finally, let A be a totally multiplicative Boolean ring contained as an ideal with $\mathfrak{a}'=\mathfrak{o}$ in a Boolean ring A_0; and let $\mathfrak{P}$ and $\mathfrak{N}$ be the Boolean rings of the principal ideals and of the normal ideals, respectively, in A. Then the correspondance $\mathfrak{a}\to\mathfrak{a}(a)\mathfrak{a}$ carries A isomorphically into $\mathfrak{P}$ and A_0 isomorphically into a subring $\mathfrak{A}_0$ of $\mathfrak{N}$; $\mathfrak{P}$ is an ideal in $\mathfrak{N}$ with the property that $\mathfrak{P}'=\mathfrak{O}$; and $\mathfrak{A}_0$ contains $\mathfrak{P}$. In particular, if $A=A_0$, this correspondence imbeds A as the ideal $\mathfrak{P}$ in $\mathfrak{N}$, with $\mathfrak{P}'=\mathfrak{O}$. In order that A_0 be totally multiplicative, it is necessary and sufficient that $\mathfrak{A}_0$ be an ideal in $\mathfrak{N}$. In order that A_0 be totally additive, it is necessary and sufficient that $\mathfrak{A}_0=\mathfrak{N}$; and A_0 is then isomorphic to $\mathfrak{N}$. Consequently, if A_1 and A_2 are totally multiplicative Boolean rings contained as ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$, with $\mathfrak{a}_1'=\mathfrak{o}$ and $\mathfrak{a}_2'=\mathfrak{o}$, in totally additive Boolean rings B_1 and B_2 respectively, then any isomorphism $A_1\leftrightarrow A_2$ can be extended to an isomorphism $B_1\leftrightarrow B_2$. Taken together the preceding results characterize the totally multiplicative Boolean rings as the ideals $\mathfrak{a}$, with $\mathfrak{a}'=\mathfrak{o}$, in totally additive Boolean rings; and show further that, except for isomorphisms, a totally multiplicative Boolean ring has essentially only one representation as such an ideal.*

If A is contained as the ideal $\mathfrak{a}$ in the totally additive Boolean ring B, we show as follows that A is totally multiplicative: if $\mathfrak{b}$ is any ideal in A, it is an ideal in $\mathfrak{a}$ and hence also in B; if $\mathfrak{b}$ is

[9]) This result is given by Mac Neille, The Theory of Partially Ordered Sets, Harvard doctoral dissertation (1935); a summary is given in Proceedings of the National Academy, U. S. A., vol. 22 (1936), pp. 45-50. From letters, I understand that Tarski has obtained this result independently.

normal relative to A, then $\mathfrak{b}=\mathfrak{b}''\mathfrak{a}$; since $\mathfrak{b}''$ is normal in B, it must be principal in accordance with Th. 2.2; by Th. 1.3 it must therefore be simple relative to A; and hence A is totally multiplicative in accordance with Th. 2.3 (2). Considered as a ring, the ideal $\mathfrak{a}''$ is totally multiplicative by the result just established. As a normal ideal in B, it is principal and therefore has its generating element as unit. By Th. 2.2, it is a totally additive ring. Obviously A is contained as an ideal $\mathfrak{a}$ in $\mathfrak{a}''$; and its orthocomplement relative to $\mathfrak{a}''$ is $\mathfrak{a}'\mathfrak{a}''=\mathfrak{o}$. Hence the first part of the theorem is established.

The second part of the theorem offers no difficulty. If A is an ideal $\mathfrak{a}$, with $\mathfrak{a}'=\mathfrak{o}$, in a totally additive Boolean ring B, then A is totally multiplicative, as we have already seen. If now A_0 is a subring $\mathfrak{a}_0$ of B such that $\mathfrak{a}_0\supset\mathfrak{a}$, the class $\mathfrak{a}=\mathfrak{a}\mathfrak{a}_0$ is an ideal in $\mathfrak{a}_0$; and the relation $\mathfrak{a}_0\supset\mathfrak{a}$ permits us to calculate the orthocomplement of $\mathfrak{a}$ relative to $\mathfrak{a}_0$ as $\mathfrak{a}'\mathfrak{a}_0=\mathfrak{o}$, as we see by direct use of R Def. 7.

The third part of the theorem remains to be discussed. Under the assumptions made, we proceed as follows. By Th. 1.3, the ideal $\mathfrak{a}(a)\mathfrak{a}$ is normal relative to $\mathfrak{a}$. The correspondence $a\to\mathfrak{a}(a)\mathfrak{a}$ therefore carries A_0 into a subclass $\mathfrak{A}_0$ of $\mathfrak{N}$; and, since $a\,\epsilon\,\mathfrak{a}$ implies $\mathfrak{a}(a)\subset\mathfrak{a}$ or $\mathfrak{a}(a)\mathfrak{a}=\mathfrak{a}(a)$, it carries A, in particular, into $\mathfrak{P}$. Thus $\mathfrak{A}_0$ contains $\mathfrak{P}$. The relations $\mathfrak{a}(a)\mathfrak{a}\vee\mathfrak{a}(b)\mathfrak{a}=\mathfrak{a}(a\vee b)\mathfrak{a}$, $\mathfrak{a}(a)\mathfrak{a}\cdot\mathfrak{a}(b)\mathfrak{a}=\mathfrak{a}(ab)\mathfrak{a}$ show [10]) that the correspondence from A_0 to $\mathfrak{A}_0$ is a homomorphism in accordance with R Th. 42. It follows that $\mathfrak{A}_0$ is a subring of $\mathfrak{N}$. In order to show that the homomorphism $A_0\to\mathfrak{A}_0$ is actually an isomorphism, we have only to observe that $a\to\mathfrak{a}(a)\mathfrak{a}=\mathfrak{o}$ implies $\mathfrak{a}(a)\subset\mathfrak{a}'=\mathfrak{o}$ and hence $a=0$. We must now verify the assertion that the orthocomplement of $\mathfrak{P}$ in $\mathfrak{N}$ is the class $\mathfrak{O}$ consisting of the zero element $\mathfrak{o}$ alone. If $\mathfrak{b}$ is in $\mathfrak{P}'$, it is a normal ideal relative to $\mathfrak{a}$ and satisfies the relation $\mathfrak{b}\mathfrak{a}(a)=\mathfrak{o}$ for every a in $\mathfrak{a}$. Thus, taking a as an arbitrary element b of $\mathfrak{b}$, we find that $\mathfrak{a}(b)=\mathfrak{b}\mathfrak{a}(b)=\mathfrak{o}$, $b=0$ and hence conclude that $\mathfrak{b}=\mathfrak{o}$, as we wished to show. Up to this point we have not used the hypothesis that A is totally multiplicative. Now, in order to prove that $\mathfrak{P}$ is an ideal in $\mathfrak{N}$, we bring it into play. When A is totally multiplicative, Th. 2.3 shows that $\mathfrak{N}$ coincides with the Boolean ring $\mathfrak{S}$ of all simple ideals in A. From R Th. 30, we know that $\mathfrak{P}$ is an ideal in $\mathfrak{N}=\mathfrak{S}$. Since all our hypotheses are fulfilled by taking A as a totally multiplicative ring and putting $A_0=A$, we can then represent A isomorphically as the ideal $\mathfrak{P}$, with $\mathfrak{P}'=\mathfrak{O}$, in the ring $\mathfrak{N}=\mathfrak{S}$; and Th. 2.4 shows that $\mathfrak{N}$ is totally multiplicative. We thus obtain the imbedding theorem described above. Now, in general, if $\mathfrak{A}_0$ is an ideal in $\mathfrak{N}$, it is totally multiplicative, by results established above; and its isomorph A_0 is also totally multiplicative. On the other hand, if A_0 is totally multiplicative, we can show that $\mathfrak{A}_0$ is an ideal in $\mathfrak{N}$. Since $\mathfrak{A}_0$ is a subring, we have to prove that, if $\mathfrak{a}(a)\mathfrak{a}$ is an arbitrary element of $\mathfrak{A}_0$ and $\mathfrak{b}$ an arbitrary element of $\mathfrak{N}$, then $\mathfrak{a}(a)\mathfrak{a}\mathfrak{b}$ is an element of $\mathfrak{A}_0$ — that is, is representable in the form $\mathfrak{a}(b)\mathfrak{a}$ where $b\,\epsilon\,A_0$. Since $\mathfrak{b}$ is a normal ideal in $\mathfrak{a}$, it is an ideal in A_0 with $\mathfrak{a}\mathfrak{b}=\mathfrak{b}$. Thus we may regard $\mathfrak{a}(a)\mathfrak{b}=\mathfrak{a}(a)\mathfrak{a}\mathfrak{b}$ as an ideal in the principal ideal $\mathfrak{a}(a)$ in A_0. By combining our hypothesis with Th. 2.3 (3), we see that the class $\mathfrak{a}(a)\mathfrak{b}$ has a sum b in $\mathfrak{a}(a)$. According to Th. 2.1 the principal ideal $\mathfrak{a}(b)$, considered in $\mathfrak{a}(a)$, is given by $\mathfrak{a}(b)=\mathfrak{a}(a)\mathfrak{b}''$, since $\mathfrak{a}(a)\mathfrak{b}''$ is the second orthocomplement of $\mathfrak{a}(a)\mathfrak{b}$ relative to $\mathfrak{a}(a)$ in accordance with R Th. 22. Since $\mathfrak{b}$ is normal relative to $\mathfrak{a}$, we have $\mathfrak{b}=\mathfrak{b}''\mathfrak{a}$. Thus we find that $\mathfrak{a}(b)\mathfrak{a}=\mathfrak{a}(a)\mathfrak{b}''\mathfrak{a}=\mathfrak{a}(a)\mathfrak{b}=\mathfrak{a}(a)\mathfrak{a}\mathfrak{b}$, as we wished to prove. Using the result just established, it is easy to determine under what circumstances A_0 is totally additive. By Th. 2.2 (3), we see that A_0 is totally additive if and only if $\mathfrak{A}_0$ has a unit and is an ideal in $\mathfrak{N}$. Obviously $\mathfrak{A}_0$ has these properties if and only if it is a principal ideal in $\mathfrak{N}$. If $\mathfrak{A}_0$ is a principal ideal in $\mathfrak{N}$, then the relation $\mathfrak{A}_0\supset\mathfrak{P}$ implies $\mathfrak{A}_0=\mathfrak{A}_0''\supset\mathfrak{P}''=\mathfrak{O}'=\mathfrak{N}$ and hence $\mathfrak{A}_0=\mathfrak{N}$; and, on the other hand, if $\mathfrak{A}_0=\mathfrak{N}$, then $\mathfrak{A}_0$ is obviously a principal ideal in $\mathfrak{N}$, with $\mathfrak{e}$ as its generating element. Thus A_0 is totally additive if and only if $\mathfrak{A}_0=\mathfrak{N}$. When the latter relation holds, A_0 and $\mathfrak{N}$ are evidently isomorphic. Now let us consider the case of two totally multiplicative Boolean rings A_1 and A_2 contained as ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$, with $\mathfrak{a}_1'=\mathfrak{o}$ and $\mathfrak{a}_2'=\mathfrak{o}$, in totally additive rings B_1 and B_2 respectively. We denote by $\mathfrak{P}(A_1)$, $\mathfrak{P}(A_2)$, $\mathfrak{N}(A_1)$, and $\mathfrak{N}(A_2)$ the associated rings of ideals. By the preceding results there exist isomorphisms $B_1\leftrightarrow\mathfrak{N}(A_1)$ and $B_2\leftrightarrow\mathfrak{N}(A_2)$ carrying A_1 into $\mathfrak{P}(A_1)$ and A_2 into $\mathfrak{P}(A_2)$ respectively. Now any isomorphism $A_1\leftrightarrow A_2$ obviously establishes an isomorphism $\mathfrak{P}(A_1)\leftrightarrow\mathfrak{P}(A_2)$ and an extension of it to an isomorphism $\mathfrak{N}(A_1)\leftrightarrow\mathfrak{N}(A_2)$. Thus if we combine the isomorphisms $B_1\leftrightarrow\mathfrak{N}(A_1)$, $B_2\leftrightarrow\mathfrak{N}(A_2)$ and $\mathfrak{N}(A_1)\leftrightarrow\mathfrak{N}(A_2)$, we obtain an isomorphism $B_1\leftrightarrow B_2$ which carries A_1 into A_2 in the same way as the postulated isomorphism $A_1\leftrightarrow A_2$.

[10]) It must be observed that the sum in $\mathfrak{N}$ is the normalized sum of R Def. 9; but, when, as here, the sum of two ideals is normal, it is equal to their normalized sum.

The characterization of all totally multiplicative Boolean rings follows at once from the preceding results, as stated in the theorem. So likewise does the essential uniqueness of the representation in terms of ideals.

The results up to this point are sufficient to settle the logical status of the types (α), (β_1), (β_2), (β_3); and are also essential to an analysis of the composite types formed from them. Before proceeding to this further analysis, it is convenient to state the information now available about the simple types. We have:

Theorem 2.6. *The four types (α), (β_1), (β_2), (β_3) of Definitions 2.5 and 2.6 are distinct and exhaust the infinite totally multiplicative Boolean rings. A Boolean ring A belongs to type (α) if and only if it is an infinite totally multiplicative ring with unit. It belongs to one of the three types (β_1), (β_2), (β_3) if and only if it is a totally multiplicative ring without unit. It belongs to the type (β_1) if and only if it satisfies one of the following two equivalent criteria:*

(1) *$\mathfrak{P}$ is a prime ideal in $\mathfrak{N}$;*

(2) $\mathfrak{P}+\mathfrak{P}^*=\mathfrak{S}=\mathfrak{N}$.

It belongs to the type (β_3) if and only if it has an infinite atomic basis.

If a ring A belongs to any one of the four types (α), (β_1), (β_2), (β_3), then Th. 2.2 (3) and Th. 2.5 show that A is totally multiplicative. If it belongs to type (α) it has a unit, by Th. 2.2 (3), and is infinite. If it belongs to any one of the types (β_1), (β_2), (β_3), it is isomorphic to a certain non-normal ideal $\mathfrak{a}$. Since $\mathfrak{a}$ is not normal, it is not principal and therefore has no unit. Consequently A has no unit and, by R Th. 1, must be infinite. We thus see that the type (α) is distinct from the aggregate of the three types (β_1), (β_2), (β_3); and that in studying these types we may confine our attention to infinite totally multiplicative rings.

If A is such a ring, the condition that it be infinite being automatically satisfied if it has no unit, we shall show that it belongs to just one of the four types. First, if A has a unit, then Th. 2.2 shows at once that it is of type (α). With this the characterization of type (α) is completed; and it is evident that the only rings of types (β_1), (β_2), (β_3) are totally multiplicative rings without unit. If A is such a ring, we proceed to imbed it as an ideal $\mathfrak{a}$ with $\mathfrak{a}'=\mathfrak{o}$ in a totally additive ring B in accordance with Th. 2.5. We can do so in essentially only one way; and the most convenient way is to identify A with the ideal $\mathfrak{P}$ in $\mathfrak{N}$. Since A and its isomorph $\mathfrak{P}$ are infinite, $\mathfrak{N}$ is also infinite and hence of type (α). Since A has no unit, $\mathfrak{P}$ is not principal in $\mathfrak{N}$ and, by Th. 2.2 (2), cannot be normal in $\mathfrak{N}$. The preliminary conditions of Def. 2.6 are thus satisfied. Now it is evident that the given ring A is of type (β_1) if and only if $\mathfrak{P}$ is a prime ideal in $\mathfrak{N}$; moreover, if we recall our earlier result that a ring of type (β_1) is necessarily a totally multiplicative ring without unit, we see that the necessity of criterion (1) for type (β_1) is now established. On the other hand the sufficiency of this criterion is so far established only for totally multiplicative rings without unit. A brief digression will enable us to remove this difficulty. If A is any ring with $\mathfrak{P}$ a prime ideal in $\mathfrak{N}$, the known relation $\mathfrak{P}'=\mathfrak{O}$ shows that $\mathfrak{P}$ is not normal and, in particular, not principal. Thus A has no unit and $\mathfrak{P}\neq\mathfrak{P}^*$. By R Th. 32 we know that $\mathfrak{P}^*$ is a subring of $\mathfrak{N}$. Since it contains the prime ideal $\mathfrak{P}$ but does not coincide with it, we must have $\mathfrak{P}^*=\mathfrak{N}$. In consequence $\mathfrak{S}=\mathfrak{N}$, and A must be totally multiplicative by virtue of Th. 2.3 (2). It now follows that A is of type (β_1). In this proof, we have found that criterion (1) implies the relations $\mathfrak{P}+\mathfrak{P}^*=\mathfrak{S}=\mathfrak{N}$ of criterion (2); but, conversely, these relations show by R Th. 32 that $\mathfrak{P}$ is a prime ideal in $\mathfrak{P}^*$ and hence also in $\mathfrak{N}$. Having justified our two criteria for type (β_1), we resume our discussion of the case of a totally multiplicative ring A without unit. Our next step is to show that if A has an atomic basis, it is not of type (β_1). Turning our attention to the ideal $\mathfrak{P}$ in $\mathfrak{N}$, we see that $\mathfrak{P}$ has an atomic basis $\mathfrak{B}$. Obviously the ideal generated by $\mathfrak{B}$ in $\mathfrak{N}$ is $\mathfrak{P}$. By R Th. 20, we see that $\mathfrak{B}''\supset\mathfrak{P}$ and hence that $\mathfrak{B}'\subset\mathfrak{P}'=\mathfrak{O}$, $\mathfrak{B}'=\mathfrak{O}$. By R Def. 5 and 7, the latter relation means that $\mathfrak{B}$ is a complete atomic system in $\mathfrak{N}$. Since every normal ideal in $\mathfrak{N}$ is principal, R Th. 62 shows that $\mathfrak{N}$ is isomorphic to the Boolean ring of all subclasses of a fixed (here necessarily infinite) class E, the correspondent of $\mathfrak{B}$ being the system of all one-element subclasses of E. If we now choose $\mathfrak{b}$ and $\mathfrak{c}$ as elements of N corresponding to two disjoint infinite subclasses of E, we see that $\mathfrak{b}\mathfrak{c}=\mathfrak{o}$ but that neither $\mathfrak{b}$ nor $\mathfrak{c}$ is in $\mathfrak{P}$. Hence $\mathfrak{P}$ is not a prime ideal in $\mathfrak{N}$, and A is not of type (β_1). It follows that the types (β_1) and (β_3) are distinct. It follows also that in a ring of type (β_3) the relation $\mathfrak{P}^*\neq\mathfrak{S}$ must hold: for the relation $\mathfrak{P}^*=\mathfrak{S}=\mathfrak{N}$ would imply that $\mathfrak{P}$ is prime in $\mathfrak{N}=\mathfrak{P}^*$. To determine whether A is of type (β_3) or not, we therefore have only to ascertain whether A has an atomic basis or not. Obviously a totally multiplicative ring without unit, being infinite, cannot have a finite atomic basis. On the other hand, any ring with an atomic basis is totally multiplicative by Th. 2.3; and, if the atomic basis is infinite, the ring obviously cannot have a unit. Hence any ring with infinite atomic basis is of type (β_3). The characterization of rings of type (β_3) is thereby completed. Finally we observe that the remaining type (β_2) was so defined as to take in all those totally multiplicative rings without unit which are not of types (β_1) or (β_3). The proof of the theorem is thus brought to a close.

The study of the various composite types depends not only upon the preceding results but also upon some further information, which we shall present next. We first give a few elementary properties of direct sums.

Theorem 2.7. *If a Boolean ring A is represented as a direct sum $A_1 \vee A_2$, then:*

(1) *A_1 and A_2 are simple ideals in A;*

(2) *A is totally multiplicative if and only if A_1 and A_2 are totally multiplicative;*

(3) *A has a unit if and only if A_1 and A_2 both have units;*

(4) *A is totally additive if and only if A_1 and A_2 are totally additive.*

If a Boolean ring A is represented as a direct sum $A_1^* \vee A_2$ where A_1^* is obtained from a ring A_1 without unit by the adjunction of a unit, then:

(1) *A_1, A_2, A_1^*, and $A_1 \vee A_2$ are ideals in A; the ideals A_1^* and A_2 are simple; the ideals A_1 and $A_1 \vee A_2$ are non-normal; and the ideal $A_1 \vee A_2$ is prime;*

(2) *A is totally multiplicative if and only if A_1 is of type (β_1) and A_2 is totally multiplicative;*

(3) *A has a unit if and only if A_2 has a unit;*

(4) *A is totally additive if and only if A_1 is of type (β_1) and A_2 is totally additive.*

In particular, the ring A^* obtained by adjunction of a unit to a ring A without unit is totally additive if and only if A is of type (β_1).

By R Th. 51, the summands A_1 and A_2 are simple ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$ respectively in the direct sum $A=A_1 \vee A_2$. If $\mathfrak{a}$ is an arbitrary ideal in A, then $\mathfrak{a}=\mathfrak{a}\mathfrak{a}_1 \vee \mathfrak{a}\mathfrak{a}_2$. If $\mathfrak{a}$ is normal in A, then $\mathfrak{a}\mathfrak{a}_1$ and $\mathfrak{a}\mathfrak{a}_2$ are also normal in A by Th. 1.2. Conversely, if $\mathfrak{a}\mathfrak{a}_1$ and $\mathfrak{a}\mathfrak{a}_2$ are normal in A, then so is $\mathfrak{a}$. To prove this, we begin by calculating $(\mathfrak{a}\mathfrak{a}_1)''$ and $(\mathfrak{a}\mathfrak{a}_2)''$. We write $(\mathfrak{a}\mathfrak{a}_1)''=(\mathfrak{a}\mathfrak{a}_1)''\mathfrak{a}_1 \vee (\mathfrak{a}\mathfrak{a}_1)''\mathfrak{a}_2$; here we may regard the first term on the right as the second orthocomplement of the ideal $\mathfrak{a}\mathfrak{a}_1=(\mathfrak{a}\mathfrak{a}_1)\mathfrak{a}_1$ relative to $\mathfrak{a}_1$ in accordance with R Th. 22 and hence find that $(\mathfrak{a}\mathfrak{a}_1)''\mathfrak{a}_1=\mathfrak{a}''\mathfrak{a}_1$; and similarly we may regard the remaining term as the second orthocomplement of the ideal $\mathfrak{o}=(\mathfrak{a}\mathfrak{a}_1)\mathfrak{a}_2$ relative to $\mathfrak{a}_2$ and hence find that $(\mathfrak{a}\mathfrak{a}_1)''\mathfrak{a}_2=\mathfrak{o}$. Thus we have $(\mathfrak{a}\mathfrak{a}_1)''=\mathfrak{a}''\mathfrak{a}_1$; and, in the same way, $(\mathfrak{a}\mathfrak{a}_2)''=\mathfrak{a}''\mathfrak{a}_2$. Now, if $\mathfrak{a}\mathfrak{a}_1$ and $\mathfrak{a}\mathfrak{a}_2$ are normal, we have $\mathfrak{a}=\mathfrak{a}\mathfrak{a}_1 \vee \mathfrak{a}\mathfrak{a}_2=(\mathfrak{a}\mathfrak{a}_1)'' \vee (\mathfrak{a}\mathfrak{a}_2)''= =\mathfrak{a}''\mathfrak{a}_1 \vee \mathfrak{a}''\mathfrak{a}_2=\mathfrak{a}''$, so that $\mathfrak{a}$ is also normal. Next Ths. 1.2 and 1.3 show that $\mathfrak{a}\mathfrak{a}_1$ is normal in A if and only if it is normal relative to the simple ideal $\mathfrak{a}_1$ containing it; and likewise that $\mathfrak{a}\mathfrak{a}_2$ is normal in A if and only if it is normal relative to $\mathfrak{a}_2$. Thus, $\mathfrak{a}$ is normal if and only if $\mathfrak{a}\mathfrak{a}_1$ and $\mathfrak{a}\mathfrak{a}_2$ are normal relative to $\mathfrak{a}_1$ and $\mathfrak{a}_2$ respectively. It is now easy to discuss the conditions under which A is totally multiplicative, using the test of Th. 2.3 (2). If $\mathfrak{a}$ is any ideal contained in $\mathfrak{a}_1$ and normal relative to $\mathfrak{a}_1$, then the preceding results show that $\mathfrak{a}=\mathfrak{a}\mathfrak{a}_1 \vee \mathfrak{a}\mathfrak{a}_2$ is normal in A. Hence, if A is totally multiplicative, $\mathfrak{a}$ is simple in A and, according to Th. 1.2, is also simple relative to $\mathfrak{a}_1$. Thus $\mathfrak{a}_1$, and $\mathfrak{a}_2$ likewise, are totally multiplicative rings. On the other hand, if $\mathfrak{a}$ is any normal ideal in A, $\mathfrak{a}\mathfrak{a}_1$ and $\mathfrak{a}\mathfrak{a}_2$ are normal relative to $\mathfrak{a}_1$ and $\mathfrak{a}_2$ respectively. Hence, if $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are totally multiplicative, $\mathfrak{a}\mathfrak{a}_1$ and $\mathfrak{a}\mathfrak{a}_2$ are simple relative to $\mathfrak{a}_1$ and $\mathfrak{a}_2$ respectively and, according to Th. 1.3, are simple also in A. It follows that $\mathfrak{a}=\mathfrak{a}\mathfrak{a}_1 \vee \mathfrak{a}\mathfrak{a}_2$ is simple in A. Thus A is totally multiplicative. We have thereby proved that A is totally multiplicative if and only if A_1 and A_2 both are. If A has a unit, both simple ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are principal so that A_1 and A_2 both have units; and, if A_1 and A_2 both have units, then so does their direct sum A. An easy application of Th. 1.2 (2) to the preceding results now shows that A is totally additive if and only if A_1 and A_2 both are.

In the direct sum $A_1^* \vee A_2$, the summands A_1^* and A_2 are simple ideals as before. Moreover, by R Th. 37, A_1 is a non-normal prime ideal in A_1^*. Hence we see that A_1 is a non-normal ideal in A; and also that $A_1 \vee A_2$ is a non-normal ideal in A, by virtue of the results established in the preceding paragraph.

To show that $A_1 \vee A_2$ is prime, we proceed as follows. Let a and b be elements of A with $ab \in A_1 \vee A_2$. We can then write $a = a_1 + b_1$, $b = a_2 + b_2$ where a_1, a_2 are in A_1^* and b_1, b_2 are in A_2. It is then clear that $ab = a_1 a_2 + b_1 b_2$ where $a_1 a_2$ is in A_1 and $b_1 b_2$ in A_2. By virtue of the fact that A_1 is prime in A_1^*, at least one of the elements a_1, a_2 is in A_1; and then the corresponding element a or b must belong to $A_1 \vee A_2$. Accordingly $A_1 \vee A_2$ is a prime ideal in A. If we apply the results of the preceding paragraph, we see that $A_1^* \vee A_2$ has a unit if and only if A_2 does: and, since A_1^*, being a ring with unit, is totally multiplicative if and only if it is totally additive, that A is totally multiplicative (totally additive) if and only if A_2 is totally multiplicative (totally additive) and A_1^* totally additive.

To complete our discussion we must show that a ring A^* obtained from a ring A without unit by the adjunction of a unit is totally additive (or, equivalently, totally multiplicative) if and only if A is of type (β_1). Since A is a non-normal prime ideal in A^*, the assumption that A^* is totally additive immediately identifies A as of type (β_1) in accordance with Definition 2.6 and Theorem 2.6. On the other hand, if A is of type (β_1), it can be imbedded as a non-normal prime ideal in a totally additive ring B. It is evident that B coincides with its subring generated by the prime ideal A and the unit of B. Hence R Th. 1 shows that A^* is isomorphic to B and thus totally additive.

In further applications of direct sum representations we find the following definition useful:

Definition 2.9. *If a Boolean ring A is represented in two ways as a direct sum:*

$$A = A_1 \vee A_2, \qquad A = A_3 \vee A_4$$

where

$$A_1 = B_1 \vee B_2, \qquad A_3 = B_1 \vee B_4,$$

$$A_2 = B_3 \vee B_4, \qquad A_4 = B_3 \vee B_2,$$

then each representation is said to be obtained from the other by interchange of the direct summands B_2 and B_4.

The connection between the results of Th. 2.5 and representations by direct sums is now easily discussed. We have:

Theorem 2.8. *The following assertions concerning a Boolean ring A_0 are equivalent:*

(1) *A_0 contains an ideal $\mathfrak{a}$ with $\mathfrak{a}' = \mathfrak{o}$ which, considered as a ring, is totally multiplicative and which together with an element a_0 not in $\mathfrak{a}$ generates A_0;*

(2) *there exists a totally additive Boolean ring B containing a subring B_0 which is isomorphic to A_0 and which is generated by an ideal $\mathfrak{b}$, with $\mathfrak{b}' = \mathfrak{o}$, and some element b_0 not in $\mathfrak{b}$;*

(3) *there exist totally multiplicative Boolean rings A_1 and A_2, where A_1 has no unit, such that A_0 is isomorphic to a direct sum $A_1^* \vee A_2$ of the kind discussed in Theorem 2.7.*

In (1) *the ideal $\mathfrak{a}$ is not uniquely determined in general but is necessarily prime, so that a_0 is free to vary outside $\mathfrak{a}$; and in* (2) *the ring B is necessarily isomorphic to $\mathfrak{N}(A_0)$ in such a way that the correspondent of B_0 is $\mathfrak{P}(A_0)$, but the correspondent of $\mathfrak{b}$ in $\mathfrak{P}(A_0)$ is not uniquely determined in general. The representation* (3) *is likewise not uniquely determined. It is possible, however, to pass reversibly from a representation of any kind to one of any other by the following processes: if a representation* (1) *is given, we put $B = \mathfrak{N}(A_0)$, $B_0 = \mathfrak{P}(A_0)$, $\mathfrak{b} = \mathfrak{P}(\mathfrak{a})$ and $b_0 = \mathfrak{a}(a_0)$ to obtain a representation* (2), *and $A_1 = \mathfrak{a}(a_0)\mathfrak{a}$, $A_2 = \mathfrak{a}'(a_0)$ to obtain a representation* (3); *if a representation* (2) *is given, we take $\mathfrak{a}$ and a_0 as the respective correspondents of $\mathfrak{b}$ and b_0 under the isomorphism $A_0 \leftrightarrow B_0$ to obtain a representation* (1), *and put $A_1 = \mathfrak{a}(b_0)\mathfrak{b}$, $A_2 = \mathfrak{a}'(b_0)\mathfrak{b}$ to obtain a representation* (3); *and, if a representation* (3) *is given, we put $\mathfrak{a} = A_1 \vee A_2$, $a_0 = e_1^*$, where e_1^* is the unit of A_1^*, to obtain a representation* (1), *and $B = \mathfrak{N}(A_1) \vee \mathfrak{N}(A_2)$, $\mathfrak{b} = \mathfrak{P}(A_1) \vee \mathfrak{P}(A_2)$, $b_0 = \mathfrak{e}_1$ where $\mathfrak{e}_1$ is the unit of $\mathfrak{N}(A_1)$, to obtain a representation* (2). *In order that a Boolean ring representable in these equivalent forms be totally multiplicative, the following conditions are separately necessary and sufficient:*

(1) *$\mathfrak{a}(a_0)\mathfrak{a}$ is a ring of type (β_1);*

(2) *$\mathfrak{a}(b_0)\mathfrak{b}$ is a ring of type (β_1);*

(3) *A_1 is a ring of type (β_1).*

First, let A_0 have a representation (1). Then the subring of A_0 generated by $\mathfrak{a}$ and a_0 consists of all elements a and $a_0 + a$, where $a \in \mathfrak{a}$; and coincides with A_0. Consequently $A_0/\mathfrak{a}$ is a two-element ring, and $\mathfrak{a}$ is prime in A. The relation $\mathfrak{a}' = \mathfrak{o}$ shows that $\mathfrak{a}$ is not normal.

By Th. 2.5 the correspondence $a\to\mathfrak{a}(a)\mathfrak{a}$ carries A_0 isomorphically into a subring $\mathfrak{A}_0$ of $\mathfrak{N}(\mathfrak{a})$, $\mathfrak{a}$ isomorphically into the ideal $\mathfrak{P}(\mathfrak{a})$ in $\mathfrak{N}(\mathfrak{a})$. On the other hand the correspondence $a\to\mathfrak{a}(a)$ defines the isomorphism $A_0\leftrightarrow\mathfrak{P}(A_0)$. Combining these correspondences we obtain an isomorphism $\mathfrak{P}(A_0)\leftrightarrow\mathfrak{A}_0$ which leaves $\mathfrak{P}(\mathfrak{a})$, as a common part of $\mathfrak{P}(A_0)$ and $\mathfrak{A}_0$, invariant. We shall show now that this isomorphism can be extended to the rings $\mathfrak{N}(A_0)$, $\mathfrak{N}(\mathfrak{a})$. From Th. 1.3, we know that the correspondence $\mathfrak{b}\to\mathfrak{b}\mathfrak{a}$ carries a normal ideal $\mathfrak{b}$ in A_0 into a normal ideal relative to $\mathfrak{a}$. Moreover, if $\mathfrak{b}$ is a normal ideal relative to $\mathfrak{a}$, the relation $\mathfrak{b}=\mathfrak{b}''\mathfrak{a}$ is valid and the normal ideal $\mathfrak{b}''$ in A_0 is carried by the above correspondence into the prescribed ideal $\mathfrak{b}$ in $\mathfrak{a}$. If we apply $\mathfrak{R}$ Th. 22 to the ideal $(\mathfrak{b}_1\vee\mathfrak{b}_2)\mathfrak{a}=(\mathfrak{b}_1\mathfrak{a}\vee\mathfrak{b}_2\mathfrak{a})\mathfrak{a}$ in $\mathfrak{a}$ we find that $(\mathfrak{b}_1\vee\mathfrak{b}_2)''\mathfrak{a}=(\mathfrak{b}_1\mathfrak{a}\vee\mathfrak{b}_2\mathfrak{a})''\mathfrak{a}$. Hence the relations $\mathfrak{b}_1\to\mathfrak{b}_1\mathfrak{a}$, $\mathfrak{b}_2\to\mathfrak{b}_2\mathfrak{a}$ imply

$$\mathfrak{b}_1\mathfrak{b}_1\to\mathfrak{b}_1\mathfrak{a}\cdot\mathfrak{b}_2\mathfrak{a},\qquad(\mathfrak{b}_1\vee\mathfrak{b}_2)''\to(\mathfrak{b}_1\mathfrak{a}\vee\mathfrak{b}_2\mathfrak{a})''\mathfrak{a}.$$

In words the second relation becomes: the normalized sum is carried by the above correspondence into the normalized sum, relative to $\mathfrak{a}$, of the correspondents of the original summands. This correspondence therefore defines a homomorphism $\mathfrak{N}(A)_0\to\mathfrak{N}(\mathfrak{a})$. If $\mathfrak{b}\to\mathfrak{b}\mathfrak{a}=\mathfrak{o}$, then $\mathfrak{b}\subset\mathfrak{a}'=\mathfrak{o}$, $\mathfrak{b}=\mathfrak{o}$. Hence the indicated homomorphism is an isomorphism, as we wished to prove. We now see that $\mathfrak{P}(\mathfrak{a})$ is an ideal in $\mathfrak{N}(A_0)$ as well as in $\mathfrak{P}(A_0)$. Hence we obtain a representation (2) for A_0 by putting $B=\mathfrak{N}(A_0)$, $B_0=\mathfrak{P}(A_0)$, $\mathfrak{b}=\mathfrak{P}(\mathfrak{a})$, $b_0=\mathfrak{a}(a_0)\epsilon\mathfrak{N}(A_0)$.

Assuming still that A_0 has a representation (1), we note that $\mathfrak{a}$, being prime, must contain $\mathfrak{a}'(a_0)$ in accordance with $\mathfrak{R}$ Th. 41. Thus we may represent A_0 as the direct sum $\mathfrak{a}(a_0)\vee\mathfrak{a}'(a_0)$ and $\mathfrak{a}$ as the direct sum $\mathfrak{a}(a_0)\mathfrak{a}\vee\mathfrak{a}'(a_0)=\mathfrak{a}(a_0)\mathfrak{a}\vee\mathfrak{a}'(a_0)\mathfrak{a}$. Since $\mathfrak{a}$ is totally multiplicative, both $\mathfrak{a}(a_0)\mathfrak{a}$ and $\mathfrak{a}'(a_0)$ are totally multiplicative by Th. 2.7. Since $\mathfrak{a}$ is prime and non-normal in A_0, Th. 1.7 shows that $\mathfrak{a}(a_0)\mathfrak{a}$ is prime and non-normal relative to $\mathfrak{a}(a_0)$. Hence $\mathfrak{a}(a_0)$ is generated by $\mathfrak{a}(a_0)\mathfrak{a}$ and a_0 — that is, arises from the ring $\mathfrak{a}(a_0)\mathfrak{a}$ without unit by the adjunction of the element a_0 as unit. Thus if we put $A_1=\mathfrak{a}(a_0)\mathfrak{a}$, $A_2=\mathfrak{a}'(a_0)$, we have $A_1^*=\mathfrak{a}(a_0)$; and find a representation (3) for A_0.

Next we suppose that A_0 has a representation (2). According to Th. 2.5, the representation of B_0 in terms of $\mathfrak{b}$ and b_0 is a representation (1); and there is an isomorphism $B\leftrightarrow\mathfrak{N}(\mathfrak{b})$ carrying $\mathfrak{b}$ into $\mathfrak{P}(\mathfrak{b})$ and B_0 into a subring $\mathfrak{B}_0$ of $\mathfrak{N}(\mathfrak{b})$. The results of the preceding paragraph show further that there is even an isomorphism $B\leftrightarrow\mathfrak{N}(B_0)$ carrying B_0 into $\mathfrak{P}(B_0)$ and $\mathfrak{b}$ into $\mathfrak{P}(\mathfrak{b})$, where $\mathfrak{P}(\mathfrak{b})$ is an ideal not only in $\mathfrak{P}(B_0)$ but also in $\mathfrak{N}(B_0)$. The isomorphism $A_0\leftrightarrow B_0$ shows that the correspondents $\mathfrak{a}$ and a_0 of $\mathfrak{b}$ and b_0 respectively provide a representation (1) for A_0. This isomorphism induces an isomorphism $\mathfrak{N}(A_0)\leftrightarrow\mathfrak{N}(B_0)$ carrying $\mathfrak{P}(A_0)$ into $\mathfrak{P}(B_0)$ and thus leads to an isomorphism $B\leftrightarrow\mathfrak{N}(A_0)$ carrying B_0 into $\mathfrak{P}(A_0)$ and $\mathfrak{b}$ into the ideal $\mathfrak{P}(\mathfrak{a})$. We therefore see that the given representation (2) is isomorphic to the one constructed from A_0 in the manner described in the preceding paragraph. Since B_0 is not an ideal in B, except in special cases, we have $\mathfrak{a}(b_0)B_0\vee\mathfrak{a}'(b_0)B_0\subset[\mathfrak{a}(b_0)\vee\mathfrak{a}'(b_0)]B_0=B_0$ by $\mathfrak{R}$ Th. 15 (5), but cannot replace the inclusion by equality without further argument. Since $\mathfrak{a}(b_0)B_0\vee\mathfrak{a}'(b_0)B_0$ contains $\mathfrak{a}(b_0)\mathfrak{b}\vee\mathfrak{a}'(b_0)\mathfrak{b}=\mathfrak{b}$ and also b_0, it is evident that it contains, and hence coincides with, B_0. Accordingly we see that B_0 is represented as the direct sum $\mathfrak{a}(b_0)B_0\vee\mathfrak{a}'(b_0)B_0$ where $\mathfrak{a}(b_0)B_0$ has b_0 as its unit. The ideal $\mathfrak{b}$ in B_0 is represented at the same time as the direct sum $\mathfrak{a}(b_0)\mathfrak{b}\vee\mathfrak{a}'(b_0)\mathfrak{b}$. Since $\mathfrak{b}$ and b_0 provide a representation (1) for B_0, the results of the preceding paragraph show that, on putting $A_1=\mathfrak{a}(b_0)\mathfrak{b}$, $A_2=\mathfrak{a}'(b_0)\mathfrak{b}$, we obtain $A_1^*=\mathfrak{a}(b_0)B_0$ and $B_0=A_1^*\vee A_2$. Thus B_0 and its isomorph A_0 have a representation (3).

We start now with the assumption that A_0 has a representation (3). We may, without loss of generality, identify A_0 with $A_1^*\vee A_2$. Th. 2.7 shows immediately that on putting $\mathfrak{a}=A_1\vee A_2$ and $a_0=e_1^*$, where e_1^* is the unit of A_1^*, we obtain a representation (1) for A_0. If we now use this representation to reconstruct a representation (3) of A_0 as in the preceding paragraphs, it is obvious that we recover the given representation (3). By Th. 2.5 we know that $A_0=A_1^*\vee A_2$ is isomorphic to the subring of $\mathfrak{N}(\mathfrak{a})=\mathfrak{N}(A_1\vee A_2)$ generated by the ideal $\mathfrak{P}(\mathfrak{a})=\mathfrak{P}(A_1\vee A_2)$ and the element $\mathfrak{a}(a_0)=\mathfrak{a}(e_1^*)\epsilon\mathfrak{N}(A_1\vee A_2)$. Since A_1 and A_2 are simple ideals in the direct sum $A_1\vee A_2$, they may be regarded as elements $\mathfrak{a}_1$ and $\mathfrak{a}_2$ of $\mathfrak{N}(A_1\vee A_2)$. Since they satisfy the relations $\mathfrak{a}_1\vee\mathfrak{a}_2=\mathfrak{e}=A_1\vee A_2$ and $\mathfrak{a}_1\mathfrak{a}_2=\mathfrak{o}$, they define a direct sum representation of $\mathfrak{N}(A_1\vee A_2)$. By Th. 2.7 and our present hypothesis, we know that $\mathfrak{N}(A_1\vee A_2)=\mathfrak{S}(A_1\vee A_2)$, $\mathfrak{N}(A_1)=\mathfrak{S}(A_1)$, $\mathfrak{N}(A_2)=\mathfrak{S}(A_2)$. Now the direct sum representation of $\mathfrak{S}(A_1\vee A_2)$ is easily calculated on the basis of $\mathfrak{R}$ Th. 51 and Ths. 1.2, 1.3, and 1.4. We find that each element of $\mathfrak{S}(A_1\vee A_2)$ is represented as the sum of components which are unrestricted elements of $\mathfrak{S}(A_1)$ and $\mathfrak{S}(A_2)$ respectively. In particular, an element of $\mathfrak{P}(A_1\vee A_2)$ is represented as the sum of components which are unrestricted elements of $\mathfrak{P}(A_1)$ and $\mathfrak{P}(A_2)$ respectively. Thus we find that $\mathfrak{N}(A_1 \vee A_2) = \mathfrak{N}(A_1) \vee \mathfrak{N}(A_2)$, $\mathfrak{P}(A_1 \vee A_2) = \mathfrak{P}(A_1) \vee \mathfrak{P}(A_2)$. Moreover the element $\mathfrak{a}(a_0) = \mathfrak{a}(e_1^*) \epsilon \mathfrak{N}(A_1 \vee A)$ is easily identified with the unit $\mathfrak{e}_1$ of $\mathfrak{N}(A_1)$. Hence, if we put $B = \mathfrak{N}(A_1) \vee \mathfrak{N}(A_2)$, $\mathfrak{b} = \mathfrak{P}(A_1) \vee \mathfrak{P}(A_2)$, $b_0 = \mathfrak{e}_1$, we obtain a representation (2) for A_0. Since we obviously have $\mathfrak{a}(b_0)\mathfrak{b} = \mathfrak{P}(A_1)$ and $\mathfrak{a}'(b_0)\mathfrak{b} = \mathfrak{P}(A_2)$, the reconstruction of a representation (3) from the representation (2) just found yields a representation essentially the same as that assumed at the outset.

In view of Th. 2.7 we see that a ring A_0 represented in the form (3) is totally multiplicative if and only if A_1 is of type (β_1). The correspondences between the three different representations thus lead to the equivalent conditions stated above for A_0 to be totally multiplicative.

The effect of Th. 2.8 is to show that all the Boolean rings A_0 obtainable from totally multiplicative rings without unit by a least possible proper extension can be constructed in either of two equivalent ways: as specific subrings of infinite totally additive rings or as direct sums of the form (3). Naturally, our interest now centers on the rings A_0 of this form which are not totally multiplicative. We have:

Theorem 2.9. *A non-totally-multiplicative Boolean ring A_0 which is representable in any of the three equivalent forms* (1), (2), (3) *of Th. 2.8 has in each case a representation which is unique in the following sense:*

(1) *for a representation* (1), *the ideal $\mathfrak{a}$ is uniquely determined;*

(2) *for a representation* (2), *the isomorphism $B \leftrightarrow \mathfrak{N}(A_0)$ carrying B_0 into $\mathfrak{P}(A_0)$ carries $\mathfrak{b}$ into a uniquely determined ideal in $\mathfrak{N}(A_0)$.*

(3) *for two representations* (3), $A_0 = A_1^* \vee A_2 = A_3^* \vee A_4$, *the direct sums $A_1 \vee A_2$ and $A_3 \vee A_4$ are obtained from each other by an exchange of totally additive direct summands.*

In case A_0 also has a unit, the various representations can be reduced to unique forms by the following normalizing conditions:

(1) *for a representation* (1), $a_0 = e$;

(2) *for a representation* (2), *b_0 is the unit in B;*

(3) *for a representation* (3), *A_2 is a one-element ring so that*

$$A_0 = A_1^* = A_1^* \vee A_2.$$

Let there be given two representations (1) for a ring A_0 in terms of ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$ respectively; and assume that A_0 is not totally multiplicative. Then by Th. 2.3 there exists a non-simple normal ideal $\mathfrak{c}$ in A_0. According to Th. 1.3, the ideals $\mathfrak{a}_1\mathfrak{c}$, $\mathfrak{a}_2\mathfrak{c}$ are normal, and hence simple, relative to the totally multiplicative rings $\mathfrak{a}_1$ and $\mathfrak{a}_2$ respectively. Thus we have the relations:

$$\mathfrak{c} \vee \mathfrak{c}' \neq \mathfrak{e}, \qquad \mathfrak{a}_1\mathfrak{c} \vee \mathfrak{a}_1\mathfrak{c}' = \mathfrak{a}_1, \qquad \mathfrak{a}_2\mathfrak{c} \vee \mathfrak{a}_2\mathfrak{c}' = \mathfrak{a}_2.$$

Since $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are both prime we conclude that $\mathfrak{a}_1 = \mathfrak{c} \vee \mathfrak{c}' = \mathfrak{a}_2$. With this result we have established the uniqueness-assertion of the theorem. In view of the results of Th. 2.8, the uniqueness assertion for representations of the form (2) follows immediately; and that for representations of the form (3) is proved in part, to the extent that $A_0 = A_1^* \vee A_2 = A_3^* \vee A_4$ is now seen to imply $\mathfrak{a} = A_1 \vee A_2 = A_3 \vee A_4$. Since the two representations $A_1^* \vee A_2$ and $A_3^* \vee A_4$ correspond to representations (1) given by $\mathfrak{a}$, a_0 and $\mathfrak{a}$, b_0 where a_0 and b_0 are any suitable elements not in $\mathfrak{a}$, we have:

$$A_1 = \mathfrak{a}(a_0)\mathfrak{a}, \qquad A_3 = \mathfrak{a}(b_0)\mathfrak{a},$$
$$A_2 = \mathfrak{a}'(a_0), \qquad A_4 = \mathfrak{a}'(b_0),$$

by the results of Th. 2.8. If we now introduce

$$B_1 = \mathfrak{a}(a_0)\mathfrak{a}(b_0)\mathfrak{a}, \qquad B_3 = \mathfrak{a}'(a_0)\mathfrak{a}'(b_0),$$
$$B_2 = \mathfrak{a}(a_0)\mathfrak{a}'(b_0), \qquad B_4 = \mathfrak{a}'(a_0)\mathfrak{a}(b_0),$$

we see immediately that $A_1 \vee A_2$ is obtained from $A_3 \vee A_4$ by exchange of the direct summands B_2 and B_3 in accordance with Def. 2.9. By Th. 1.2, B_2 and B_4 are principal ideals in A_0 and in $\mathfrak{a}$; hence both are totally additive rings by Th. 2.3 (3). Thus the uniqueness-assertion for representations of the form (3) is established. In view of Th. 2.7, any exchange of totally additive direct summands within $\mathfrak{a} = A_1 \vee A_2$ is permissible.

In case A_0 has a unit we may obviously take $a_0 = e$ in the representation (1). It then follows that in any representation (2), the subring B_0 contains the unit of B: for the isomorphism $B \leftrightarrow \mathfrak{N}(A_0)$ carrying B_0 into the subring $\mathfrak{P}(A_0)$ of $\mathfrak{N}(A_0)$ obviously takes the unit of B into the unit $\mathfrak{a}(e) \epsilon \mathfrak{P}(A_0)$ of $\mathfrak{N}(A_0)$. Accordingly, we may take b_0 as the unit of B. Finally the representation (3) corresponding to the choice $a_0 = e$ is evidently given by $A_0 = A_1^* = A_1^* \vee A_2$ where $A_1 = \mathfrak{a}(e)\mathfrak{a} = \mathfrak{a}$ and $A_2 = \mathfrak{a}'(e) = \mathfrak{a}(0) = \mathfrak{o}$.

We can now give our series of fundamental types in full. We have:

Theorem 2.10. *The infinite Boolean rings constructed from (infinite) totally additive Boolean rings by the selection of ideals, finite direct summation, and at most one unit-adjunction applied to a ring without unit fall into exactly nine distinct types — namely, the types* (α), (β_1), (β_2), (β_3), (β_2^*), (β_3^*), (β_2^*, β_1), (β_2^*, β_2), *and* (β_3^*, β_3). *A ring of type* (α), (β_1), (β_2), (β_3), (β_2^*), *or* (β_3^*) *has a unique representation as a member of this type; a ring of type* (β_2^*, β_1) *or* (β_2^*, β_2) *has a representation as a member of this type which is unique except for an exchange of totally additive direct summands between the two underlying components; and a ring of type* (β_3^*, β_3) *has a representation as a member of this type which is unique except for an exchange of finite direct summands between the two underlying components. As a special case under type* (β_2), *we note the type* (β_1, β_1); *any ring of this type has a representation as a member of this type which is unique except for an exchange of totally additive direct summands between the underlying components. Other constructions of these types of ring are given in Th. 2.8.*

The types (α), (β_1), (β_2), (β_3) have already been discussed in Th. 2.6. Since they exhaust the infinite totally multiplicative rings, we have to show that the five remaining types are distinct and exhaust the rings treated in Th. 2.9. Th. 2.7 enables us to discuss the rings of type $(*, **)$ more exactly. They are all infinite and totally multiplicative. Such a ring is of type (α) if and only if it is of type (α, α). Similarly such a ring is of type (β_1) if and only if it is of type (β_1, α). It is easily seen that such a ring is of type (β_3) if and only if each of its components has an atomic basis; in other words, if and only if the ring is of type (β_3, β_3). Consequently, the rings of types (β_2, α), (β_3, α), (β_1, β_1), (β_2, β_1), (β_3, β_1), (β_2, β_2), (β_2, β_3) are all included under the type (β_2). It is obvious that we could retain some of these special types for the purpose of subdividing the type (β_2). As none of them except the type (β_1, β_1) has any further interest as an individual type in this paper, we shall not do so. So far as the type (β_1, β_1) is concerned, it appears as a special case under types discussed in the following section. We shall therefore prove the uniqueness-assertion concerning the representation as a member of this special type. If $A_1 \vee A_2 = A_3 \vee A_4$ are two representations of a ring A of type (β_1, β_1), we put:

$$A_1 = B_1 \vee B_2, \qquad A_3 = B_1 \vee B_4,$$
$$A_2 = B_3 \vee B_4, \qquad A_4 = B_3 \vee B_2,$$

where

$$B_1 = A_1 A_3, \qquad B_2 = A_1 A_4,$$
$$B_3 = A_2 A_4, \qquad B_4 = A_2 A_3.$$

Since A_3 and A_4 are simple ideals in A their intersections B_1, B_2, B_3, B_4 with the simple ideals A_1 and A_2 are simple in A and also in whichever of the ideals A_1, A_2 contains them, as we see by reference to Ths. 1.2 and 1.3. Now according to Th. 2.6, the fact that A_1 and A_2 are rings of type (β_1) implies that the ideals B_1, B_2, B_3, B_4 are semi-principal in whichever of the ideals A_1, A_2 contains them. Since neither A_1 nor A_2 has a unit, only one of the two ideals contained in each of A_1 and A_2 can be principal. Our notation can be so adjusted, by proper assignment of the indices 3 and 4, that B_2 is principal in A_1 and B_4 in A_2. Thus B_2 and B_4 are totally additive rings in accordance with Th. 2.3 (3). We have therefore proved that each of the given representations arises from the other by an exchange of totally additive direct summands. Th. 2.7 shows furthermore that any such exchange is permissible, in that it leaves each component of type (β_1).

Next we consider the types (β_1^*), (β_2^*), (β_3^*). Th. 2.7 shows that a ring of one of these types is totally multiplicative if and only if it is of type (β_1^*); and is then of type (α). Thus Th. 2.9 shows that the two types (β_2^*) and (β_3^*) are distinct and together exhaust the rings with unit considered there.

We now consider the composite types (β_2^*, β_1), (β_2^*, β_2), (β_2^*, β_3), (β_3^*, β_1), (β_3^*, β_2), (β_3^*, β_3). Th. 2.7 shows that no ring of any of these types is totally multiplicative or has a unit. It is obvious therefore that these types exhaust the rings without unit considered under Th. 2.9. Since the representation of one of the latter rings in the form $A_1^* \vee A_2$ is unique except for exchanges of totally additive direct summands between the components A_1 and A_2; and since Th. 2.7 shows any such exchange to be permissible — we find that the type (β_3^*, β_1) is included under the type (β_2^*, β_1), the types (β_2^*, β_3) and (β_3^*, β_2) under the type (β_2^*, β_2). We discuss as typical the case of type (β_3^*, β_1). Shifting a direct summand of type (α) from the component A_2 of type (β_1) to the component A_1 of type (β_3), we obtain a new representation in which, according to Th. 2.7, the components A_1 and A_2 are now of types (β_3, α) and (β_1) respectively. Since the type (β_3, α) is included under type (β_2), the new representation exhibits the original ring as a member of type (β_2^*, β_1). Th. 2.7 shows that no ring of type (β_2^*, β_1) is a member of either of the types (β_2^*, β_2) and (β_3^*, β_3): for an exchange of totally additive direct summands between the components of a direct sum replaces a component of type (β_1) by one of type (β_1). For a similar reason no ring of type (β_3^*, β_3) is a member of either of the types (β_2^*, β_2) or (β_2^*, β_1). In fact, a totally additive direct summand in a ring of type (β_3) is necessarily a principal ideal in the ring and must therefore be a finite ring. Hence the only exchange of totally additive direct summands between the underlying components of a ring of type (β_3^*, β_3) is an exchange of finite direct summands; and such an exchange obviously leaves both components with finite atomic bases. We thus conclude that the three types (β_2^*, β_1), (β_2^*, β_2), (β_3^*, β_3) are distinct and together exhaust the rings without unit considered in Th. 2.9. We have proved incidentally that the representation of a ring as a member of any of these three types is unique in the sense described above.

The results established in the preceding paragraphs show that the operations admitted here cannot produce any composite type other than the nine explicitly investigated: for example the composite type $(((\beta_1, \beta_2)^*, \beta_3), (\alpha, (\beta_3, (\beta_1, \beta_2))))$ is seen to reduce first to the type $((\beta_2^*, \beta_3), (\alpha, (\beta_3, \beta_2)))$, then to the type $((\beta_2^*, \beta_3), (\alpha, \beta_2))$, then to the type $((\beta_2^*, \beta_3), \beta_2)$ or $(\beta_2^*, (\beta_3, \beta_2))$, and thus finally to the type (β_2^*, β_2). It is essential, of course, that not more than one unit-adjunction is allowed in the process of composition.

A further refinement of type could be introduced by the consideration of the systems of atomic elements, if any exists, in totally multiplicative rings. While we shall not consider such a refinement in the remaining sections of the paper, it seems appropriate to complete our investigations of totally multiplicative and related rings by examining the part played by atomic systems. We have:

Theorem 2.11. *With respect to the existence of atomic elements, the totally multiplicative Boolean rings may be classified as follows:*

(1) *rings without atomic element;*

(2) *rings with complete atomic systems;*

(3) *direct sums of the preceding two kinds of ring.*

Any totally multiplicative ring A containing an incomplete atomic system $\mathfrak{s}$ is uniquely representable as a direct sum $\mathfrak{s}' \vee \mathfrak{s}''$ where $\mathfrak{s}'$ and $\mathfrak{s}''$ are totally multiplicative rings belonging to classes (1) *and* (2) *respectively.*

The proof consists in establishing the last statement of the theorem. By $\mathfrak{R}$ Th. 19 the classes $\mathfrak{s}'$ and $\mathfrak{s}''$ in A are ideals; by $\mathfrak{R}$ Ths. 20 and 27 both are normal in A; and by Th. 2.3 (2) both are simple in A. By hypothesis $\mathfrak{s}' \neq \mathfrak{o}$, $\mathfrak{s}'' \neq \mathfrak{o}$. It is clear that $\mathfrak{s}'$ contains no atomic element; and it is easily shown that the orthocomplement of $\mathfrak{s}$ relative to $\mathfrak{s}'' \supset \mathfrak{s}$ is $\mathfrak{s}'\mathfrak{s}'' = \mathfrak{o}$ and hence that $\mathfrak{s}$ is a complete atomic system in $\mathfrak{s}''$. Since $\mathfrak{s}'$ and $\mathfrak{s}''$ are simple ideals in A, we can represent A as the desired direct sum $A = \mathfrak{s}' \vee \mathfrak{s}''$ in accordance with $\mathfrak{R}$ Th. 51 and Th. 2.7.

We conclude by recalling a known algebraic criterion for the existence of a complete atomic system in a totally additive Boolean ring [11]).

Theorem 2.12. *In a totally additive Boolean ring A the following properties are equivalent:*

(1) *A contains a complete atomic system $\mathfrak{s}$;*

(2) *if $\mathfrak{A}$ is any non-void family of two-element subclasses $\mathfrak{a}$ of A and if $\mathfrak{B}$ is the family of all those subclasses $\mathfrak{b}$ of A which are contained in $\sum_{\mathfrak{a} \in \mathfrak{A}} \mathfrak{a}$ and have exactly one element in common with each $\mathfrak{a}$ in $\mathfrak{A}$, then the distributive law is valid:*

$$\underset{\mathfrak{a} \in \mathfrak{A}}{P} \; \underset{a \in \mathfrak{a}}{S} \, a = \underset{\mathfrak{b} \in \mathfrak{B}}{S} \; \underset{a \in \mathfrak{b}}{P} \, a.$$

In order that a Boolean ring A be isomorphic to the Boolean ring of all subclasses of a fixed class E, it is necessary and sufficient that it be totally additive and have the equivalent properties (1) *and* (2). *In such a Boolean ring, the following general form of the distributive law is valid: if $\mathfrak{A}$ is any non-void family of non-void subclasses $\mathfrak{a}$ of A and if $\mathfrak{B}$ is the family of all those subclasses $\mathfrak{b}$ of A which are contained in $\sum_{\mathfrak{a} \in \mathfrak{A}} \mathfrak{a}$ and which have exactly one element in common with each $\mathfrak{a}$ in $\mathfrak{A}$, then*

$$\underset{\mathfrak{a} \in \mathfrak{A}}{P} \; \underset{a \in \mathfrak{a}}{S} \, a = \underset{\mathfrak{b} \in \mathfrak{B}}{S} \; \underset{a \in \mathfrak{b}}{P} \, a.$$

We here use the symbols S and P to indicate product and sum as defined in Def. 2.1 and 2.2. First let us show that (2) implies (1). Let each class $\mathfrak{a}$ consist of an element a and the corresponding element a', and define the family $\mathfrak{A}$ by permitting a to run over the entire ring A. Of course each class $\mathfrak{a}$ occurs twice in $\mathfrak{A}$.

[11]) A. Tarski, Fund. Math. 24 (1935), pp. 177-198 especially p. 196. The result was found in collaboration with A. Lindenbaum.

We then have $e=\underset{\mathfrak{a}\in\mathfrak{A}}{P}\ \underset{a\in\mathfrak{a}}{S}a=\underset{\mathfrak{b}\in\mathfrak{B}}{S}\ \underset{a\in\mathfrak{b}}{P}a$. Hence the subclass $\mathfrak{s}$ of A consisting of those elements b such that $b=\underset{a\in\mathfrak{b}}{P}a\neq 0$ for $\mathfrak{b}\in\mathfrak{B}$ is certainly not void. We shall show that $\mathfrak{s}$ is a complete atomic system. From our definition of the classes $\mathfrak{b}$, we see immediately that, if c is any element of A and $\mathfrak{b}$ any member of $\mathfrak{B}$, then $\mathfrak{b}$ contains either c or c'; and that, accordingly, either $c\cdot\underset{a\in\mathfrak{b}}{P}a=\underset{a\in\mathfrak{b}}{P}a$ or $c\cdot\underset{a\in\mathfrak{b}}{P}a=0$. In particular, if $c\in A$ and $b\in\mathfrak{s}$, then either $cb=b$ or $cb=0$. Hence $\mathfrak{s}$ is an atomic system in A in accordance with $\mathfrak{R}$ Def. 3. Now it is evident that $e=\underset{b\in\mathfrak{s}}{S}b$. By Th. 2.1 we have $\mathfrak{s}''=\mathfrak{e}$, $\mathfrak{s}'=\mathfrak{o}$. It follows that the atomic system $\mathfrak{s}$ is complete in accordance with $\mathfrak{R}$ Def. 5.

By $\mathfrak{R}$ Th. 62, a Boolean ring A which is totally additive and satisfies condition (1) is isomorphic to the Boolean ring of all subclasses of a fixed class E; and conversely, in such a ring the final statement of the theorem is easily proved: an element of E belongs to $\underset{\mathfrak{a}\in\mathfrak{A}}{P}\ \underset{a\in\mathfrak{a}}{S}a$ if and only if it belongs to some a in each class $\mathfrak{a}$; an element of E belongs to $\underset{\mathfrak{b}\in\mathfrak{B}}{S}\ \underset{a\in\mathfrak{b}}{P}a$ if and only if it belongs to every a in some class $\mathfrak{b}$; and thus the definition of the classes $\mathfrak{b}$ in terms of the classes $\mathfrak{a}$ implies the desired equality. It follows that in a totally additive Boolean ring (1) implies (2) as a special case of the equality just proved.

§ 3. Barrier Ideals and Associated Types of Boolean Ring.

In the present section we shall introduce a new class of ideals — the barrier ideals — and two associated special types of Boolean ring. Our choice of terminology is dictated by topological reasons which will be developed in the following section. The two definitions fundamental for the purposes of the present section are:

Definition 3.1. *An ideal $\mathfrak{a}$ in a Boolean ring A is said to be a barrier ideal if $\mathfrak{a}\neq\mathfrak{e}$, $\mathfrak{a}'=\mathfrak{o}$, and there exist normal ideals $\mathfrak{b}$ and $\mathfrak{c}$ such that $\mathfrak{a}=\mathfrak{b}\vee\mathfrak{c}$, $\mathfrak{b}\mathfrak{c}=\mathfrak{o}$. The class of all barrier ideals in A is denoted by $\mathfrak{B}$, the class of all other ideals in A by $\mathfrak{C}$.*

Definition 3.2. *A Boolean ring A is said to be of type (ω) if it is isomorphic to a prime ideal $\mathfrak{p}$, in a Boolean ring with unit, such that $\mathfrak{p}$ non $\in\mathfrak{N}$, $\mathfrak{p}\in\mathfrak{C}$.*

We begin with the consideration of the properties of barrier ideals.

Theorem 3.1. *In order that $\mathfrak{a}$ be a barrier ideal it is necessary and sufficient that $\mathfrak{a}=\mathfrak{b}\vee\mathfrak{b}'$ where $\mathfrak{b}$ is a non-simple normal ideal; and it is necessary that $\mathfrak{a}$ be non-normal.*

If $\mathfrak{a}=\mathfrak{b}\vee\mathfrak{b}'$ where $\mathfrak{b}$ is a non-simple normal ideal, then $\mathfrak{a}\neq\mathfrak{e}$, $\mathfrak{a}'=(\mathfrak{b}\vee\mathfrak{b}')'=\mathfrak{b}'\mathfrak{b}''=\mathfrak{o}$, and $\mathfrak{a}=\mathfrak{b}\vee\mathfrak{c}$, $\mathfrak{b}\mathfrak{c}=\mathfrak{o}$ where $\mathfrak{b}\in\mathfrak{N}$ and $\mathfrak{c}=\mathfrak{b}'\in\mathfrak{N}$. Hence $\mathfrak{a}\in\mathfrak{B}$. On the other hand, if $\mathfrak{a}$ is a barrier ideal, the properties given in Def. 3.1 show that $\mathfrak{b}\mathfrak{c}=\mathfrak{o}$, $\mathfrak{b}'\mathfrak{c}'=(\mathfrak{b}\vee\mathfrak{c})'=\mathfrak{a}'=\mathfrak{o}$ and hence that $\mathfrak{c}\subset\mathfrak{b}'\subset\mathfrak{c}''=\mathfrak{c}$, $\mathfrak{c}=\mathfrak{b}'$; and we therefore have $\mathfrak{a}=\mathfrak{b}\vee\mathfrak{b}'\neq\mathfrak{e}$ so that $\mathfrak{b}\in\mathfrak{N}$ but $\mathfrak{b}$ non $\in\mathfrak{S}$. The relations $\mathfrak{a}\neq\mathfrak{e}$, $\mathfrak{a}'=\mathfrak{o}$ show that $\mathfrak{a}''\neq\mathfrak{a}$ and hence that $\mathfrak{a}$ is not normal.

Theorem 3.2. *The classes $\mathfrak{B}$ and $\mathfrak{C}$ satisfy the inclusion relations $\mathfrak{N}\subset\mathfrak{C}$, $\mathfrak{B}\subset\mathfrak{J}\Delta\mathfrak{N}$; and the class $\mathfrak{B}$ is void if and only if A is a totally multiplicative Boolean ring.*

The present theorem is an immediate consequence of Th. 3.1 and Th. 2.3 (2).

Theorem 3.3. *If $\mathfrak{a}_1$ is a barrier ideal and $\mathfrak{a}_2$ a simple ideal such that $\mathfrak{a}=\mathfrak{a}_1\vee\mathfrak{a}_2\neq\mathfrak{e}$, then $\mathfrak{a}$ is also a barrier ideal. In particular, if $\mathfrak{a}$ is any non-prime barrier ideal then there exists a barrier ideal $\mathfrak{b}$ such that $\mathfrak{b}\supset\mathfrak{a}$, $\mathfrak{b}\neq\mathfrak{a}$.*

Let $\mathfrak{a}_1=\mathfrak{b}_1\vee\mathfrak{b}_1'\neq\mathfrak{e}$ where $\mathfrak{b}_1\in\mathfrak{N}$; and let $\mathfrak{a}=\mathfrak{a}_1\vee\mathfrak{a}_2\neq\mathfrak{e}$ where $\mathfrak{a}_2\in\mathfrak{S}$. The ideals $\mathfrak{b}=\mathfrak{b}_1\vee\mathfrak{a}_2$ and $\mathfrak{c}=\mathfrak{a}_2'\mathfrak{b}_1'$ then belong to $\mathfrak{N}$ by virtue of Th. 1.2. Moreover they satisfy the relations

$$\mathfrak{a}=(\mathfrak{b}_1\vee\mathfrak{b}_1')\vee\mathfrak{a}_2=(\mathfrak{b}_1\vee\mathfrak{a}_2)\vee\mathfrak{b}_1'(\mathfrak{a}_2\vee\mathfrak{a}_2')=\mathfrak{b}_1\vee\mathfrak{a}_2\vee\mathfrak{b}_1'\mathfrak{a}_2'=\mathfrak{b}\vee\mathfrak{c},\qquad \mathfrak{b}\mathfrak{c}=\mathfrak{o}.$$

By hypothesis, $\mathfrak{a}\neq\mathfrak{e}$; and we have also $\mathfrak{a}'=(\mathfrak{a}_1\vee\mathfrak{a}_2)'=\mathfrak{a}_1'\mathfrak{a}_2'=\mathfrak{o}$. Thus $\mathfrak{a}$ is a barrier ideal in accordance with Def. 3.1. If $\mathfrak{a}$ is a non-prime barrier ideal, there exists an element a not in $\mathfrak{a}$ such that $\mathfrak{b}=\mathfrak{a}\vee\mathfrak{a}(a)\neq\mathfrak{e}$. Then $\mathfrak{b}$ is a barrier ideal by the preceding results, and $\mathfrak{b}$ contains $\mathfrak{a}$ as a proper subclass.

Theorem 3.4. *If $\mathfrak{a}_n$, $n=1,\dots,N$, are barrier ideals such that $\mathfrak{a}_m\vee\mathfrak{a}_n=\mathfrak{e}$ for $m\neq n$, $m,n=1,\dots,N$, then $\mathfrak{a}_1\dots\mathfrak{a}_N$ is a barrier ideal.*

We begin with the case $N=2$. By Th. 3.1 we can write $\mathfrak{a}_1=\mathfrak{b}_1\vee\mathfrak{b}_1'$, $\mathfrak{a}_2=\mathfrak{b}_2\vee\mathfrak{b}_2'$ where $\mathfrak{b}_1$ and $\mathfrak{b}_2$ are normal. We introduce the ideals $\mathfrak{a}=\mathfrak{a}_1\mathfrak{a}_2$, $\mathfrak{b}=\mathfrak{b}_1\mathfrak{b}_2'\vee\mathfrak{b}_1'\mathfrak{b}_2$, $\mathfrak{c}=\mathfrak{b}_1\mathfrak{b}_2\vee\mathfrak{b}_1'\mathfrak{b}_2'$. Since $\mathfrak{a}\subset\mathfrak{a}_1\neq\mathfrak{e}$, we have $\mathfrak{a}\neq\mathfrak{e}$; and the relations $\mathfrak{a}_1'=\mathfrak{a}_2'=\mathfrak{o}$ imply $\mathfrak{a}'=\mathfrak{o}$ in accordance with R Th. 28. It is evident that

$$\mathfrak{b}\vee\mathfrak{c}=(\mathfrak{b}_1\vee\mathfrak{b}_1')(\mathfrak{b}_2\vee\mathfrak{b}_2')=\mathfrak{a}_1\mathfrak{a}_2=\mathfrak{a},\qquad \mathfrak{b}\mathfrak{c}=\mathfrak{o}.$$

Hence it is sufficient for us to prove that $\mathfrak{b}$ and $\mathfrak{c}$ are normal. We recall that the orthocomplement of any ideal product $\mathfrak{c}_1\mathfrak{c}_2$ relative to $\mathfrak{c}_2$ may be calculated either as $(\mathfrak{c}_1\mathfrak{c}_2)'\mathfrak{c}_2$ or as $\mathfrak{c}_1'\mathfrak{c}_2$ so that $(\mathfrak{c}_1\mathfrak{c}_2)'\mathfrak{c}_2=\mathfrak{c}_1'\mathfrak{c}_2$; in particular, $\mathfrak{c}_1\mathfrak{c}_2=\mathfrak{o}$ implies $\mathfrak{c}_1'\mathfrak{c}_2=\mathfrak{c}_2$. Applying these remarks we obtain the relations

$$\begin{aligned}
&\mathfrak{b}_1\mathfrak{b}_2'=(\mathfrak{b}_1\mathfrak{b}_2)'\mathfrak{b}_1=(\mathfrak{b}_1'\mathfrak{b}_2')'\mathfrak{b}_2', &\quad& \mathfrak{b}_1\mathfrak{b}_2=(\mathfrak{b}_1'\mathfrak{b}_2)'\mathfrak{b}_2=(\mathfrak{b}_1\mathfrak{b}_2')'\mathfrak{b}_1,\\
&\mathfrak{b}_1'\mathfrak{b}_2=(\mathfrak{b}_1\mathfrak{b}_2)'\mathfrak{b}_2=(\mathfrak{b}_1'\mathfrak{b}_2')'\mathfrak{b}_1', && \mathfrak{b}_1'\mathfrak{b}_2'=(\mathfrak{b}_1'\mathfrak{b}_2)'\mathfrak{b}_1'=(\mathfrak{b}_1\mathfrak{b}_2')\mathfrak{b}_2',\\
&(\mathfrak{b}_1'\mathfrak{b}_2')'(\mathfrak{b}_1\vee\mathfrak{b}_2)=\mathfrak{b}_1\vee\mathfrak{b}_2, && (\mathfrak{b}_1\mathfrak{b}_2')'(\mathfrak{b}_1'\vee\mathfrak{b}_2)=\mathfrak{b}_1'\vee\mathfrak{b}_2,\\
&(\mathfrak{b}_1\mathfrak{b}_2)'(\mathfrak{b}_1'\vee\mathfrak{b}_2')=\mathfrak{b}_1'\vee\mathfrak{b}_2', && (\mathfrak{b}_1'\mathfrak{b}_2)'(\mathfrak{b}_1\vee\mathfrak{b}_2')=\mathfrak{b}_1\vee\mathfrak{b}_2'.
\end{aligned}$$

Combining the four sets of equations on the left, we have

$$\begin{aligned}
\mathfrak{b}&=(\mathfrak{b}_1\mathfrak{b}_2'\vee\mathfrak{b}_1'\mathfrak{b}_2)\vee(\mathfrak{b}_1\mathfrak{b}_2'\vee\mathfrak{b}_1'\mathfrak{b}_2)=(\mathfrak{b}_1\mathfrak{b}_2)'(\mathfrak{b}_1\vee\mathfrak{b}_2)\vee(\mathfrak{b}_1'\mathfrak{b}_2')'(\mathfrak{b}_1'\vee\mathfrak{b}_2')\\
&=(\mathfrak{b}_1\mathfrak{b}_2)'(\mathfrak{b}_1'\mathfrak{b}_2')'(\mathfrak{b}_1\vee\mathfrak{b}_1'\vee\mathfrak{b}_2\vee\mathfrak{b}_2')=\mathfrak{c}'(\mathfrak{a}_1\vee\mathfrak{a}_2)=\mathfrak{c}';
\end{aligned}$$

and combining the four sets of equations on the right we have, similarly, $\mathfrak{c}=\mathfrak{b}'$. Thus $\mathfrak{b}=\mathfrak{c}'=\mathfrak{b}''$, $\mathfrak{c}=\mathfrak{b}'=\mathfrak{c}''$, as we wished to prove.

If the theorem holds for $N=2,\ldots,M$, it holds also for $N=M+1$. In fact, the ideal $\mathfrak{a}_1\ldots\mathfrak{a}_M$ is a barrier ideal; and the relation $\mathfrak{a}_1\ldots\mathfrak{a}_M\vee\mathfrak{a}_{M+1}=(\mathfrak{a}_1\vee\mathfrak{a}_{M+1})\ldots(\mathfrak{a}_M\vee\mathfrak{a}_{M+1})=\mathfrak{e}$ is valid. Hence the result of the preceding paragraph shows that $\mathfrak{a}_1\ldots\mathfrak{a}_M\mathfrak{a}_{M+1}$ is a barrier ideal. The theorem is therefore established for $N=2,3,\ldots$.

It is also possible to establish a partial converse of the preceding result. We have:

Theorem 3.5. *If $\mathfrak{a}_1,\ldots,\mathfrak{a}_N$ are ideals such that $\mathfrak{a}_n\neq\mathfrak{e}$ for $n=1,\ldots,N$ and if $\mathfrak{a}_1\ldots\mathfrak{a}_N$ is a barrier ideal, then the condition*

(1) *for $m\neq n$ there exists a simple ideal $\mathfrak{d}_{mn}$ satisfying the relations*

$$\mathfrak{a}_m\vee\mathfrak{d}_{mn}=\mathfrak{e},\qquad \mathfrak{a}_n\supset\mathfrak{d}_{mn}$$

implies that $\mathfrak{a}_1,\ldots,\mathfrak{a}_N$ are barrier ideals.

The condition (1) *is equivalent to the condition*

(1′) *there exist simple ideals $\mathfrak{d}_n$, $n=1,\ldots,N$, such that $\mathfrak{a}_n\vee\mathfrak{d}_n=\mathfrak{e}$,*

$$\mathfrak{d}_1\vee\ldots\vee\mathfrak{d}_N=\mathfrak{e},\quad \mathfrak{d}_m\mathfrak{d}_n=\mathfrak{o}\quad \textit{for}\quad m\neq n,\quad m,n=1,\ldots,N.$$

The conditions (1) *and* (1′) *are satisfied if the ideals $\mathfrak{a}_1,\ldots,\mathfrak{a}_N$ are distinct prime ideals; and also if A is a ring with unit and the ideals $\mathfrak{a}_1,\ldots,\mathfrak{a}_N$ have the property that $\mathfrak{a}_m\vee\mathfrak{a}_n=\mathfrak{e}$ for $m\neq n$, $m,n=1,\ldots,N$.*

First let us prove the equivalence of the conditions (1) and (1′). If (1′) holds, we may put $\mathfrak{d}_{mn}=\mathfrak{d}_m$ since $\mathfrak{a}_m\vee\mathfrak{d}_m=\mathfrak{e}$ and

$$\mathfrak{d}_m=(\mathfrak{a}_n\vee\mathfrak{d}_n)\mathfrak{d}_m=\mathfrak{a}_n\mathfrak{d}_m\subset\mathfrak{a}_n\qquad\text{for}\quad m\neq n,\quad m,n=1,\ldots,N;$$

and we thus obtain (1). On the other hand, if (1) holds, we obtain (1′) by induction. For $N=2$ we obtain the desired result by setting $\mathfrak{d}_1=\mathfrak{d}_{12}$, $\mathfrak{d}_2=\mathfrak{d}_{12}'$: for we obviously have

$$\begin{aligned}
\mathfrak{a}_1\vee\mathfrak{d}_1&=\mathfrak{a}_1\vee\mathfrak{d}_{12}=\mathfrak{e}, &\qquad \mathfrak{a}_2\vee\mathfrak{d}_2&\supset\mathfrak{d}_{12}\vee\mathfrak{d}_{12}'=\mathfrak{e},\\
\mathfrak{d}_1\vee\mathfrak{d}_2&=\mathfrak{e}, & \mathfrak{d}_1\mathfrak{d}_2&=\mathfrak{o}.
\end{aligned}$$

If we have established the desired result for $N=2,\ldots,M$, we treat the case $N=M+1$ as follows. By hypothesis there exist simple ideals $\mathfrak{c}_1,\ldots,\mathfrak{c}_M$ such that $\mathfrak{a}_n\vee\mathfrak{c}_n=\mathfrak{e}$, $\mathfrak{c}_1\vee\ldots\vee\mathfrak{c}_M=\mathfrak{e}$, $\mathfrak{c}_m\mathfrak{c}_n=\mathfrak{o}$ for $m\neq n$, $m,n=1,\ldots,M$. We introduce the ideals $\mathfrak{d}=\mathfrak{d}_{1,M+1}\vee\ldots\vee\mathfrak{d}_{M,M+1}$, $\mathfrak{d}_n=\mathfrak{c}_n\mathfrak{d}$ for $n=1,\ldots,M$, $\mathfrak{d}_{M+1}=\mathfrak{d}'$. By Th. 1.2, the ideals, $\mathfrak{d},\mathfrak{d}',\mathfrak{d}_n$, $n=1,\ldots,M$ are all simple. It is evident that

$$\mathfrak{d}_1\vee\ldots\vee\mathfrak{d}_{M+1}=(\mathfrak{c}_1\vee\ldots\vee\mathfrak{c}_M)\mathfrak{d}\vee\mathfrak{d}'=\mathfrak{d}\vee\mathfrak{d}'=\mathfrak{e}$$

and that $\mathfrak{d}_m\mathfrak{d}_n=\mathfrak{o}$ for $m\neq n$, $m,n=1,\ldots,M+1$. Since $\mathfrak{d}_{n,M+1}\subset\mathfrak{d}$ for $n=1,\ldots,M$, we have $\mathfrak{a}_n\vee\mathfrak{d}_n=(\mathfrak{a}_n\vee\mathfrak{c}_n)(\mathfrak{a}_n\vee\mathfrak{d})=\mathfrak{a}_n\vee\mathfrak{d}\supset\mathfrak{a}_n\vee\mathfrak{d}_{n,M+1}=\mathfrak{e}$, $\mathfrak{a}_{M+1}\vee\mathfrak{d}_{M+1}\supset\mathfrak{d}\vee\mathfrak{d}'=\mathfrak{e}$. Hence (1′) follows from (1) in the case $N=M+1$. The proof of the equivalence of (1) and (1′) is now complete.

If the condition (1), or the equivalent condition (1′), is satisfied and $\mathfrak{a}_1\ldots\mathfrak{a}_N$ is a barrier ideal, we can write $\mathfrak{a}_1\ldots\mathfrak{a}_N=\mathfrak{b}\vee\mathfrak{c}$, $\mathfrak{b}\mathfrak{c}=\mathfrak{o}$, $\mathfrak{b}\in\mathfrak{N}$, $\mathfrak{c}\in\mathfrak{N}$ in accordance with Def. 3.1. Using condition (1′) we put $\mathfrak{b}_n=\mathfrak{b}\vee\mathfrak{d}_n'$, $\mathfrak{c}_n=\mathfrak{c}\mathfrak{d}_n$ for $n=1,\ldots,N$. By Th. 1.2 it is clear that $\mathfrak{b}_n$ and $\mathfrak{c}_n$ are normal ideals. It is obvious that

$$\mathfrak{b}_n\vee\mathfrak{c}_n=\mathfrak{b}\vee\mathfrak{d}_n'\vee\mathfrak{c}\mathfrak{d}_n=\mathfrak{b}\vee\mathfrak{c}\vee\mathfrak{d}_n'=\mathfrak{a}_1\ldots\mathfrak{a}_N\vee\mathfrak{d}_n'=(\mathfrak{a}_1\vee\mathfrak{d}_n')\ldots(\mathfrak{a}_N\vee\mathfrak{d}_n');$$

and hence the relations $\mathfrak{a}_n\vee\mathfrak{d}_n'=(\mathfrak{a}_n\vee\mathfrak{d}_n)(\mathfrak{a}_n\vee\mathfrak{d}_n')=\mathfrak{a}_n\vee\mathfrak{d}_n\mathfrak{d}_n'=\mathfrak{a}_n$ and $\mathfrak{a}_m\vee\mathfrak{d}_n'\supset\mathfrak{a}_m\vee\mathfrak{d}_m=\mathfrak{e}$, $m\neq n$, imply $\mathfrak{b}_n\vee\mathfrak{c}_n=\mathfrak{a}_n$. It is evident that $\mathfrak{b}_n\mathfrak{c}_n=\mathfrak{o}$. Since $\mathfrak{a}_n\supset\mathfrak{a}_1\ldots\mathfrak{a}_N$, we have $\mathfrak{a}_n'\subset(\mathfrak{a}_1\ldots\mathfrak{a}_N)'=\mathfrak{o}$, $\mathfrak{a}_n'=\mathfrak{o}$. Accordingly, $\mathfrak{a}_n$ is a barrier ideal if $\mathfrak{a}_n\neq\mathfrak{e}$, $n=1,\ldots,N$.

If $\mathfrak{a}_1,\ldots,\mathfrak{a}_N$ are distinct prime ideals, then for $m\neq n$ there exists an element a_{mn} belonging to $\mathfrak{a}_n$ but not to $\mathfrak{a}_m$. If we set $\mathfrak{d}_{mn}=\mathfrak{a}(a_{mn})$, we have $\mathfrak{a}_m\vee\mathfrak{d}_{mn}=\mathfrak{e}$, $\mathfrak{a}_n\supset\mathfrak{d}_{mn}$ for $m\neq n$, $m,n=1,\ldots,N$. Thus condition (1) is satisfied in this case.

If A has a unit e and the ideals $\mathfrak{a}_1,\ldots,\mathfrak{a}_N$ satisfy the relations $\mathfrak{a}_m\vee\mathfrak{a}_n=\mathfrak{e}$ for $m\neq n$, $m,n=1,\ldots,N$, we know that for $m\neq n$ there exist elements a_m,a_n such that $a_m\vee a_n=e$, $a_m\in\mathfrak{a}_m$, $a_n\in\mathfrak{a}_n$. If we put $\mathfrak{d}_{mn}=\mathfrak{a}(a_n)$, we have $\mathfrak{a}_m\vee\mathfrak{d}_{mn}\supset\mathfrak{a}(a_m)\vee\mathfrak{a}(a_n)=\mathfrak{a}(e)=\mathfrak{e}$, $\mathfrak{a}_n\supset\mathfrak{d}_{mn}$. Hence condition (1) is satisfied in this case also.

Ths. 3.1-3.5 give us information about the existence and construction of barrier ideals. While it will not be applied in detail, it provides a background for the further discussion.

We shall now characterize the Boolean rings of type (ω). We have:

Theorem 3.6. *The following properties of a Boolean ring A are equivalent:*

(1) *A is a ring without unit in which every simple ideal is semiprincipal;*

(2) *A is a ring of type (ω).*

A ring of type (ω) is totally multiplicative if and only if it is of type (β_1).

Starting with the remark that a ring of type (ω) cannot have a unit, we may proceed under either of the conditions (1) and (2) to adjoin a unit to A as in R Th. 1; in this way we obtain a ring A^* which is uniquely determined except for isomorphisms and which contains A as a prime ideal $\mathfrak{p}$ in accordance with R Th. 38. If (1) holds, then R Th. 39 shows that $\mathfrak{p}$ is not normal; and if (2) holds then by definition $\mathfrak{p}$ is not normal. If (1) holds, then $\mathfrak{p}$ cannot be a barrier ideal. If it were we could write $\mathfrak{p}=\mathfrak{b}\vee\mathfrak{b}'$, $\mathfrak{b}\in\mathfrak{N}$, in accordance with Th. 3.1. Thus $\mathfrak{b}$ would be simple relative to $\mathfrak{p}$ and by (1) would thus be semiprincipal relative to $\mathfrak{p}$. Accordingly, one of the ideals $\mathfrak{b}$ and $\mathfrak{b}'$ would be principal in $\mathfrak{p}$ and hence also in A^*. The fact that $\mathfrak{b}=\mathfrak{b}''$ and the fact that A^* has a unit would thus imply that both $\mathfrak{b}$ and $\mathfrak{b}'$ were principal ideals. In this way we would reach the contradiction that $\mathfrak{b}\vee\mathfrak{b}'=\mathfrak{e}$. On the other hand, if $\mathfrak{p}$ is not a barrier ideal, then (1) holds. If $\mathfrak{b}\subset\mathfrak{p}$ is any ideal simple relative to $\mathfrak{p}$, then $\mathfrak{b}$ is an ideal in A^* and

$$\mathfrak{b}''\vee\mathfrak{b}'\supset(\mathfrak{b}''\vee\mathfrak{b}')\mathfrak{p}=\mathfrak{b}''\mathfrak{p}\vee\mathfrak{b}'\mathfrak{p}=\mathfrak{b}\vee\mathfrak{b}'\mathfrak{p}=\mathfrak{p}$$

by virtue of R Th. 22. Since $\mathfrak{p}$ is not a barrier ideal, Th. 3.1 shows that $\mathfrak{b}''\vee\mathfrak{b}'\neq\mathfrak{p}$. The fact that $\mathfrak{p}$ is prime therefore implies $\mathfrak{b}''\vee\mathfrak{b}'=\mathfrak{e}$. Accordingly the ideals $\mathfrak{b}''$ and $\mathfrak{b}'$ are simple, and hence principal, in the ring A^* with unit. By R Th. 41, we have either $\mathfrak{b}''\subset\mathfrak{p}$ or $\mathfrak{b}'\subset\mathfrak{p}$. In the first alternative, $\mathfrak{b}''=\mathfrak{b}''\mathfrak{p}=\mathfrak{b}''(\mathfrak{b}\vee\mathfrak{b}'\mathfrak{p})=\mathfrak{b}''\mathfrak{b}\vee\mathfrak{b}''\mathfrak{b}'\mathfrak{p}=\mathfrak{b}$ so that $\mathfrak{b}$ is principal both in A^* and in $\mathfrak{p}$. In the second alternative, $\mathfrak{b}'=\mathfrak{b}'\mathfrak{p}$ is principal both in A^* and in $\mathfrak{p}$. Hence we see that $\mathfrak{b}$ is semiprincipal relative to $\mathfrak{p}$. The equivalence of (1) and (2) is thus established.

By combining the preceding results with Ths. 2.3 (2) and 2.6, we obtain the final assertion of the present theorem.

A characterization of rings of type (ω,ω) is given by the following result:

Theorem 3.7. *The following properties of a Boolean ring A are equivalent:*

(1) *A is isomorphic to the product of prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ in a ring with unit neither of which is a normal ideal or a barrier ideal;*

(2) *A is of type (ω,ω).*

A ring satisfying (1) or (2) has no unit. A ring of type (ω,ω) is totally multiplicative if and only if it is of type (β_1,β_1).

First we show that (1) implies (2). Identifying A with the product $\mathfrak{p}_1\mathfrak{p}_2$ in the given ring A^* with unit, we select an element a of A^* belonging to $\mathfrak{p}_2$ but not to $\mathfrak{p}_1$, and note that by R Th. 41 the element a' belongs to $\mathfrak{p}_1$ but not to $\mathfrak{p}_2$. We can then represent A^* as the direct sum $\mathfrak{a}(a)\vee\mathfrak{a}(a')$, and A as the direct sum

$$\mathfrak{p}_1\mathfrak{p}_2(\mathfrak{a}(a)\vee\mathfrak{a}(a'))=\mathfrak{p}_1\mathfrak{a}(a)\vee\mathfrak{p}_2\mathfrak{a}(a'),$$

in accordance with R Th. 51 and Th. 1.3. We wish to prove that the rings $\mathfrak{p}_1\mathfrak{a}(a)$ and $\mathfrak{p}_2\mathfrak{a}(a')$ are of type (ω). Evidently it is enough to treat the first. Here we have to show that $\mathfrak{p}_1\mathfrak{a}(a)$ is an ideal which is prime in $\mathfrak{a}(a)$ but is neither a normal ideal nor a barrier ideal relative to $\mathfrak{a}(a)$. Th. 1.7 shows that $\mathfrak{p}_1\mathfrak{a}(a)$ is prime and non-normal relative to $\mathfrak{a}(a)$. If $\mathfrak{p}_1\mathfrak{a}(a)$ were a barrier ideal in $\mathfrak{a}(a)$, we could write $\mathfrak{p}_1=\mathfrak{b}\vee\mathfrak{c}$, $\mathfrak{b}\mathfrak{c}=\mathfrak{o}$, where $\mathfrak{b}$ and $\mathfrak{c}$ are normal relative to $\mathfrak{a}(a)$. We should then have $\mathfrak{p}_1=\mathfrak{p}_1(\mathfrak{a}(a)\vee\mathfrak{a}(a'))=\mathfrak{p}_1\mathfrak{a}(a)\vee\mathfrak{a}(a')=\mathfrak{b}\vee(\mathfrak{c}\vee\mathfrak{a}(a'))$ where $\mathfrak{b},\mathfrak{c}$, and $\mathfrak{c}\vee\mathfrak{a}(a')$ are normal relative to A^*, by virtue of Ths. 1.4 and 1.1. Since $\mathfrak{b}(\mathfrak{c}\vee\mathfrak{a}(a'))=\mathfrak{o}$, the ideal $\mathfrak{p}_1$ would be a barrier ideal in A^*, contrary to hypothesis. The proof that (1) implies (2) is therefore complete.

On the other hand (2) implies (1). If $A=A_1\vee A_2$ where A_1 and A_2 are of type (ω), we adjoin units to A_1 and A_2 obtaining rings A_1^* and A_2^*. We then introduce the ring with unit $A^*=A_1^*\vee A_2^*$ and the subrings $\mathfrak{p}_1=A_1\vee A_2^*$, $\mathfrak{p}_2=A_1^*\vee A_2$. By Th. 2.7 both $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are non-normal prime ideals in A^*. Their product is obviously the given ring A. We thus have to prove that neither $\mathfrak{p}_1$ nor $\mathfrak{p}_2$ is a barrier ideal in A^*.

It is enough to treat $\mathfrak{p}_1$. If $\mathfrak{p}_1$ were a barrier ideal, we could write $\mathfrak{p}_1 = \mathfrak{b} \vee \mathfrak{c}$, $\mathfrak{b}\mathfrak{c} = \mathfrak{o}$, $\mathfrak{b} \in \mathfrak{N}$, $\mathfrak{c} \in \mathfrak{N}$. Considering A_1^* as a principal ideal $\mathfrak{a}(a)$ in A^*, the ideals $\mathfrak{b}\mathfrak{a}(a)$ and $\mathfrak{c}\mathfrak{a}(a)$ would be normal relative to $\mathfrak{a}(a)$ by Th. 1.3. Moreover they would satisfy the relations $\mathfrak{p}_1\mathfrak{a}(a) = \mathfrak{b}\mathfrak{a}(a) \vee \mathfrak{c}\mathfrak{a}(a)$, $\mathfrak{b}\mathfrak{a}(a) \cdot \mathfrak{c}\mathfrak{a}(a) = \mathfrak{o}$. Since $\mathfrak{p}_1\mathfrak{a}(a)$ coincides with A_1, it is a non-normal prime ideal in A_1^*. We thus conclude that $\mathfrak{p}_1\mathfrak{a}(a)$ would be a barrier ideal relative to $\mathfrak{a}(a)$, against our assumption concerning A_1 as an ideal in A_1^*. Hence $\mathfrak{p}_1$ cannot be a barrier ideal in A^*. With this we complete our proof that (2) implies (1).

It is evident from Ths. 2.7 and 3.6 that no ring of type (ω, ω) has a unit; and, from Ths. 2.7 and 3.6, it is clear that a ring of type (ω, ω) is totally multiplicative if and only if it is of type (β_1, β_1). According to Th. 2.10 the rings of type (β_1, β_1) appear as a special case under type (β_2).

The types (ω) and (ω, ω) are by no means so sharply limited as the various types discussed in § 2, as we see by virtue of the following result:

Theorem 3.8. *If A_1 is a Boolean ring of type (ω) or (ω, ω), and if A_2 is an arbitrary Boolean ring with unit, then the ring $A = A_1 \vee A_2$ is of the same type as A_1.*

First, let A_1 be of type (ω). By Th. 2.7, the ring $A = A_1 \vee A_2$ is a non-normal prime ideal in the direct sum $A_1^* \vee A_2$. If it were a barrier ideal, then the arguments used in the second paragraph of the proof of Th. 3.7 could be applied, since A_2 has a unit, to show that A_1 is a barrier ideal in A_1^*; but by hypothesis A_1 is of type (ω) and therefore cannot be a barrier ideal in A_1^*. Hence A is of type (ω). If A_1 is of type (ω, ω), then $A_1 = A_{11} \vee A_{12}$ where A_{11} and A_{12} are of type (ω). If we write $A = A_1 \vee A_2 = A_{11} \vee (A_{12} \vee A_2)$ and apply the result just proved, we find that A is also of type (ω, ω).

Later, in § 8, we give a further characterization of rings of type (ω, ω). For the present we content ourselves with the following remark:

Theorem 3.9. *The types (ω) and (ω, ω) are distinct.*

By definition a ring A of type (ω, ω) is the direct sum $A_1 \vee A_2$ of rings of type (ω). Since neither A_1 nor A_2 has a unit, each is a non-semiprincipal simple ideal in A by virtue of Th. 2.7. By Th. 3.6 the ring A cannot be of type (ω).

§ 4. Topological Aspects of the Two Preceding Sections.

In the present section, we shall add some topological comments upon the general concepts introduced in §§ 2, 3. We shall presuppose an acquaintance with our paper A establishing the fundamental connections between Boolean rings and general topology.

We first indicate the topological origin of the totally additive rings.

Theorem 4.1. *If $A_{\mathfrak{R}}$ is the complete basic ring of a T_0-space $\mathfrak{R}$ and $\mathfrak{a}_{\mathfrak{R}}$ is the ideal of nowhere dense sets in $A_{\mathfrak{R}}$, then the quotient-ring $A(\mathfrak{R}) = A_{\mathfrak{R}}/\mathfrak{a}_{\mathfrak{R}}$ is totally additive. Conversely, if A is a totally additive ring, then there exists at least one T_0-space $\mathfrak{R}$ such that $A(\mathfrak{R})$ is isomorphic to A.*

The complete basic ring $A_{\mathfrak{R}}$ is the ring generated by $\mathfrak{a}_{\mathfrak{R}}$ and the open sets in $\mathfrak{R}$. It obviously contains $\mathfrak{a}_{\mathfrak{R}}$ as an ideal[12]. If $\mathfrak{a}$ is any non-void subclass of $A_{\mathfrak{R}}$, we define a corresponding open set a_0 as $\sum_{a \in \mathfrak{a}} a'^{-\prime}$. Since $a \equiv a'^{-\prime} \pmod{\mathfrak{a}_{\mathfrak{R}}}$, the relation $a_0 a'^{-\prime} = a'^{-\prime}$ implies $a_0 a \equiv a \pmod{\mathfrak{a}_{\mathfrak{R}}}$, for every a in $\mathfrak{a}$. If b is an element of $A_{\mathfrak{R}}$ such that $ba \equiv a \pmod{\mathfrak{a}_{\mathfrak{R}}}$ for every a in $\mathfrak{a}$, we have $b'a \equiv b'ba \equiv 0 \pmod{\mathfrak{a}_{\mathfrak{R}}}$, $b^{-\prime} a'^{-\prime} \equiv (b')'^{-\prime} a'^{-\prime} \equiv b'a \equiv 0 \pmod{\mathfrak{a}_{\mathfrak{R}}}$ for every a in $\mathfrak{a}$. Since $b^{-\prime}$ and $a'^{-\prime}$ are open sets the relation $b^{-\prime} a'^{-\prime} \equiv 0 \pmod{\mathfrak{a}_{\mathfrak{R}}}$ implies $b^{-\prime} a'^{-\prime} = 0$. Hence $b^{-\prime} a_0 = 0$; and, since $b^{-\prime} \equiv b' \pmod{\mathfrak{a}_{\mathfrak{R}}}$, $b' a_0 \equiv 0 \pmod{\mathfrak{a}_{\mathfrak{R}}}$ or, equivalently, $b a_0 \equiv a_0 \pmod{\mathfrak{a}_{\mathfrak{R}}}$. Now if $\mathfrak{a}(\mathfrak{R})$ is a non-void subclass of $A(\mathfrak{R})$, its antecedent $\mathfrak{a}$ in $A_{\mathfrak{R}}$ under the homomorphism $A_{\mathfrak{R}} \to A_{\mathfrak{R}}/\mathfrak{a}_{\mathfrak{R}} = A(\mathfrak{R})$ defines an element a_0 in the manner just described. Let $a_0(\mathfrak{R})$ be the image of a_0 in $A(\mathfrak{R})$. We shall show that $a_0(\mathfrak{R})$ is the sum of $\mathfrak{a}(\mathfrak{R})$ in the sense of Def. 2.1. If $a(\mathfrak{R}) \in \mathfrak{a}(\mathfrak{R})$, its antecedents a in $A_{\mathfrak{R}}$ satisfy the relation $a_0 a \equiv a \pmod{\mathfrak{a}_{\mathfrak{R}}}$; and we see therefore that $a_0(\mathfrak{R}) a(\mathfrak{R}) = a(\mathfrak{R})$ or, equivalently, $a_0(\mathfrak{R}) > a(\mathfrak{R})$ for every $a(\mathfrak{R})$ in $\mathfrak{a}(\mathfrak{R})$. If $b(\mathfrak{R})$ is an element of $A(\mathfrak{R})$ such that $b(\mathfrak{R}) > a(\mathfrak{R})$ or, equivalently, $b(\mathfrak{R}) a(\mathfrak{R}) = a(\mathfrak{R})$ for every $a(\mathfrak{R})$ in $\mathfrak{a}(\mathfrak{R})$, then its antecedents b in $A_{\mathfrak{R}}$ satisfy the relation $ba \equiv a \pmod{\mathfrak{a}_{\mathfrak{R}}}$ for every a in $\mathfrak{a}$. By the preceding results $b a_0 \equiv a_0 \pmod{\mathfrak{a}_{\mathfrak{R}}}$. Hence we find that $b(\mathfrak{R}) a_0(\mathfrak{R}) = a_0(\mathfrak{R})$ or, equivalently, $b(\mathfrak{R}) > a_0(\mathfrak{R})$. Thus $a_0(\mathfrak{R})$ is the sum of $\mathfrak{a}(\mathfrak{R})$; and $A(\mathfrak{R})$ is a totally additive ring.

---

[12]) For a detailed discussion of the complete basic ring, see A Ch. II, Ths. 24 and 25.

We now let A be an arbitrary totally additive Boolean ring, and take $\mathfrak{R}$ as its representative bicompact Boolean space[13]). Then A is isomorphic to the Boolean ring of all closed-and-open sets in $\mathfrak{R}$; and the ideals in A are represented by the open sets in $\mathfrak{R}$. The normal ideals in A are precisely those represented by regular open sets in $\mathfrak{R}$. Now Th. 2.2 (2) shows that all the normal ideals in A are principal. Hence the space $\mathfrak{R}$ has the property that its regular open sets are precisely the closed-and-open sets. Thus we see that, in $A_{\mathfrak{R}}$, the regular open sets constitute a subring isomorphic to A. On the other hand, we know that each residual class (mod $\mathfrak{a}_{\mathfrak{R}}$) in $A_{\mathfrak{R}}$ contains exactly one regular open set. It follows immediately that $A(\mathfrak{R})=A_{\mathfrak{R}}/\mathfrak{a}_{\mathfrak{R}}$ is isomorphic to the ring of regular open sets in $\mathfrak{R}$ and hence to the given ring A.

Th. 4.1 provides a method of construction capable of yielding all totally additive Boolean rings. It also raises the interesting problem of determining all the different T_0-spaces such that the corresponding rings $A(\mathfrak{R})$ are isomorphic to a given totally additive ring A. Our paper A provides methods which should suffice for a deep investigation, if not actually for a complete solution, of this problem.

In connection with his theory of continuous geometries, v. Neumann[14]) has had occasion to point out that the ring $A(\mathfrak{R})$ is totally additive, although he was not in a position to prove that all totally additive rings could be obtained as rings $A(\mathfrak{R})$. He has observed also that a similar construction — namely, reduction of the ring of all measurable sets on the unit interval modulo the ideal of null sets — yields a totally additive Boolean ring. When this construction is generalized in the obvious way, it provides a means of obtaining a variety of totally additive Boolean rings. That it cannot suffice for the treatment of the entire category of totally additive rings was also pointed out by v. Neumann. In fact, it is easily seen that when $\mathfrak{R}$ is the unit interval the ring $A(\mathfrak{R})$ cannot support the kind of numerical measure which would have to be defined in it if it were obtainable by the construction under consideration. Thus an interesting problem arises in this connection: to determine which totally additive Boolean rings can be constructed in terms of the general theory of measure as indicated above.

[13]) See A Ch. I, §§ 1, 2 especially Ths. 1, 2, 5.

[14]) J. v. Neumann, Proceedings of the National Academy of Sciences, U. S. A., 22 (1936), pp. 92-108.

We turn now to a topological study of the concept of barrier ideals introduced in § 3. Our results throw new light on the nature of totally additive Boolean rings. We begin with some general theorems of topology which are parallel to those given in the first part of § 3 and do, in fact, contain the latter as special cases.

Definition 4.1. *A set $\mathfrak{F}$ in a T_0-space $\mathfrak{R}$ is said to be a barrier if it is the non-void common boundary of two disjoint open sets $\mathfrak{G}_1$ and $\mathfrak{G}_2$.*

Theorem 4.2. *If $\mathfrak{F}$ is a barrier in a T_0-space $\mathfrak{R}$ and $\mathfrak{G}_1$, $\mathfrak{G}_2$, are associated open sets, then $\mathfrak{F}$ is closed, $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are non-void regular open sets, and $\mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{F} = (\mathfrak{G}_1 \cup \mathfrak{G}_2)^-$ is closed. The associated open sets may be so chosen that $(\mathfrak{G}_1 \cup \mathfrak{G}_2)^- = \mathfrak{R}$. Conversely, if $\mathfrak{G}_1^-$ and $\mathfrak{G}_2$ are disjoint non-void regular open sets such that $(\mathfrak{G}_1 \cup \mathfrak{G}_2)^- = \mathfrak{R}$, they have a common boundary $\mathfrak{F}$ which is a barrier in $\mathfrak{R}$ whenever it is non-void.*

If $\mathfrak{F}$ is a barrier and $\mathfrak{G}_1$, $\mathfrak{G}_2$ are associated open sets we have $\mathfrak{F} = \mathfrak{G}_1^- \mathfrak{G}_1' = \mathfrak{G}_2^- \mathfrak{G}_2' \neq 0$, and hence $\mathfrak{G}_1 \neq 0$, $\mathfrak{G}_2 \neq 0$. Since $\mathfrak{G}_1$ and $\mathfrak{G}_1'$ are closed, $\mathfrak{F}$ is also closed. The relation $\mathfrak{G}_1 \subset \mathfrak{G}_1^-$ implies $\mathfrak{G}_1 \subset \mathfrak{G}_1^{-'-'}$. Hence to prove the equality $\mathfrak{G}_1 = \mathfrak{G}_1^{-'-'}$, which identifies $\mathfrak{G}_1$ as a regular open set, it is sufficient to prove the inclusion relation $\mathfrak{G}_1 \supset \mathfrak{G}_1^{-'-'}$. Since $\mathfrak{G}_2 \mathfrak{G}_1 = \mathfrak{G}_2 \mathfrak{F} = 0$, we have

$$\begin{aligned}
&\mathfrak{G}_1^- = \mathfrak{G}_1^- \mathfrak{G}_1 \cup \mathfrak{G}_1^- \mathfrak{G}_1' = \mathfrak{G}_1 \cup \mathfrak{F} = (\mathfrak{G}_1 \cup \mathfrak{F}) \mathfrak{G}_2',\\
&\mathfrak{G}_1^{-'} = \mathfrak{G}_1' \mathfrak{F}' \cup \mathfrak{G}_2,\\
&\mathfrak{G}^{-'-} = (\mathfrak{G}_1' \mathfrak{F}')^- \cup \mathfrak{G}_2^- \supset \mathfrak{G}_1' \mathfrak{F}' \cup \mathfrak{G}_2 \cup \mathfrak{F},\\
&\mathfrak{G}_1^{-'-'} \subset (\mathfrak{G}_1 \cup \mathfrak{F}) \mathfrak{G}_2' \mathfrak{F}' = \mathfrak{G}_1 \mathfrak{G}_2' \mathfrak{F}' \subset \mathfrak{G}_1.
\end{aligned}$$

Hence $\mathfrak{G}_1$ is a regular open set. The same argument applies to show that $\mathfrak{G}_2$ is a regular open set. If is evident that

$$\mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{F} = (\mathfrak{G}_1 \cup \mathfrak{F}) \cup (\mathfrak{G}_2 \cup \mathfrak{F}) = \mathfrak{G}_1^- \cup \mathfrak{G}_2^- = (\mathfrak{G}_1 \cup \mathfrak{G}_2)^-$$

is a closed set.

If $\mathfrak{G}$ is the open set $(\mathfrak{G}_1 \cup \mathfrak{G}_2)^{-'}$, we shall show that the open sets $\mathfrak{G}_1$ and $\mathfrak{G}_3 = \mathfrak{G}_2 \cup \mathfrak{G}$ are an associated pair for $\mathfrak{F}$. It is evident that $\mathfrak{G}_1 \mathfrak{G}_3 = \mathfrak{G}_1 \mathfrak{G}_2 \cup \mathfrak{G}_1 \mathfrak{G} = \mathfrak{G}_1 \mathfrak{G} = \mathfrak{G}_1 \mathfrak{G}_1' \mathfrak{G}_2' \mathfrak{F}' = 0$. Now

$$\mathfrak{G}^- = (\mathfrak{G}_1 \cup \mathfrak{G}_2)^{-'-} \subset (\mathfrak{G}_1 \cup \mathfrak{G}_2)'^- = (\mathfrak{G}_1' \mathfrak{G}_2')^- = \mathfrak{G}_1' \mathfrak{G}_2' = \mathfrak{G} \cup \mathfrak{F}.$$

Hence

$$\mathfrak{G}_3^- = \mathfrak{G}_2^- \cup \mathfrak{G}^- = \mathfrak{F}\cup\mathfrak{G}_2\cup\mathfrak{G} = \mathfrak{F}\cup\mathfrak{G}_3,$$
$$\mathfrak{G}_3^-\mathfrak{G}_3' = \mathfrak{F}\mathfrak{G}_3' = \mathfrak{F}\mathfrak{G}_2'\mathfrak{G}' = \mathfrak{F}\mathfrak{G}' = \mathfrak{F}(\mathfrak{G}_1\cup\mathfrak{G}_2\cup\mathfrak{F}) = \mathfrak{F}.$$

Thus the open sets $\mathfrak{G}_1, \mathfrak{G}_3$ are disjoint and have $\mathfrak{F}$ as their common boundary. Moreover

$$(\mathfrak{G}_1\cup\mathfrak{G}_3)^- = \mathfrak{G}_1^-\cup\mathfrak{G}_3^- = \mathfrak{G}_1\cup\mathfrak{G}_3\cup\mathfrak{F} = \mathfrak{G}_1\cup\mathfrak{G}_2\cup\mathfrak{G}\cup\mathfrak{F} = \mathfrak{R}.$$

We have thereby proved that when $\mathfrak{F}$ is a barrier the associated open sets may be chosen as stated in the theorem.

We now prove the converse part of the theorem. If we start with open sets $\mathfrak{G}_1$ and $\mathfrak{G}_2$ having the indicated properties, we first prove that $\mathfrak{G}_1^-\supset\mathfrak{G}_2'$, $\mathfrak{G}_2^-\supset\mathfrak{G}_1'$. By symmetry, it is enough to establish the first relation. Since $\mathfrak{G}_1^-\cup\mathfrak{G}_2^- = (\mathfrak{G}_1\cup\mathfrak{G}_2)^- = \mathfrak{R}$, we have $\mathfrak{G}_1^-\supset\mathfrak{G}_2^{-\prime}$ and hence $\mathfrak{G}_1^-\supset\mathfrak{G}_2^{-\prime-}$. Since $\mathfrak{G}_2$ is a regular open set we have $\mathfrak{G}_2^{-\prime-\prime} = \mathfrak{G}_2$, $\mathfrak{G}_2^{-\prime-} = \mathfrak{G}_2'$. Hence the relation $\mathfrak{G}_1^-\supset\mathfrak{G}_2'$ is valid. Since $\mathfrak{G}_1\mathfrak{G}_2=0$, we have $\mathfrak{G}_1\cup\mathfrak{G}_1'\mathfrak{G}_2' = (\mathfrak{G}_1\cup\mathfrak{G}_1')(\mathfrak{G}_1\cup\mathfrak{G}_2') = \mathfrak{G}_2'$, and similarly $\mathfrak{G}_2\cup\mathfrak{G}_1'\mathfrak{G}_2' = \mathfrak{G}_1'$. Hence we see that the set $\mathfrak{G}_1\cup\mathfrak{G}_1'\mathfrak{G}_2' = \mathfrak{G}_2'$ is closed and contains $\mathfrak{G}_1$. Thus we have $\mathfrak{G}_1^-\subset\mathfrak{G}_1\cup\mathfrak{G}_1'\mathfrak{G}_2'$. On the other hand, we also have $\mathfrak{G}_1\cup\mathfrak{G}_1'\mathfrak{G}_2'\subset\mathfrak{G}_1^-\cup\mathfrak{G}_1'\mathfrak{G}_1^- = \mathfrak{G}_1^-$ by the earlier results. We therefore find that $\mathfrak{G}_1^- = \mathfrak{G}_1\cup\mathfrak{G}_1'\mathfrak{G}_2'$; and, in similar fashion, that $\mathfrak{G}_2^- = \mathfrak{G}_2\cup\mathfrak{G}_1'\mathfrak{G}_2'$. Hence we conclude that $\mathfrak{G}_1^-\mathfrak{G}_1' = \mathfrak{G}_1'\mathfrak{G}_2' = \mathfrak{G}_2^-\mathfrak{G}_2'$. Thus the closed set $\mathfrak{F} = \mathfrak{G}_1'\mathfrak{G}_2'$ is the common boundary of $\mathfrak{G}_1$ and $\mathfrak{G}_2$. Obviously $\mathfrak{F}=0$ if and only if $\mathfrak{G}_1\cup\mathfrak{G}_2 = \mathfrak{F}' = \mathfrak{R}$; that is, if and only if $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are closed as well as open. When $\mathfrak{F}$ is not void, it is a barrier by Def. 4.1.

Theorem 4.3. *If $\mathfrak{F}$ is a barrier in the T_0-space $\mathfrak{R}$ and $\mathfrak{G}$ is a closed-and-open set such that $\mathfrak{F}\mathfrak{G}\neq 0$, then $\mathfrak{F}\mathfrak{G}$ is a barrier in $\mathfrak{R}$.*

Let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ be open sets associated with $\mathfrak{F}$ and introduce the open sets $\mathfrak{G}_3=\mathfrak{G}_1\mathfrak{G}$, $\mathfrak{G}_4=\mathfrak{G}_2\mathfrak{G}$. Since $(\mathfrak{G}_1\mathfrak{G})^-\subset\mathfrak{G}_1^-\mathfrak{G}^- = \mathfrak{G}_1^-\mathfrak{G}$ and $(\mathfrak{G}_1\mathfrak{G}')^-\subset\mathfrak{G}_1^-\mathfrak{G}'^- = \mathfrak{G}_1^-\mathfrak{G}'$, we have

$$(\mathfrak{G}_1\mathfrak{G})^-\mathfrak{G}'=0,\qquad (\mathfrak{G}_1\mathfrak{G}')^-\mathfrak{G}=0.$$

$$\begin{aligned}\mathfrak{G}_1^-\mathfrak{G} &= (\mathfrak{G}_1\mathfrak{G}\cup\mathfrak{G}_1\mathfrak{G}')^-\mathfrak{G} = (\mathfrak{G}_1\mathfrak{G})^-\mathfrak{G}\cup(\mathfrak{G}_1\mathfrak{G}')^-\mathfrak{G}\\ &= (\mathfrak{G}_1\mathfrak{G})^-\mathfrak{G} = (\mathfrak{G}_1\mathfrak{G})^-(\mathfrak{G}\cup\mathfrak{G}') = (\mathfrak{G}_1\mathfrak{G})^-.\end{aligned}$$

Hence $\mathfrak{G}_3^-\mathfrak{G}_3' = (\mathfrak{G}_1\mathfrak{G})^-(\mathfrak{G}_1'\cup\mathfrak{G}') = \mathfrak{G}_1^-\mathfrak{G}(\mathfrak{G}_1'\cup\mathfrak{G}') = \mathfrak{G}_1^-\mathfrak{G}_1'\mathfrak{G} = \mathfrak{F}\mathfrak{G}$. In the same way $\mathfrak{G}_4^-\mathfrak{G}_4' = \mathfrak{F}\mathfrak{G}$. Thus, as the common boundary of $\mathfrak{G}_3$ and $\mathfrak{G}_4$, the set $\mathfrak{F}\mathfrak{G}$ is a barrier when it is non-void.

Theorem 4.4. *If $\mathfrak{F}_1,\dots,\mathfrak{F}_N$ are disjoint barriers in a T_0-space $\mathfrak{R}$, then $\mathfrak{F}_1\cup\dots\cup\mathfrak{F}_N$ is a barrier in $\mathfrak{R}$.*

It is evidently sufficient to treat the case $N=2$, since an obvious induction then yields the general case without difficulty. When $N=2$, let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ be open sets associated with $\mathfrak{F}_1$, $\mathfrak{G}_3$ and $\mathfrak{G}_4$ open sets associated with $\mathfrak{F}_2$. By virtue of Th. 4.2 we may suppose that $\mathfrak{G}_1\cup\mathfrak{G}_2\cup\mathfrak{F}_1 = \mathfrak{G}_3\cup\mathfrak{G}_4\cup\mathfrak{F}_2 = \mathfrak{R}$. We now introduce the open sets $\mathfrak{G}_5=\mathfrak{G}_1\mathfrak{G}_3\cup\mathfrak{G}_2\mathfrak{G}_4$ and $\mathfrak{G}_6=\mathfrak{G}_1\mathfrak{G}_4\cup\mathfrak{G}_2\mathfrak{G}_3$, the closed set $\mathfrak{F}=\mathfrak{F}_1\cup\mathfrak{F}_2$. Using the relations

$$\mathfrak{G}_1\mathfrak{G}_2=\mathfrak{G}_1\mathfrak{F}_1=\mathfrak{G}_2\mathfrak{F}_1=\mathfrak{G}_3\mathfrak{G}_4=\mathfrak{G}_3\mathfrak{F}_2=\mathfrak{G}_4\mathfrak{F}_2=0$$

we find that $\mathfrak{G}_5\mathfrak{G}_6=\mathfrak{G}_5\mathfrak{F}=\mathfrak{G}_6\mathfrak{F}=0$; and by similar reckoning that $\mathfrak{G}_5\cup\mathfrak{G}_6\cup\mathfrak{F}=\mathfrak{R}$. From the relations

$$\mathfrak{G}_5^-\subset\mathfrak{G}_1^-\mathfrak{G}_3^-\cup\mathfrak{G}_2^-\mathfrak{G}_4^- = (\mathfrak{G}_1\cup\mathfrak{F}_1)(\mathfrak{G}_3\cup\mathfrak{F}_2)\cup(\mathfrak{G}_2\cup\mathfrak{F}_1)(\mathfrak{G}_4\cup\mathfrak{F}_2) = \mathfrak{G}_5\cup\mathfrak{F}$$

and $\mathfrak{F}\subset\mathfrak{G}_5^-$, to be proved below, we conclude that

$$\mathfrak{G}_5^-=\mathfrak{G}_5\cup\mathfrak{F},\qquad \mathfrak{G}_5^-\mathfrak{G}_5'=\mathfrak{F}\mathfrak{G}_5'=\mathfrak{F}(\mathfrak{G}_6\cup\mathfrak{F})=\mathfrak{F}.$$

By symmetry, we have $\mathfrak{G}_6^-\mathfrak{G}_6'=\mathfrak{F}$. Thus $\mathfrak{F}$, as the common boundary of the disjoint open sets $\mathfrak{G}_5$ and $\mathfrak{G}_6$, is a barrier. The proof of the relation $\mathfrak{F}\subset\mathfrak{G}_5^-$ runs as follows. We have

$$\mathfrak{F}_1\mathfrak{G}_3\subset\mathfrak{G}_1^-\mathfrak{G}_3\subset(\mathfrak{G}_1\mathfrak{G}_3\cup\mathfrak{G}_3')^-\mathfrak{G}_3\subset(\mathfrak{G}_1\mathfrak{G}_3)^-\cup\mathfrak{G}_3'^-\mathfrak{G}_3=(\mathfrak{G}_1\mathfrak{G}_3)^-$$

and, similarly, $\mathfrak{F}_1\mathfrak{G}_4\subset(\mathfrak{G}_2\mathfrak{G}_4)^-$. Hence we find that

$$\mathfrak{F}_1=\mathfrak{F}_1\mathfrak{F}_2'=\mathfrak{F}_1(\mathfrak{G}_3\cup\mathfrak{G}_4)\subset(\mathfrak{G}_1\mathfrak{G}_3)^-\cup(\mathfrak{G}_2\mathfrak{G}_4)^-=\mathfrak{G}_5^-.$$

In much the same way, we show that $\mathfrak{F}_2\subset\mathfrak{G}_5^-$ and thus arrive at the desired relation $\mathfrak{F}=\mathfrak{F}_1\cup\mathfrak{F}_2\subset\mathfrak{G}_5^-$.

Theorem 4.5. *If $\mathfrak{F}_1,\dots,\mathfrak{F}_N$ are non-void sets in a T_0-space $\mathfrak{R}$ and if $\mathfrak{F}_1\cup\dots\cup\mathfrak{F}_N$ is a barrier, then the condition*

(1) *there exists a closed-and-open set $\mathfrak{D}_{mn}$ such that $\mathfrak{F}_m\subset\mathfrak{D}_{mn}$, $\mathfrak{D}_{mn}\mathfrak{F}_n=0$ for $m\neq n$, $m, n=1,\dots,N$ —*

implies that $\mathfrak{F}_1,\dots,\mathfrak{F}_N$ are barriers.

The set $\mathfrak{G}_m=\mathfrak{D}_{m1}\mathfrak{D}_{m2}\dots\mathfrak{D}_{m,m-1}\mathfrak{D}_{m,m+1}\dots\mathfrak{D}_{m,N-1}\mathfrak{D}_{m,N}$ is a closed-and-open set such that $\mathfrak{F}_m\subset\mathfrak{G}_m$, $\mathfrak{G}_m\mathfrak{F}_n=0$ for $m\neq n$, $m, n=1,\dots,N$. Hence we have $\mathfrak{F}_m=\mathfrak{G}_m(\mathfrak{F}_1\cup\dots\cup\mathfrak{F}_N)$ for $m=1,\dots,N$. Th. 4.3 now shows that, if $\mathfrak{F}_m\neq 0$ and if $\mathfrak{F}_1\cup\dots\cup\mathfrak{F}_N$ is a barrier, then $\mathfrak{F}_m$ is also a barrier, $m=1,\dots,N$.

We now indicate the connection between barriers and barrier ideals. We have:

Theorem 4.6. *Let $\mathfrak{a}$ be an ideal in a Boolean ring A; and let $\mathfrak{B}(A)$ be a representative Boolean space for A, $\mathfrak{G}(\mathfrak{a})$ the representative open set for $\mathfrak{a}$ in $\mathfrak{B}(A)$. Then $\mathfrak{a}$ is a barrier ideal if and only if $\mathfrak{G}'(\mathfrak{a})$ is a barrier.*

We base the proof on results of our paper A and Th. 4.2 above. If $\mathfrak{a}$ is a barrier ideal, we have $\mathfrak{a} \neq \mathfrak{e}$, $\mathfrak{a}' = \mathfrak{o}$, $\mathfrak{a} = \mathfrak{b} \vee \mathfrak{c}$, $\mathfrak{b}\mathfrak{c} = \mathfrak{o}$, $\mathfrak{b} \in \mathfrak{N}$, $\mathfrak{c} \in \mathfrak{N}$. Thus the corresponding representative open sets $\mathfrak{G}(\mathfrak{a})$, $\mathfrak{G}(\mathfrak{b})$, $\mathfrak{G}(\mathfrak{c})$ in $\mathfrak{B}(A)$ have the properties

$$\mathfrak{G}'(\mathfrak{a}) \neq 0, \quad \mathfrak{G}^{-\prime}(\mathfrak{a}) = 0, \quad \mathfrak{G}(\mathfrak{a}) = \mathfrak{G}(\mathfrak{b}) \cup \mathfrak{G}(\mathfrak{c}), \quad \mathfrak{G}(\mathfrak{b})\mathfrak{G}(\mathfrak{c}) = 0;$$

and $\mathfrak{G}(\mathfrak{b})$ and $\mathfrak{G}(\mathfrak{c})$ are regular open sets. From Th. 4.2 we see at once that $\mathfrak{G}'(\mathfrak{a}) = \mathfrak{G}'(\mathfrak{b})\mathfrak{G}'(\mathfrak{c})$ is the common boundary of $\mathfrak{G}(\mathfrak{b})$ and $\mathfrak{G}(\mathfrak{c})$ and is a barrier in $\mathfrak{B}(A)$. On the other hand if $\mathfrak{G}'(\mathfrak{a})$ is a barrier in $\mathfrak{B}(A)$ there exist associated open sets $\mathfrak{G}_1$, $\mathfrak{G}_2$, which we may suppose, by virtue of Th. 4.2, to satisfy the relation $\mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{G}'(\mathfrak{a}) = \mathfrak{B}(A)$. Hence there exist ideals $\mathfrak{b}$ and $\mathfrak{c}$ in A such that $\mathfrak{G}(\mathfrak{b}) = \mathfrak{G}_1$, $\mathfrak{G}(\mathfrak{c}) = \mathfrak{G}_2$. The relations $\mathfrak{G}_1 \cup \mathfrak{G}_2 = \mathfrak{G}(\mathfrak{a})$ and $\mathfrak{G}_1\mathfrak{G}_2 = 0$ imply that $\mathfrak{a} = \mathfrak{b} \vee \mathfrak{c}$, $\mathfrak{b}\mathfrak{c} = \mathfrak{o}$. Since $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are regular open sets by Th. 4.2, the ideals $\mathfrak{b}$ and $\mathfrak{c}$ are normal. Since $\mathfrak{G}'(\mathfrak{a}) \neq 0$, we have $\mathfrak{a} \neq \mathfrak{e}$; and since

$$\mathfrak{G}^{-\prime}(\mathfrak{a}) = (\mathfrak{G}_1 \cup \mathfrak{G}_2)^{-\prime} = (\mathfrak{G}_1^- \cup \mathfrak{G}_2^-)' = 0$$

in accordance with Th. 4.2, we see also that $\mathfrak{a}' = \mathfrak{o}$. Hence $\mathfrak{a}$ is a barrier ideal. We may remark that if $\mathfrak{F}$ is any barrier in $\mathfrak{B}(A)$, then the open set $\mathfrak{F}'$ is the representative of a barrier ideal in A.

As a consequence of Ths. 3.2 and 4.6 we immediately have the following result, the proof of which is obvious:

Theorem 4.7. *The following properties of a Boolean space $\mathfrak{B}$ are equivalent:*

(1) *there exists no barrier in $\mathfrak{B}$;*

(2) *$\mathfrak{B}$ is the representative space for a totally multiplicative Boolean ring.*

It is plain that the presence of a barrier in a T_0-space $\mathfrak{R}$ indicates some sort of connectedness for $\mathfrak{R}$. Accordingly, Th. 4.7 proves that the Boolean spaces, except for those which are the representatives of totally multiplicative Boolean rings, still show traces of connectedness in spite of the fact that they are totally disconnected in the usual sense (given any two distinct points of the space, there exist disjoint open sets, one about each point, of which the entire space is the union). In consequence a certain interest attaches to the problem of determining the T_0-spaces which exhibit the extreme of disconnectedness implied by the absence of barriers. It is easily seen that an H-space without barriers is totally disconnected in the usual sense; and the methods of our paper A appear sufficient to characterize the semiregular H-spaces without barrier, topologically, as the dense subsets of bicompact Boolean spaces without barrier.

The preceding results throw a certain amount of light on the facts developed in § 3. Thus Ths. 3.3, 3.4, and 3.5 follow directly from the corresponding Ths. 4.3, 4.4, and 4.5 with the help of Th. 4.6. Furthermore the behavior of prime ideals can be deduced from Th. 4.6, if we recall that an open set in $\mathfrak{B}(A)$ represents a prime ideal in A if and only if its complement consists of a single point. If the removal of this point divides $\mathfrak{B}(A)$ into two disjoint non-void sets open in $\mathfrak{B}(A)$ which have it as their common point of accumulation, then the corresponding prime ideal is a barrier ideal. If the removal of this point does not so divide $\mathfrak{B}(A)$, then the corresponding prime ideal is not a barrier ideal. Thus we see that the rings of type (ω) have representative Boolean spaces which are obtained by the removal of single points of the latter kind from bicompact Boolean spaces; and are characterized by this property. We see also that the rings of type (ω, ω) arise similarly by removal of two such points.

§ 5. Algebraic Characterizations of Finite Rings.

In this section we shall give some equivalent algebraic characterizations of the finite Boolean rings.

We first prove a theorem about infinite rings.

Theorem 5.1. *The following assertions concerning a Boolean ring A are equivalent:*

(1) *A is infinite — that is, A contains a sequence $\{a_n\}$ such that $a_m \neq a_n$ for arbitrarily great m and n;*

(2) *A contains a sequence $\{b_n\}$ such that $b_n < b_{n+1}$, $b_n \neq b_{n+1}$ for $n = 1, 2, 3, \ldots$;*

(3) *A contains a sequence $\{c_n\}$ such that $c_m c_n = 0$ for $m \neq n$, $c_n \neq 0$, $m, n = 1, 2, 3, \ldots$.*

It is evident that (1) follows from (2) or (3). It is easily seen that (2) follows from (3): we have only to define $b_n = c_1 + \dots + c_n = c_1 \vee \dots \vee c_n$ to obtain the desired sequence. Now suppose that (1) implies (2) for the case of a ring with unit. We then deduce (3), and hence (2) also, for the general case in the following manner. Starting with an arbitrary infinite ring A we adjoin a unit, obtaining an infinite ring A^* which contains A as a prime ideal. This process can be carried out whether A happens to have a unit or not. Then, under our present hypothesis, A^* contains a sequence $\{b_n\}$ with the properties given in (2). If we define $c_n = b_{n+1} + b_n$ we have $c_n \neq 0$ and

$$c_m c_n = (b_{m+1} + b_m)(b_{n+1} + b_n) = b_{m+1} b_{n+1} + b_{m+1} b_n + b_m b_{n+1} + b_m b_n = \\ = b_{m+1} + b_{m+1} + b_m + b_m = 0$$

for $m+1 \leqslant n$ by virtue of the relations $b_m < b_{m+1} < b_n < b_{n+1}$. Consequently the sequence $\{c_n\}$ has the properties desired under (3); but its elements are elements of A^* rather than of A. We can show however that at most one member of the sequence $\{c_n\}$ fails to belong to A. If the element c_m is not in A, the relation $c_m c_n = 0$ for $m \neq n$ and the fact that A is a prime ideal together show that $c_n \epsilon A$ for $n \neq m$. Hence, by rejecting at most one member of the sequence $\{c_n\}$, we obtain a sequence in A which has all the properties desired under (3). Thus the truth of the theorem depends upon the deduction of (2) from (1) under the assumption that A has a unit. We proceed now to this remaining step, constructing the desired sequence $\{b_n\}$ from the given sequence $\{a_n\}$.

The given sequence has the property (P) of containing infinitely many unequal elements. In our construction we shall make repeated use of the following principle: if c is any element of a ring with unit and $\{d_n\}$ is any sequence with the property (P), then at least one of the two sequences $\{cd_n\}$, $\{c'd_n\}$ has the property (P). The proof of this principle is obvious: if $cd_n = cd_N$ and $c'd_n = c'd_N$ for $n \geqslant N$, then $d_n = cd_n \vee c'd_n = cd_N \vee c'd_N = d_N$ for $n \geqslant N$, contrary to hypothesis. Thus, starting with the sequence $\{a_n\}$ we see that at least one of the sequences $\{a_1 a_n\}$, $\{a_1' a_n\}$ has the property (P). If both have the property (P), we define $b_1 = a_1$; if one fails to have the property (P), we define b_1 as equal to the associated element a_1 or a_1'. Suppose now that $b_1, \dots, b_k$ have been determined so that:

(α) $b_1, \dots, b_k$ are unequal;

(β) $b_1 < b_2 < \dots < b_{k-1} < b_k$;

(γ) the sequence $\{b_k' a_n\}$ has the property (P).

By (γ) there exists a least integer l such that $b_k' a_l \neq 0$, $b_k' a_l \neq b_k'$. At least one of the sequences $\{a_l b_k' a_n\}$, $\{a_l' b_k' a_n\}$ has the property (P). If both do, we define $c_k = a_l b_k'$; and if one fails to have the property (P), we define c_k as equal to the associated element $a_l b_k'$ or $a_l' b_k'$. We then define $b_{k+1} = b_k \vee c_k$. Since $b_k a_l b_k' = b_k a_l' b_k' = 0$, we have $b_k c_k = 0$. Moreover $c_k \neq 0$: for $a_l b_k' \neq 0$ by our determination of l; and $a_l' b_k' = 0$ would imply $a_l b_k' = a_l b_k' \vee a_l' b_k' = b_k'$ contrary to our determination of l. It is thus obvious that the properties (α) and (β) above can be extended to the elements $b_1, \dots, b_{k+1}$: they are unequal and satisfy the relations $b_1 < b_2 < \dots < b_k < b_{k+1}$. The property ($\gamma$) can also be extended to b_{k+1}: the sequence $\{b_{k+1} a_n\}$ has the property (P). If $c_k = a_l b_k'$, we have $b_{k+1} a_n b^k = c_k' a^n = b_k'(a_l' \vee b_k) a_n = a_l' b_k' a_n$; and, if $c_k = a_l' b_k'$, we have similarly $b_{k+1}' a_n = a_l b_k a_n$. By virtue of our choice of c_k, we see that the sequence $(b_{k+1}' a_n)$ has the property (P). The principle of mathematical induction therefore establishes the existence of the $\{b_n\}$ desired under (2); and our proof is brought to a close.

With the aid of this theorem it is easy to characterize the finite Boolean rings.

Theorem 5.2. *A Boolean ring A is finite if and only if it has one of the following four equivalent properties:*

(1) *A is a ring with unit in which every ideal is normal;*

(2) *every ideal in A is principal;*

(3) *A has a finite atomic basis or consists of the element* 0 *alone;*

(4) *A is isomorphic to the Boolean ring of all subclasses of a fixed finite class.*

The equivalence of (1) and (2) follows at once from R Ths. 23, 24, 25. The equivalence of (3), (4), and the property of finiteness follows from R Ths. 11, 12, 13. If A is finite every ideal in A is a finite Boolean ring and hence has a unit by R Th. 1; and hence every ideal in A is principal. On the other hand (2) implies that A is finite. If A were not finite, there would exist a sequence $\{b_n\}$ in A of the kind described in Th. 5.1 (2). The ideal generated by the class consisting of all elements of $\{b_n\}$ would then be a principal ideal $\mathfrak{a}(a)$, by hypothesis. Obviously, we would then have $b_n < a$ for $n = 1, 2, 3, \dots$; and, on the other hand, R Th. 17 shows that for some integer k we must have $a < b_1 \vee \dots \vee b_k = b_k$. We thus reach the contradiction that $b_n < a < b_k$ for $n > k$. With this the proof of the theorem is complete.

We shall consider briefly the part played by the divisor chain condition ("Teilerkettensatz") and by the maximal and minimal conditions on ideals. The great emphasis which has been laid upon such conditions in the general theory of rings justifies an explicit formulation of the facts in the special case before us.

Theorem 5.3. *In order that a Boolean ring A be finite, each of the following four assertions is both necessary and sufficient:*

(1) *A satisfies the ascending chain condition;*
(2) *A satisfies the maximal condition;*
(3) *A satisfies the descending chain condition;*
(4) *A satisfies the minimal condition* [15]).

It is obvious that a finite Boolean ring has properties (1), (2), (3), (4) since its ideals are finite in number. We can show at once that (1) and (3) separately imply the finiteness of A. If A were infinite, then Th. 5.1 (2) would provide us with a sequence $\{b_n\}$ such that $b_n < b_{n+1}, b_n \neq b_{n+1}$. The corresponding sequences $\{\mathfrak{a}(b_n)\}$ and $\{\mathfrak{a}'(b_n)\}$ would then have the properties:

$$\mathfrak{a}(b_n) \subset \mathfrak{a}(b_{n+1}), \qquad \mathfrak{a}(b_n) \neq \mathfrak{a}(b_{n+1}), \qquad n = 1, 2, \dots,$$
$$\mathfrak{a}'(b_n) \supset \mathfrak{a}'(b_{n+1}), \qquad \mathfrak{a}'(b_n) \neq \mathfrak{a}'(b_{n+1}), \qquad n = 1, 2, \dots$$

These properties contradict (1) and (3) respectively. Since (2) implies (1) and (4) implies (3), in obvious ways, our proof is complete.

§ 6. Algebraic Characterizations of Boolean Rings of Types (α), (β_1), (β_2), (β_3). In the present section we shall give characterizations of the indicated types of ring, including some already obtained in earlier sections and some new ones.

We shall begin with the rings of type (β_3), finding

Theorem 6.1. *A Boolean ring A is of type (β_3) if and only if it has one of the following four equivalent properties:*

(1) *A is a ring without unit in which every ideal is normal;*
(2) *A is a ring without unit in which every ideal is simple;*
(3) *A has an infinite atomic basis;*
(4) *A is isomorphic to the Boolean ring of all finite subclasses of a fixed infinite class.*

[15]) For definitions of the terms used, see B. L. van der Waerden ,*Moderne Algebra* II (Berlin 1931), pp. 23—30, 151—152.

In a Boolean ring of type (β_3) there exist simple ideals which are not semiprincipal.

The equivalence of (1) and (2) follows at once from R Ths. 23 and 24. The equivalence of (3) and (4) is proved in R Th. 11. A ring of type (β_3) has property (3) by Th. 2.6. By the same theorem, a ring with property (3) is of type (β_3). R Th. 26 enables us to deduce (2) from (3). For, if $\mathfrak{a}$ is any ideal in A and $\mathfrak{a}(a)$ any principal ideal, we see that $\mathfrak{a}(a)$ is a finite ring generated by the atomic elements b, finite in number, such that $ba \neq 0$; and hence that the ideal $\mathfrak{a}\mathfrak{a}(a)$, being a finite ring, has a unit and is principal. On the other hand, we can deduce (3) from (2). If $\mathfrak{a}(a)$ is any principal ideal in A and $\mathfrak{a}$ any ideal in $\mathfrak{a}(a)$, then $\mathfrak{a}$, considered as an ideal in A, must be simple by (2). Th. 1.1 shows that $\mathfrak{a}$ is principal in A and hence in $\mathfrak{a}(a)$ also. Thus Th. 6.1 shows that $\mathfrak{a}(a)$ has a finite atomic basis. It follows that every element a in A is a finite sum of atomic elements in A. Hence A has an atomic basis. If the basis were finite, A would be finite by Th. 6.1 and would have a unit contrary to (2). Hence A has an infinite atomic basis, as we wished to prove. In the demonstration of Th. 2.6 we have already shown that a ring of type (β_3) contains a non-semiprincipal simple ideal.

Theorem 6.2. *A Boolean ring A is of type (α) if and only if* $\mathfrak{P} = \mathfrak{P}^* = \mathfrak{S} = \mathfrak{N} \neq \mathfrak{J}$.

By Th. 2.2 (2), the totally additive Boolean rings are characterized by the equation $\mathfrak{P} = \mathfrak{N}$. By Th. 5.2, the finite rings among these are characterized by the further equation $\mathfrak{N} = \mathfrak{J}$. Hence the rings of type (α) are characterized by the relations $\mathfrak{P} = \mathfrak{N} \neq \mathfrak{J}$. The relations $\mathfrak{P} = \mathfrak{P}^* = \mathfrak{S} = \mathfrak{N}$ then follow in accordance with R Th. 23.

Theorem 6.3. *A Boolean ring A is of type (β_1) if and only if* $\mathfrak{P} \neq \mathfrak{P}^* = \mathfrak{S} = \mathfrak{N} \neq \mathfrak{J}$.

In Th. 2.6 we proved that A is of type (β_1) if and only if $\mathfrak{P} \neq \mathfrak{P}^* = \mathfrak{S} = \mathfrak{N}$. We must show that these relations imply $\mathfrak{N} \neq \mathfrak{J}$. By reference to Ths. 5.2 and 6.1 we see that the relation $\mathfrak{N} = \mathfrak{J}$ would imply either $\mathfrak{P} = \mathfrak{J}$ or $\mathfrak{P}^* \neq \mathfrak{S}$; and it therefore follows that $\mathfrak{N} \neq \mathfrak{J}$ in the present case.

Theorem 6.4. *A Boolean ring A is of type (β_2) if and only if* $\mathfrak{P} \neq \mathfrak{P}^* \neq \mathfrak{S} = \mathfrak{N} \neq \mathfrak{J}$.

By Th. 2.3 (2) the totally multiplicative rings are characterized by the equation $\mathfrak{S}=\mathfrak{N}$. The totally multiplicative rings are all distributed among the finite rings, and the rings of distinct types (α), (β_1), (β_2), (β_3). Since among these, the finite rings and the rings of type (β_3) are characterized by the equation $\mathfrak{N}=\mathfrak{J}$ in accordance with Th. 5.2 and 6.1, the rings belonging to one of the three types (α), (β_1), (β_2), are characterized by the relations $\mathfrak{S}=\mathfrak{N}\neq\mathfrak{J}$. According to Th. 6.2 and 6.3 the relations $\mathfrak{P}^*=\mathfrak{S}=\mathfrak{N}\neq\mathfrak{J}$ characterize the rings belonging to one of the types (α), (β_1). Hence the rings of type (β_2) are characterized by the relations $\mathfrak{P}^*\neq\mathfrak{S}=N\neq\mathfrak{J}$; and these relations imply that $\mathfrak{P}\neq\mathfrak{P}^*$ by R Th. 25. With this the proof of the present theorem is complete.

§ 7. Algebraic Characterizations of Boolean Rings of Types (β_2^*), (β_3^*), (β_2^*,β_1), (β_2^*,β_2), (β_3^*,β_3). In this section we continue the studies of §§ 5, 6, 7, obtaining new characterizations of some of the indicated types, either individually or in groups.

We begin with a discussion of types (β_3^*) and (β_3^*,β_3).

Theorem 7.1. *The Boolean rings of types (β_3^*) and (β_3^*,β_3) are collectively characterized as those rings A with $\mathfrak{J}\neq\mathfrak{S}$ possessing one of the following four equivalent properties:*

(1) *there exists a prime ideal $\mathfrak{p}$ in A such that $\mathfrak{p}\subset\mathfrak{a}\vee\mathfrak{a}'$ whatever the ideal $\mathfrak{a}$;*

(2) *there exists a prime ideal $\mathfrak{q}$ in A such that every ideal contained in $\mathfrak{q}$ is simple relative to $\mathfrak{q}$;*

(3) *there exists a non-void class $\mathfrak{s}$ of atomic elements in A such that the ideal $\mathfrak{a}(\mathfrak{s})$ is prime;*

(4) *the ring A is isomorphic to the Boolean ring generated by all the finite subclasses of a fixed infinite class E and a single infinite subclass Γ of E.*

In such a ring, the relations $\mathfrak{p}=\mathfrak{q}=\mathfrak{a}(\mathfrak{s})$ hold, $\mathfrak{s}$ is a complete atomic system, and the ideal $\mathfrak{p}$ is not normal; and the ring A belongs to type (β_3^*) or (β_3^*,β_3) according as A has or has not a unit. In the representation given by (4), the ring A has a unit if and only if the class $E\mathbin{\Delta}\Gamma$ is finite; and in that case is characterized by the cardinal number of E. When A has no unit, it is characterized by the cardinal numbers of the two classes $E\mathbin{\Delta}\Gamma$ and Γ, both infinite. In a ring of type (β_3^*), the relations $\mathfrak{P}=\mathfrak{P}^*=\mathfrak{S}\neq\mathfrak{N}\neq\mathfrak{J}$ are valid; and in a ring of type (β_3^*,β_3), the relations $\mathfrak{P}\neq\mathfrak{P}^*\neq\mathfrak{S}\neq\mathfrak{N}\neq\mathfrak{J}$.

We may remark that in a ring with $\mathfrak{S}=\mathfrak{J}$, the properties (1) and (2) hold for arbitrary prime ideals $\mathfrak{p}$ and $\mathfrak{q}$, as we see by reference to Ths. 5.2 and 6.1. Similarly, in a finite ring or a ring of type (β_3), we can satisfy condition (3) by defining $\mathfrak{s}$ through the suppression of exactly one element of the atomic basis. Thus the condition $\mathfrak{J}\neq\mathfrak{S}$ is essential in connection with (1), (2), and (3), if we wish to eliminate types already discussed. Condition (4), on the other hand, implies $\mathfrak{J}\neq\mathfrak{S}$ as we shall see; hence the condition $\mathfrak{J}\neq\mathfrak{S}$ is superfluous so far as (4) is concerned. By Th. 2.8, we know that the rings of type (β_3^*) or type (β_3^*,β_3) are not totally multiplicative. Thus we have for them the relations $\mathfrak{N}\neq\mathfrak{S}$ and hence $\mathfrak{S}\neq\mathfrak{J}$, $\mathfrak{N}\neq\mathfrak{J}$. Since a ring of type (β_3^*) has a unit, the relations $\mathfrak{P}=\mathfrak{P}^*=\mathfrak{S}\neq\mathfrak{N}\neq\mathfrak{J}$ must hold for it. In a ring of type (β_3^*,β_3), we have $\mathfrak{P}\neq\mathfrak{P}^*$ and $\mathfrak{S}\neq\mathfrak{N}\neq\mathfrak{J}$; and we shall see presently that $\mathfrak{P}^*\neq\mathfrak{S}$ also.

We first show that (1), (2), and (3) are equivalent and that the condition $\mathfrak{J}\neq\mathfrak{S}$ implies the uniqueness and equality of the ideals $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{a}(\mathfrak{s})$. To prove that (1) implies (2), we consider an arbitrary ideal $\mathfrak{a}$ in the ring $\mathfrak{p}$. Since $\mathfrak{a}$ is then an ideal in A and the relation $\mathfrak{a}\vee\mathfrak{a}'\supset\mathfrak{p}$ is valid by (1), we have $\mathfrak{a}\vee\mathfrak{a}'\mathfrak{p}=(\mathfrak{a}\vee\mathfrak{a}')\mathfrak{p}=\mathfrak{p}$. Thus $\mathfrak{a}$ is simple relative to $\mathfrak{p}$, its orthocomplement relative to $\mathfrak{p}$ being the ideal $\mathfrak{a}'\mathfrak{p}$. Accordingly, (1) implies (2) with $\mathfrak{q}=\mathfrak{p}$. Next (2) implies (3) with $\mathfrak{a}(\mathfrak{s})=\mathfrak{q}$. For Th. 6.1 shows that $\mathfrak{q}$ has an atomic basis $\mathfrak{s}$; and the ideal $\mathfrak{a}(\mathfrak{s})$ generated by $\mathfrak{s}$, considered as an atomic system in A, obviously coincides with $\mathfrak{q}$. Now (3) implies (1) with $\mathfrak{p}=\mathfrak{a}(\mathfrak{s})$. The ideal generated by a non-void class of atomic elements $\mathfrak{s}$ is easily seen to consist of all finite sums of elements of $\mathfrak{s}$, in accordance with R Def. 3 and R Th. 17. We know from Ths. 5.2 and 6.1 that every ideal in $\mathfrak{a}(\mathfrak{s})$ is simple relative to $\mathfrak{a}(\mathfrak{s})$. Hence, if $\mathfrak{a}$ is any ideal in A, we have $\mathfrak{a}\mathfrak{a}(\mathfrak{s})\vee\mathfrak{a}'\mathfrak{a}(\mathfrak{s})=\mathfrak{a}(\mathfrak{s})$ or, equivalently, $\mathfrak{a}\vee\mathfrak{a}'\supset\mathfrak{a}(\mathfrak{s})$. Consequently (1) holds for $\mathfrak{p}=\mathfrak{a}(\mathfrak{s})$. Our discussion shows that, if any one of the ideals $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{a}(\mathfrak{s})$ is not uniquely determined, then the others are not. Now the condition $\mathfrak{J}\neq\mathfrak{S}$ implies that the ideal $\mathfrak{p}$ of (1), and hence also the ideals $\mathfrak{q}$ and $\mathfrak{a}(\mathfrak{s})$, is uniquely determined. We note first that the ideal $\mathfrak{p}$ must be non-normal under the present circumstances: for there exists an ideal $\mathfrak{a}$ with $\mathfrak{a}\vee\mathfrak{a}'\neq\mathfrak{e}$. The relation $\mathfrak{a}\vee\mathfrak{a}'\supset\mathfrak{p}$ therefore implies $\mathfrak{p}=\mathfrak{a}\vee\mathfrak{a}'$, $\mathfrak{p}'=\mathfrak{a}'\mathfrak{a}''=\mathfrak{o}$. According to R Th. 38, the prime ideal $\mathfrak{p}$ is not normal. Thus if $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are two ideals with the properties required in (1), we have $\mathfrak{p}_1\subset\mathfrak{p}_2\vee\mathfrak{p}_2'=\mathfrak{p}_2$, $\mathfrak{p}_2\subset\mathfrak{p}_1\vee\mathfrak{p}_1'=\mathfrak{p}_1$, and $\mathfrak{p}_1=\mathfrak{p}_2$. We conclude therefore that $\mathfrak{J}\neq\mathfrak{S}$ implies the uniqueness and also the equality of $\mathfrak{p}$, $\mathfrak{q}$, and $\mathfrak{a}(\mathfrak{s})$.

In particular, we have shown incidentally that $\mathfrak{s}$ is an atomic basis for $\mathfrak{a}(\mathfrak{s})$ and that the ideals $\mathfrak{p}=\mathfrak{q}=\mathfrak{a}(\mathfrak{s})$ are non-normal when $\mathfrak{J}\neq\mathfrak{S}$.

We prove next that a ring is of type (β_3^*) or (β_3^*,β_3) if and only if it is a ring with $\mathfrak{J}\neq\mathfrak{S}$ satisfying the equivalent conditions (1), (2), and (3). By Ths. 2.8—2.10 a ring of type (β_3^*) or (β_3^*,β_3) is representable as a ring A containing a prime ideal $\mathfrak{a}$ with $\mathfrak{a}'=\mathfrak{o}$ which is of type (β_3). According to Th. 6.1, the ideal $\mathfrak{a}$ is then a prime ideal with the properties demanded of $\mathfrak{q}$ in (2). We have already observed that a ring of type (β_3^*) or (β_3^*,β_3) satisfies the condition $\mathfrak{J}\neq\mathfrak{S}$. On the other hand, if a ring A with $\mathfrak{J}\neq\mathfrak{S}$ contains an ideal $\mathfrak{q}$ with the properties demanded in (2), the fact that $\mathfrak{q}$ is not normal shows that $\mathfrak{q}$ has no unit and is of type (β_3) in accordance with Th. 6.1. Thus A must be either a totally multiplicative ring or a ring of one of the types (β_3^*), (β_3^*,β_3), by virtue of Ths. 2.8—2.10. If A were totally multiplicative, then we could obtain a contradiction as follows. Let a be an element of A not in $\mathfrak{q}$. Then the principal ideal $\mathfrak{a}(a)$ would be totally additive by Th. 2.3; and $\mathfrak{a}(a)\mathfrak{q}$ would be a non-normal prime ideal relative to $\mathfrak{a}(a)$. Thus $\mathfrak{a}(a)\mathfrak{q}$ would be a ring of type (β_1). On the other hand $\mathfrak{q}$ has an atomic basis by the preceding results. It follows that $\mathfrak{a}(a)\mathfrak{q}$ has an atomic basis likewise, as an ideal in $\mathfrak{q}$. Thus $\mathfrak{a}(a)\mathfrak{q}$ is of type (β_3): it is a ring without unit, being a non-normal prime ideal in $\mathfrak{a}(a)$, and has an atomic basis. Since the types (β_1) and (β_3) are distinct, we have the desired contradiction. We see therefore that A is of type (β_3^*) or of type (β_3^*,β_3).

The characteristic representation described in (4) is now easily established. First, we shall prove that, when $\mathfrak{J}\neq\mathfrak{S}$, (3) implies (4). From what has been proved already, we know that $\mathfrak{a}(\mathfrak{s})$ is a non-normal prime ideal in A with $\mathfrak{s}$ as an atomic basis. In particular we have $\mathfrak{a}'(\mathfrak{s})=\mathfrak{o}$. Since $\mathfrak{a}(\mathfrak{s})$ is a totally multiplicative ring of type (β_3), Ths. 2.5 and 6.1 show that $\mathfrak{a}(\mathfrak{s})$ is isomorphic to the ring of all finite subclasses of a fixed infinite class E; and that A is isomorphic to the subring of the totally additive ring of all subclasses of E generated by a fixed subclass Γ and the finite subclasses. Conversely, we show that (4) implies (3). A ring A of the kind described in (4) is obtained by the process analyzed in Th. 2.5: in the totally additive ring of all subclasses of E, the one-element subclasses constitute a complete atomic system $\mathfrak{s}$; the ideal $\mathfrak{a}(\mathfrak{s})$ generated by $\mathfrak{s}$ is the system of all finite subclasses of E and obviously has the property that $\mathfrak{a}'(\mathfrak{s})=\mathfrak{o}$; and A arises as the subring generated by Γ and $\mathfrak{a}(\mathfrak{s})$. Hence we see that $\mathfrak{a}(\mathfrak{s})$ is a non-normal prime ideal in A

and that it is generated by the atomic system $\mathfrak{s}$, regarded now as a subclass of A. In view of the fact that $\mathfrak{a}(\mathfrak{s})$ is non-normal, we have $\mathfrak{J}\neq\mathfrak{N}$, $\mathfrak{J}\neq\mathfrak{S}$. Thus A is a ring with $\mathfrak{J}\neq\mathfrak{S}$ satisfying (3). Incidentally, the equivalence just shown for (3) and (4) implies that, when $\mathfrak{J}\neq\mathfrak{S}$, the atomic system $\mathfrak{s}$ in A is complete and serves as an atomic basis for $\mathfrak{a}(\mathfrak{s})$.

It is obvious that a ring A represented in the form (4) has a unit if and only if the class E is a member of it; but E is a member if and only if $E\equiv\Gamma \pmod{\mathfrak{a}(\mathfrak{s})}$ or, equivalently, if and only if $E\Delta\Gamma$ is finite. Since such a ring is generated by E and its finite subclasses, it is completely characterized by the cardinal number of E. The theory of cardinal numbers shows that E and A have the same infinite cardinal number. In order that A should have no unit, it is necessary and sufficient that $E\Delta\Gamma$ be infinite. In this case, the theory of cardinal numbers shows that A is characterized by the infinite cardinal numbers of Γ and $E\Delta\Gamma$; and also that A and E have the same cardinal number, equal to the greater of the cardinal numbers of Γ and $E\Delta\Gamma$.

It is trivial that, under the foregoing conditions, a ring A belongs to type (β_3^*) or to type (β_3^*,β_3) according as it has a unit or not.

We now wish to show that in a ring A of type (β_3^*,β_3), the relation $\mathfrak{P}^*\neq\mathfrak{S}$ is valid. Since such a ring is representable in the form $A_1^*\vee A_2$ where A_1 and A_2 are of type (β_3), and since A_2 can be represented in the form $A_3\vee A_4$ where A_3 and A_4 are without unit, by virtue of the fact that A_2 contains simple ideals which are not semiprincipal; — we see that A is represented as a direct sum of the rings $A_1^*\vee A_3$ and A_4, neither of which has a unit. By R Th. 51, the two summands in this representation are non-semiprincipal simple ideals in A.

If we make use of the existence and divisibility properties of prime ideals, we can add to the list of equivalent properties set forth in the preceding theorem. We have:

Theorem 7.2. *In a Boolean ring A with $\mathfrak{J}\neq\mathfrak{S}$, the following properties are equivalent:*

(1) *there exists a prime ideal $\mathfrak{p}$ in A, such that $\mathfrak{p}\subset\mathfrak{a}\vee\mathfrak{a}'$, whatever the ideal $\mathfrak{a}$;*

(2) *there exists a prime ideal $\mathfrak{q}$ in A such that $\mathfrak{q}\subset\mathfrak{a}''\vee\mathfrak{a}'$, whatever the ideal $\mathfrak{a}$, and $\mathfrak{q}\supset\mathfrak{a}\vee\mathfrak{a}'$, whatever the non-simple ideal $\mathfrak{a}$;*

(3) *there exists exactly one non-normal prime ideal $\mathfrak{r}$ in A;*

(4) *the sum of all non-simple ideals in A is a prime ideal $\mathfrak{s}$.*

In such a ring A, the relations $\mathfrak{p}=\mathfrak{q}=\mathfrak{r}=\mathfrak{s}$ are valid.

It is evident that (1) implies (2) with $\mathfrak{q}=\mathfrak{p}$: for $\mathfrak{a}''\vee\mathfrak{a}'\supset\mathfrak{p}$, whatever the ideal $\mathfrak{a}$; and the relations $\mathfrak{a}\vee\mathfrak{a}'\neq\mathfrak{e}$, $\mathfrak{a}\vee\mathfrak{a}'\supset\mathfrak{p}$, holding for any non-simple ideal $\mathfrak{a}$, imply $\mathfrak{p}=\mathfrak{a}\vee\mathfrak{a}'$ and hence $\mathfrak{p}\supset\mathfrak{a}\vee\mathfrak{a}'$. It is also easily seen that (2) implies (3) with $\mathfrak{r}=\mathfrak{q}$. If $\mathfrak{r}$ is any non-normal prime ideal, we have $\mathfrak{r}'=\mathfrak{o}$, $\mathfrak{r}=\mathfrak{r}\vee\mathfrak{r}'\subset\mathfrak{q}\neq\mathfrak{e}$, and hence $\mathfrak{r}=\mathfrak{q}$. Secondly, $\mathfrak{q}$ is not normal: if it were, every prime ideal in A would be normal; hence every ideal, being the product of its prime ideal divisors (unless it is the normal ideal $\mathfrak{e}$), would be normal in A, by virtue of $\mathfrak{R}$ Ths. 29 and 66; and hence every ideal in A would be simple in accordance with $\mathfrak{R}$ Th. 24. We assumed, however, that $\mathfrak{J}\neq\mathfrak{S}$. Next we show that (3) implies (4) with $\mathfrak{s}=\mathfrak{r}$. Since the ideal $\mathfrak{r}$ is not normal and hence not simple, we see that $\mathfrak{s}$, the sum of all non-simple ideals in A, contains $\mathfrak{r}$. If $\mathfrak{a}$ is any non-simple, ideal, then the ideal $\mathfrak{a}\vee\mathfrak{a}'$ is not normal since $(\mathfrak{a}\vee\mathfrak{a}')''=(\mathfrak{a}'\mathfrak{a}'')'=$ $=\mathfrak{o}'=\mathfrak{e}\neq\mathfrak{a}\vee\mathfrak{a}'$; and if $\mathfrak{t}$ is any prime ideal divisor of $\mathfrak{a}\vee\mathfrak{a}'$, the relation $\mathfrak{a}\vee\mathfrak{a}'\subset\mathfrak{t}$ implies $\mathfrak{t}'\subset(\mathfrak{a}\vee\mathfrak{a}')'=\mathfrak{o}$, $\mathfrak{t}'=\mathfrak{o}$, so that $\mathfrak{t}$ is non-normal and must coincide with $\mathfrak{r}$. By $\mathfrak{R}$ Th. 66, we conclude that $\mathfrak{a}\subset\mathfrak{a}\vee\mathfrak{a}'=\mathfrak{r}$. Thus we find that $\mathfrak{s}\subset\mathfrak{r}$, and hence that $\mathfrak{s}=\mathfrak{r}$. Finally, we show that (4) implies (1) with $\mathfrak{p}=\mathfrak{s}$. If $\mathfrak{a}$ is any simple ideal we have $\mathfrak{a}\vee\mathfrak{a}'=$ $=\mathfrak{e}\supset\mathfrak{s}$. On the other hand, if $\mathfrak{a}$ is any non-simple ideal, we know that $\mathfrak{a}\vee\mathfrak{a}'$ is not normal. It follows that $\mathfrak{a}\vee\mathfrak{a}'\subset\mathfrak{s}$. If $\mathfrak{t}$ is any prime ideal divisor of $\mathfrak{a}\vee\mathfrak{a}'$, we have $\mathfrak{t}'=\mathfrak{o}$ so that $\mathfrak{t}$ is not normal. It follows that $\mathfrak{t}\subset\mathfrak{s}\neq\mathfrak{e}$ and hence that $\mathfrak{t}=\mathfrak{s}$. $\mathfrak{R}$ Th. 66 now shows that $\mathfrak{a}\vee\mathfrak{a}'=\mathfrak{s}$. We conclude that $\mathfrak{a}\vee\mathfrak{a}'\supset\mathfrak{s}$, whatever the ideal $\mathfrak{a}$ in A. Since the ideals $\mathfrak{r}$ and $\mathfrak{s}$ of (3) and (4) are uniquely determined, our argument shows that $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$, $\mathfrak{s}$ are unique and equal.

We proceed now with the discussion of the remaining types.

Theorem 7.3. *The Boolean rings of types* (β_2^*), (β_2,β_1) *and* (β_2^*,β_2) *are characterized as those rings* A *with* $\mathfrak{N}\neq\mathfrak{S}$, *possessing one of the following equivalent properties*:

(1) *there exists a prime ideal* $\mathfrak{p}$ *in* A *such that* $\mathfrak{p}\subset\mathfrak{a}''\vee\mathfrak{a}'$, *whatever the ideal* $\mathfrak{a}$, *while for some ideal* $\mathfrak{a}$ *the relation* $\mathfrak{p}\subset\mathfrak{a}\vee\mathfrak{a}'$ *is false*;

(2) *there exists a prime ideal* $\mathfrak{q}$ *in* A *such that every ideal in* $\mathfrak{q}$ *which is normal relative to* $\mathfrak{q}$ *is simple relative to* $\mathfrak{q}$ *while some ideal in* $\mathfrak{q}$ *is not normal relative to* $\mathfrak{q}$.

The ideals $\mathfrak{p}$ *and* $\mathfrak{q}$ *of* (1) *and* (2) *respectively are unique and equal. Among the rings of the kind described by these various equivalent properties, those of type* (β_2^*) *are characterized by the presence of a unit, and those of type* (β_2^*,β_1) *by membership in the type* (ω).

We begin by showing that (1) implies (2) with $\mathfrak{q}=\mathfrak{p}$. If $\mathfrak{a}$ is any ideal in $\mathfrak{p}$ normal relative to $\mathfrak{p}$, then

$$\mathfrak{a}=\mathfrak{a}''\mathfrak{p} \quad\text{and}\quad \mathfrak{a}\vee\mathfrak{a}'\mathfrak{p}=\mathfrak{a}''\mathfrak{p}\vee\mathfrak{a}'\mathfrak{p}=\mathfrak{p}.$$

Hence $\mathfrak{a}$ is simple relative to $\mathfrak{p}$. On the other hand, let $\mathfrak{a}$ be an ideal in A such that $\mathfrak{a}\vee\mathfrak{a}'$ does not contain $\mathfrak{p}$. Then the ideal $\mathfrak{a}\mathfrak{p}$, considered as an ideal in $\mathfrak{p}$ has the properties $\mathfrak{a}\mathfrak{p}\vee\mathfrak{a}'\mathfrak{p}\neq\mathfrak{p}$, $\mathfrak{a}''\mathfrak{p}\vee\mathfrak{a}'\mathfrak{p}=\mathfrak{p}$. Hence $\mathfrak{a}\mathfrak{p}\neq\mathfrak{a}''\mathfrak{p}$ and $\mathfrak{a}\mathfrak{p}$ is not normal relative to $\mathfrak{p}$. Likewise, (2) implies (1) with $\mathfrak{p}=\mathfrak{q}$. If $\mathfrak{a}$ is an arbitrary ideal in A, then $\mathfrak{a}\mathfrak{q}$ is an ideal in $\mathfrak{q}$ with orthocomplement $\mathfrak{a}'\mathfrak{q}$ in $\mathfrak{q}$. Since $\mathfrak{a}'\mathfrak{q}$ is normal relative to $\mathfrak{q}$, it is simple relative to $\mathfrak{q}$ so that $\mathfrak{a}''\vee\mathfrak{a}'\supset\mathfrak{a}'\mathfrak{q}\vee\mathfrak{a}''\mathfrak{q}=\mathfrak{q}$. On the other hand, if $\mathfrak{a}$ is a non-normal ideal relative to $\mathfrak{q}$, it is not simple relative to $\mathfrak{q}$ and $\mathfrak{a}\mathfrak{q}\vee\mathfrak{a}'\mathfrak{q}=\mathfrak{a}\vee\mathfrak{a}'\mathfrak{q}\neq\mathfrak{q}$. Thus $\mathfrak{a}$ is an ideal in A with the property that $\mathfrak{a}\vee\mathfrak{a}'$ does not contain $\mathfrak{q}$. Our argument shows that if either $\mathfrak{p}$ or $\mathfrak{q}$ is uniquely determined then the other is also, and $\mathfrak{p}$ and $\mathfrak{q}$ are equal. If $\mathfrak{N}\neq\mathfrak{S}$, we can show that $\mathfrak{p}$ is uniquely determined. In fact, let $\mathfrak{a}$ be any non-simple normal ideal in A. Then $\mathfrak{a}=\mathfrak{a}''$, $\mathfrak{a}\vee\mathfrak{a}'=\mathfrak{a}''\vee\mathfrak{a}'\supset\mathfrak{p}$, $\mathfrak{a}\vee\mathfrak{a}'\neq\mathfrak{e}$, so that $\mathfrak{a}\vee\mathfrak{a}'=\mathfrak{p}$. Our assertion is thus established. Incidentally, we see that the relation $\mathfrak{p}'=(\mathfrak{a}\vee\mathfrak{a}')'=\mathfrak{a}'\mathfrak{a}''=\mathfrak{o}$ implies that $\mathfrak{p}$, and hence $\mathfrak{q}$ also, is not normal.

If A is a ring of any of the types (β_2^*), (β_2^*,β_1), or (β_2^*,β_2), then A is representable in terms of a prime ideal $\mathfrak{a}$ with $\mathfrak{a}'\neq\mathfrak{o}$, where $\mathfrak{a}$ is totally multiplicative, and an element a_0 in accordance with Ths. 2.8-2.10; and this representation is essentially unique. Moreover A is not totally multiplicative, so that $\mathfrak{N}\neq\mathfrak{S}$ in accordance with Th. 2.3; and $\mathfrak{a}$ is of type (β_2) of or one of the special types (β_2,β_1), (β_2,β_2) included under (β_2) and hence is not of type (β_3). Th. 6.1 now shows that $\mathfrak{a}$ contains at least one non-normal ideal; and Th. 2.3 that every normal ideal in $\mathfrak{a}$ is simple relative to $\mathfrak{a}$. Thus we find that A is a ring with $\mathfrak{N}\neq\mathfrak{S}$ possessing property (2) with $\mathfrak{q}=\mathfrak{a}$. On the other hand, let A be a ring with $\mathfrak{N}\neq\mathfrak{S}$ possessing property (2). Then the ideal $\mathfrak{q}$ has the property that $\mathfrak{q}'=\mathfrak{o}$ as we noted above; and $\mathfrak{q}$ is a totally multiplicative ring in accordance with Th. 2.3. Since $\mathfrak{q}$ is prime in A and the relation $\mathfrak{N}\neq\mathfrak{S}$ implies that A is not totally multiplicative, we see that A must be of one of the types (β_2^*), (β_3^*), (β_2^*,β_1), (β_2^*,β_2), (β_3^*,β_3). Since the ideal is uniquely determined in A, by virtue of Th. 2.9, we see that Th. 7.1 (2) excludes the possibility of membership in either of the types (β_3^*), (β_3^*,β_3). Hence we see that A is one of the three types (β_2^*), (β_2^*,β_1), (β_2^*,β_2).

It is evident that among the rings belonging to these three types those of type (β_2^*) are characterized by the presence of a unit. Ths. 3.6 and 3.8 show that every ring of type (β_2^*, β_1), being the direct sum of a ring with unit and a ring of type (β_1), is of type (ω). On the other hand, we can show that a ring A of type (β_2^*) or of type (β_2^*, β_2) is not of type (ω). In the case in the first of these types, the presence of a unit shows at once that A is not of type (ω). In the other case, we express A as the direct sum $A_1^* \vee A_2$ of a ring with unit and a ring A_2 of type (β_2). Th. 2.6 shows that A_2 contains non-semiprincipal simple ideals. Hence, by virtue of R Th. 51, we can express A_2 as the direct sum $A_3 \vee A_3$ of rings without unit. Writing A as the direct sum $(A_1^* \vee A_3) \vee A_4$, by an exchange of direct summands, we see that neither of the new summands has a unit. Using R Th. 51, we see further that both the new summands are non-semiprincipal simple ideals in A. The relation $\mathfrak{S} \neq \mathfrak{P}^*$ now shows by Th. 3.6 that A is not of type (ω). With this the proof of the theorem is complete.

Again employing the existence and divisibility properties of prime ideals, we can add to the list of equivalent properties given in Th. 7.3. We have:

Theorem 7.4. *In a Boolean ring A with $\mathfrak{N} \neq \mathfrak{S}$, the following properties are equivalent:*

(1) *there exists a prime ideal $\mathfrak{p}$ in A such that $\mathfrak{p} \subset \mathfrak{a}'' \vee \mathfrak{a}'$, whatever the ideal $\mathfrak{a}$, while for some ideal $\mathfrak{a}$ the relation $\mathfrak{p} \subset \mathfrak{a} \vee \mathfrak{a}'$ is false;*

(2) *the sum of all non-normal ideals is the ideal $\mathfrak{e}$, the sum of all normal non-simple ideals is a prime ideal $\mathfrak{r}$.*

The ideals $\mathfrak{p}$ and $\mathfrak{r}$ in such a ring are unique and equal.

We first prove that (1) implies (2) with $\mathfrak{r} = \mathfrak{p}$. By hypothesis there exist non-simple normal ideals in A. If $\mathfrak{a}$ is such an ideal, we have $\mathfrak{p} \subset \mathfrak{a}'' \vee \mathfrak{a}' = \mathfrak{a} \vee \mathfrak{a}' \neq \mathfrak{e}$ and hence $\mathfrak{p} = \mathfrak{a} \vee \mathfrak{a}'$. Since all such ideals are contained in $\mathfrak{p}$, by the last relation, their sum also is contained in $\mathfrak{p}$. On the other hand, if $\mathfrak{a}$ is such an ideal, $\mathfrak{a}'$ is normal and the relation $\mathfrak{a} \vee \mathfrak{a}' = \mathfrak{p} \neq \mathfrak{e}$ shows that $\mathfrak{a}'$ is not simple. Since $\mathfrak{a} \vee \mathfrak{a}'$ is then contained in the sum of all non-simple normal ideals, we conclude that this sum coincides with $\mathfrak{p}$. Since $\mathfrak{J} \neq \mathfrak{S}$ follows from $\mathfrak{N} \neq \mathfrak{S}$, the assumption that there exists no non-normal prime ideal other than $\mathfrak{p}$ would imply that $\mathfrak{p} \subset \mathfrak{a} \vee \mathfrak{a}'$ for every ideal $\mathfrak{a}$, contrary to (1): for the relation $\mathfrak{p} = \mathfrak{a} \vee \mathfrak{a}'$ obtained above leads to the equation $\mathfrak{p}' = \mathfrak{a}'\mathfrak{a}'' = \mathfrak{o}$ and hence implies that $\mathfrak{p}$ is non-normal; and Th. 7.2 would then establish the desired contradiction. Hence we see that there exists a non-normal prime ideal $\mathfrak{q}$ distinct from $\mathfrak{p}$. It is now clear that the sum of all non-normal ideals in A contains $p \vee \mathfrak{q} = \mathfrak{e}$ and thus coincides with $\mathfrak{e}$.

On the other hand (2) implies (1) with $\mathfrak{p} = \mathfrak{r}$. Let a be any element of $\mathfrak{r}$ and let $\mathfrak{a}(a)$ be the principal ideal generated in A and in $\mathfrak{r}$ by a. If $\mathfrak{a}$ is any ideal in $\mathfrak{q}(a)$ normal relative to $\mathfrak{a}(a)$, then $\mathfrak{a}$ is normal in A. The relation $\mathfrak{a} \subset \mathfrak{a}(a)$ implies $\mathfrak{a}' \supset \mathfrak{a}'(a)$. R Th. 41 shows that $\mathfrak{a}'(a)$ is not contained in the prime ideal $\mathfrak{r}$. Hence $\mathfrak{a}'$ is not contained in $\mathfrak{r}$ and, being normal, must be simple by virtue of the definition of $\mathfrak{r}$. Hence $\mathfrak{a} = \mathfrak{a}''$ is also simple; but, being contained in the principal ideal $\mathfrak{a}(a)$, the ideal $\mathfrak{a}$ must even be principal by virtue of Th. 1.1. Since every normal ideal in $\mathfrak{a}(a)$ is principal, $\mathfrak{a}(a)$ is a totally additive ring by Th. 2.2 (2); and thus $\mathfrak{r}$ is a totally multiplicative ring by Th. 2.3 (3). Consequently every ideal normal relative to $\mathfrak{r}$ is simple relative to $\mathfrak{r}$ by Th. 2.3 (2). If now $\mathfrak{a}$ is an arbitrary ideal in A, then $\mathfrak{a}''$ is normal in A and $\mathfrak{a}''\mathfrak{r}$ is normal relative to $\mathfrak{r}$ in accordance with Th. 1.3. Hence we see that $\mathfrak{a}''\mathfrak{r}$ is simple relative to $\mathfrak{r}$ and conclude that $\mathfrak{a}'' \vee \mathfrak{a}' \supset \mathfrak{a}''\mathfrak{r} \vee \mathfrak{a}'\mathfrak{r} = \mathfrak{r}$. Now our assumption that the sum of all non-normal ideals in A coincides with $\mathfrak{e}$ shows that there exists at least one non-normal ideal $\mathfrak{a}$ not contained in r. Since $\mathfrak{a} \vee \mathfrak{a}' \neq \mathfrak{e}$, by virtue of the fact that $\mathfrak{a}$ is not simple, we see that the relation $\mathfrak{a} \vee \mathfrak{a}' \supset \mathfrak{r}$ is false: for otherwise we would have $\mathfrak{a} \subset \mathfrak{a} \vee \mathfrak{a}' = \mathfrak{r}$, against our choice of $\mathfrak{a}$. Hence (2) implies (1) with $\mathfrak{p} = \mathfrak{r}$.

Finally, it is evident that the ideals $\mathfrak{p}$, $\mathfrak{r}$ of (1) and (2) respectively are uniquely determined and are equal to one another.

§ 8. Algebraic Characterizations of Boolean Rings of Types (ω), (ω, ω). In this section we proceed to characterize rings of the indicated types.

In Th. 3.6 we have already proved the following result:

Theorem 8.1. *A Boolean ring A is of type (ω) if and only if $\mathfrak{P} \neq \mathfrak{P}^* = \mathfrak{S}$.*

We now prove:

Theorem 8.2. *A Boolean ring A is of type (ω, ω) if and only if it has the properties:*

(1) *A contains a non-semiprincipal simple ideal $\mathfrak{a}$;*

(2) *if $\mathfrak{a}$ is any non-semiprincipal simple ideal in A, then every ideal $\mathfrak{b}$ contained in $\mathfrak{a}$ and simple relative to $\mathfrak{a}$ is semiprincipal relative to $\mathfrak{a}$.*

According to R Th. 51, the relation $\mathfrak{P}^* \neq \mathfrak{S}$ is equivalent to the existence of a representation of A as a direct sum of two rings without unit, the summands being simple, but not semiprincipal, ideals in A. Hence we shall consider A as such a direct sum, $\mathfrak{c} \vee \mathfrak{c}'$, in the remainder of the discussion. If A is of type (ω, ω), we may choose $\mathfrak{c}$ and $\mathfrak{c}'$ as rings of type (ω). Let $\mathfrak{a}$ be a non-semiprincipal simple ideal in A; and let $\mathfrak{b}$ be an ideal in $\mathfrak{a}$, simple relative to $\mathfrak{a}$. By Th. 1.3, $\mathfrak{b}$ is simple in A; and $\mathfrak{a} \supset \mathfrak{b}$ implies $\mathfrak{b}' \supset \mathfrak{a}'$. If $\mathfrak{b}$ were principal in $\mathfrak{a}$, it would be semiprincipal relative to $\mathfrak{a}$ without further discussion. Hence we may suppose that $\mathfrak{b}$ is not principal in $\mathfrak{a}$ or in A. Now at least two of the ideals $\mathfrak{abc} = \mathfrak{bc}$, $\mathfrak{ab}'\mathfrak{c}, \mathfrak{a}'\mathfrak{b}'\mathfrak{c} = \mathfrak{a}'\mathfrak{c}$ must be principal. For example, if $\mathfrak{abc} = \mathfrak{bc}$ is not principal, it is not principal in $\mathfrak{c}$. Being a product of simple ideals, it is simple both in A and in $\mathfrak{c}$ by Ths. 1.2 and 1.3. Since $\mathfrak{c}$ is of type (ω), the orthocomplement of $\mathfrak{abc} = \mathfrak{bc}$ relative to $\mathfrak{c}$ must be principal in $\mathfrak{c}$ and hence in A. Thus $\mathfrak{b}'\mathfrak{c}$ is a principal ideal. Consequently the ideals $\mathfrak{ab}'\mathfrak{c}$, and $\mathfrak{a}'\mathfrak{b}'\mathfrak{c}$, being the products of the simple ideals $\mathfrak{a}$ and $\mathfrak{a}'$ with the principal ideal $\mathfrak{b}'\mathfrak{c}$, are principal by Th. 1.2. Similarly, if $\mathfrak{ab}'\mathfrak{c}$ is not principal, $\mathfrak{a}'\mathfrak{c} \vee \mathfrak{bc} = (\mathfrak{ab}')'\mathfrak{c}$ is principal, and $\mathfrak{a}'\mathfrak{c}$ and $\mathfrak{bc}$ are principal by Th. 1.1; and if $\mathfrak{a}'\mathfrak{c}$ is not principal, then $\mathfrak{ac} = \mathfrak{a}''\mathfrak{c}$ is principal and so also are $\mathfrak{abc}$ and $\mathfrak{ab}'\mathfrak{c}$. By similar arguments, we see that at least two of the ideals $\mathfrak{abc}' = \mathfrak{bc}'$, $\mathfrak{ab}'\mathfrak{c}'$, $\mathfrak{a}'\mathfrak{b}'\mathfrak{c}' = \mathfrak{a}'\mathfrak{c}'$ are principal. On the other hand, the ideals $\mathfrak{bc}$ and $\mathfrak{bc}'$ are not both principal since we have assumed that their sum $\mathfrak{b} = \mathfrak{bc} \vee \mathfrak{bc}'$ is non-principal. For the same reason ,the ideals $\mathfrak{a}'\mathfrak{c}$ and $\mathfrak{a}'\mathfrak{c}'$ are not both principal. We conclude therefore that the ideals $\mathfrak{ab}'\mathfrak{c}$ and $\mathfrak{ab}'\mathfrak{c}'$ are both principal. Hence the orthocomplement of $\mathfrak{b}$ relative to $\mathfrak{a}$ is found to be the principal ideal $\mathfrak{ab}' = \mathfrak{ba}'\mathfrak{c} \vee \mathfrak{ab}'\mathfrak{c}'$. Thus we have shown that $\mathfrak{b}$ is semiprincipal relative to $\mathfrak{a}$. This completes our proof of the assertion that a ring of type (ω, ω) has properties (1) and (2). On the other hand a ring with these properties can be represented as a direct sum $\mathfrak{c} \vee \mathfrak{c}'$, as we noted above. Since (2) shows that every ideal $\mathfrak{b}$ contained in $\mathfrak{c}$ or in $\mathfrak{c}'$ which is simple relative thereto is semiprincipal relative thereto, we see by reference to Th. 8.1 that $\mathfrak{c}$ and $\mathfrak{c}'$ are rings of type (ω). Hence the given ring is of type (ω, ω).

We have already given some investigations into the relation of the types (ω) and (ω, ω) to the other types considered. It is convenient to complete and summarize our results in the following terms:

Theorem 8.3. *Of the nine types* (α), (β_1), (β_2), (β_3), (β_2^*), (β_3^*), (β_2^*, β_1), (β_2^*, β_2), (β_3^*, β_3) *only* (β_1) *and* (β_2^*, β_1) *are included under type* (ω); *and only the special types* (β_1, β_1) *and* $(\beta_2^*, (\beta_1, \beta_1))$, *included under* (β_2) *and* (β_2^*, β_2) *respectively, are included under type* (ω, ω).

The rings of types $(\alpha), (\beta_3^*)$, and (β_2^*) all have units and hence belong to neither of the types (ω) and (ω, ω). We have already seen, in Ths. 3.6 and 7.3, that the rings of types (β_1) and (β_2^*, β_1) are of type (ω) while those of types (β_2), (β_3), and (β_2^*, β_2) are not. Comparison of Th. 8.1 with the result of Th. 7.1 that a ring of type (β_3^*, β_3) has the property $\mathfrak{P}^* \neq \mathfrak{S}$ shows that such a ring is not of type (ω). Since the types (ω) and (ω, ω) are distinct, by Th. 3.9, the rings of types (β_1) and (β_2^*, β_1) are not of type (ω, ω). Furthermore, Th. 3.7 shows that the rings of type (β_3) are not of type (ω, ω), and that among the rings of type (β_2) only those of the special type (β_1, β_1) are of type (ω, ω). Consequently we see that among the nine types only (β_1) and (β_2^*, β_1) are included under type (ω); and that only (β_1, β_1), (β_2^*, β_2), and (β_3^*, β_3) could possibly be included under type (ω, ω). Since the type (β_1, β_1) is included under the type (ω, ω), we have only two types left to consider.

Now let a Boolean ring A be represented in the form $A_1^* \vee A_2$ where A_2 has no unit. According to Th. 3.8, A is of type (ω, ω) if A_2 is of that type. Conversely, we shall show that, if A is of type (ω, ω), then so is A_2. First, A_2 cannot be of type (ω): for, if it were, Th. 3.8 would show that A is of type (ω), distinct from the type (ω, ω). Thus Th. 8.1 shows that A_2 contains simple ideals which are not semiprincipal. If $\mathfrak{a}$ is such an ideal in A_2, the fact that A_2 is a simple ideal in the direct sum $A_1^* \vee A_2$ shows that $\mathfrak{a}$ is simple in A by virtue of Th. 1.4. Hence, if $\mathfrak{b}$ is any ideal contained in $\mathfrak{a}$ and simple relative to $\mathfrak{a}$, then $\mathfrak{b}$ must be semiprincipal relative to $\mathfrak{a}$ by an application of Th. 8.2 together with the assumption that A is of type (ω, ω). We see therefore that A_2 has the properties (1) and (2) of Th. 8.2 and must consequently be of type (ω, ω). Now a ring of either of the types (β_2^*, β_2), (β_3^*, β_3) has an essentially unique representation in the form $A_1^* \vee A_2$ where A_1 and A_2 are totally multiplicative rings without unit. In particular A_2 is of type (β_2) or (β_3) according as A is of type (β_2^*, β_2) or (β_3^*, β_3). If A is of type (ω, ω), then so is A_2, by the preceding results. Hence Th. 3.7 shows that A cannot be of type (ω, ω) except in the case where A_2 is of type (β_1, β_1), included under type (β_2). Hence no ring of type (β_3^*, β_3) is of type (ω,ω); and the only rings of type (β_2^*,β_2) which can possibly be included under type (ω,ω) are those of the special type $(\beta_2^*,(\beta_1,\beta_1))$. Of course, Th. 3.7 and 3.8 show that the rings of types (β_1,β_1) and $(\beta_2^*,(\beta_1,\beta_1))$ are of type (ω,ω).

§ 9. Countable Boolean Rings. We shall now consider which of the various types of Boolean ring introduced in §§ 2, 3 and discussed in §§ 4—8 include countable rings. We shall find that only the types (β_3), (β_3^*) and (β_3^*,β_3) have this property.

Our fundamental result is the following

Theorem 9.1. *An infinite totally additive Boolean ring A has cardinal number not less than $2^{\aleph_0}$, the cardinal number of the continuum.*

Let A be a ring of the kind described. Th. 5.1 then shows that A contains a sequence $\{c_n\}$ such that $c_n \neq 0$, $c_m c_n = 0$ for $m \neq n$. If $\mathfrak{s}$ is any non-void subclass of this sequence its sum $a(\mathfrak{s})$ is the generating element of the principal ideal $\mathfrak{a}''(\mathfrak{s})$ in accordance with Th. 2.1. Now $c_n \epsilon \mathfrak{s}$ implies $a(\mathfrak{s})c_n = c_n$; and c_n non $\epsilon \mathfrak{s}$ implies $c_n \epsilon \mathfrak{s}' = \mathfrak{a}'(\mathfrak{s})$ and hence $a(\mathfrak{s})c_n = 0$. It follows that the class of all sums $a(\mathfrak{s})$ is in biunivocal correspondence with the non-void subclasses formed from the sequence $\{c_n\}$. Since such subclasses constitute a collection with the cardinal number $2^{\aleph_0}$, we conclude that the cardinal number of A is not less than $2^{\aleph_0}$.

As an immediate consequence of this theorem, we have:

Theorem 9.2. *A countable Boolean ring A is totally additive if and only if it is finite; a countable Boolean ring A is totally multiplicative if and only if it has a countable atomic basis.*

By Th. 9.1, a countable Boolean ring A cannot be infinite if it is totally additive. Hence the result stated here. If a Boolean ring A is countable and totally multiplicative, then every principal ideal in A is obviously countable and in addition is totally additive by virtue of Th. 2.3 (3). Hence every principal ideal in A is finite. It follows that A has a countable atomic basis, by virtue of Th. 6.1. On the other hand a ring A with countable atomic basis is finite if the basis is finite, and is isomorphic to the ring of all finite subclasses of a fixed countably infinite class E in accordance with Th. 6.1 if the basis is countably infinite. Hence A is countable and totally multiplicative.

Another fundamental result is the following

Theorem 9.3. *In a countable Boolean ring A with unit every non-normal prime ideal is a barrier ideal.*

Let $\mathfrak{p}$ be a non-normal prime ideal in a countable ring. Since $\mathfrak{p}$ is a non-principal ideal, it is a ring without unit and is therefore infinite. Hence its elements may be written in an infinite sequence $\{a_n\}$. We now construct a subsequence $\{b_n\}$ with the properties: (1) $b_n < b_{n+1}$, $b_n \neq b_{n+1}$ for $n=1,2,3,\dots$; (2) if a is any element in $\mathfrak{p}$, then $a < b_n$ for some index n. We put $b_1 = a_1$. If we have defined $b_k = a_{n_k}$ for $k=1,\dots,m$ so that $a_k < a_{n_k}$ for $k=1,\dots,m$ and $a_{n_k} < a_{n_{k+1}}$, $n_k < n_{k+1}$ for $k=1,\dots,m-1$, we then choose n_{m+1} as the first index n after n_m such that $a_m < a_n$, $a_{n_m} < a_n$. This choice is possible: for if we had either $a_m > a_n$ or $a_{n_m} > a_n$ for $n > n_m$, the element $a_1 \vee \dots \vee a_m \vee \dots \vee a_{n_m} = a$ would belong to $\mathfrak{p}$ and would have the property that $a > a_n$ for $n=1,2,3,\dots$; and we could then conclude that $\mathfrak{p}$ is a principal ideal with a as its generating element. By induction, we obtain a sequence $b_k = a_{n_k}$, $k=1,2,3,\dots$ such that $a_k < a_{n_k}$, $a_{n_k} < a_{n_{k+1}}$, $n_k < n_{k+1}$ for $k=1,2,3,\dots$. Since the relation $n_k < n_{k+1}$ implies that $a_{n_k} \neq a_{n_{k+1}}$, this sequence has the desired properties (1) and (2). By the construction given in the proof of Th. 5.1, we can now replace the sequence $\{b_n\}$ by the sequence $\{c_n\}$, where $c_1 = b_1$ and $c_{n+1} = b_{n+1} + b_n$ for $n \geqslant 1$, so as to secure the properties $c_n \neq 0$, $c_m c_n = 0$ for $m \neq n$, and $b_n = c_1 + \dots + c_n = c_1 \vee \dots \vee c_n$, for $m,n=1,2,3,\dots$. It is evident that $a \epsilon \mathfrak{p}$ implies $a < c_1 + \dots + c_n$ for some index n. Let $\mathfrak{s}$ be the class of all elements c_{2m}, for $m=1,2,3,\dots$; and $\mathfrak{t}$ the class of all elements c_{2m-1}, for $m=1,2,3,\dots$. We shall determine the ideals $\mathfrak{s}'$ and $\mathfrak{t}'$. It is evident that $\mathfrak{s}' \supset \mathfrak{t}$, $\mathfrak{t}' \supset \mathfrak{s}$. It follows that $\mathfrak{s}'' \subset \mathfrak{t}'$, $\mathfrak{t}'' \subset \mathfrak{s}'$. If an element a of A is in both $\mathfrak{s}'$ and $\mathfrak{t}'$ then $ac_n = 0$ for $n=1,2,3,\dots$. Hence, if b is any element of $\mathfrak{p}$, there is an index n such that $ab < a(c_1 \vee \dots \vee c_n) = ac_1 \vee \dots \vee ac_n = 0$. Consequently a belongs to $\mathfrak{p}'$. Since $\mathfrak{p}$ is a non-normal prime ideal, we have $\mathfrak{p}' = \mathfrak{o}$ and hence $a = 0$. Thus $\mathfrak{s}'\mathfrak{t}' = \mathfrak{o}$. It follows that $\mathfrak{s}' \subset \mathfrak{t}''$, $\mathfrak{s}'' \supset \mathfrak{t}'$. Combining these results with those obtained above, we see that $\mathfrak{s}'' = \mathfrak{t}'$, $\mathfrak{s}' = \mathfrak{t}''$. The ideal, $\mathfrak{s}' \vee \mathfrak{t}'$ contains both $\mathfrak{s}$ and $\mathfrak{t}$, and thus contains every element c_n, $n=1,2,3,\dots$. Consequently, if a is any element of $\mathfrak{p}$, we have $a < c_1 \vee \dots \vee c_n$ for a suitable index n and conclude that $a \epsilon \mathfrak{s}' \vee \mathfrak{t}'$. The relation $\mathfrak{p} \subset \mathfrak{s}' \vee \mathfrak{t}'$ is thus established.

On the other hand, let a be an element which belongs to $\mathfrak{s}'$. Then $ac_{2m}=0$ for $m=1,2,3,\ldots$, or, equivalently, $a'>c_{2m}$ for $m=1,2,3,\ldots$. If a' were an element of $\mathfrak{p}$, there would exist an index n such that $a'<c_1\vee\ldots\vee c_n$. Choosing $2m>n$ we would therefore have $c_{2m}<c_1\vee\ldots\vee c_n$ and hence

$$c_{2m}=c_{2m}(c_1\vee\ldots\vee c_n)=c_{2m}c_1\vee\ldots\vee c_{2m}c_n=0.$$

Since $c_{2m}\neq0$, we conclude that a' is not in $\mathfrak{p}$. By R Th. 36, we reach the result that $a\epsilon\mathfrak{s}'$ implies $a\epsilon\mathfrak{p}$ or, equivalently, that $\mathfrak{s}'\subset\mathfrak{p}$. In the same way we find that $\mathfrak{t}'\subset\mathfrak{p}$. Thus we must have $\mathfrak{p}=\mathfrak{s}'\vee\mathfrak{t}'$, $\mathfrak{s}'\mathfrak{t}'=\mathfrak{o}$. Since $\mathfrak{s}'$ and $\mathfrak{t}'$ are normal ideals by virtue of the relations $\mathfrak{s}'=\mathfrak{s}'''$, $\mathfrak{t}'=\mathfrak{t}'''$ of R Th. 20, we see that $\mathfrak{p}$ is a barrier ideal in accordance with Def. 3.1.

We can now obtain the chief results concerning types of countable Boolean ring. We have:

Theorem 9.4. *There exist no countable Boolean rings of any of the types (α), (β_1), (β_2), (β_2^*), (β_2^*,β_1), (β_2^*,β_2), (ω), (ω,ω). A Boolean ring of one of the three types (β_3), (β_3^*), (β_3^*,β_3) is countable if and only if in the corresponding representation as a ring of subclasses of a fixed infinite class E, described in Th. 6.1 and 7.1, the class E is countable.*

Ths. 9.1 and 9.2 show that no ring of type (α), (β_1), or (β_2) is countable; and Th. 9.3 shows that no ring of type (ω) is countable. Since the rings of types (β_2^*), (β_2^*,β_1), (β_2^*,β_2), (ω,ω) contain subrings of some of the types (β_1), (β_2), (ω), none of them can be countable. Thus the only types which can contain countable rings are the three mentioned in the statement of the theorem. The condition given for a ring of any of these three types to be countable is evident.

§ 10. The Fundamental Classification of Ideals.

In this section we shall collect some of the results of §§ 5—9 in such a form as to give a complete presentation of the possible relations of equality between the fundamental classes $\mathfrak{P},\mathfrak{P}^*,\mathfrak{S},\mathfrak{N},\mathfrak{J},\mathfrak{E}$ of ideals in a Boolean ring A. We have:

Theorem 10.1. *The possible reductions of the inclusion relations $\mathfrak{P}\subset\mathfrak{P}^*\subset\mathfrak{S}\subset\mathfrak{N}\subset\mathfrak{J}$ to equalities are summarized in the following table, in which the Boolean rings associated with any particular combination of equalities and inequalities are described at the right:*

$\mathfrak{P}=\mathfrak{P}^*=\mathfrak{S}=\mathfrak{N}=\mathfrak{J}$	Finite rings.
$\mathfrak{P}=\mathfrak{P}^*=\mathfrak{S}=\mathfrak{N}\neq\mathfrak{J}$	Rings of type (α).
$\mathfrak{P}=\mathfrak{P}^*=\mathfrak{S}\neq\mathfrak{N}\neq\mathfrak{J}$	General rings with unit including those of types (β_2^*) and (β_3^*).
$\mathfrak{P}\neq\mathfrak{P}^*=\mathfrak{S}=\mathfrak{N}\neq\mathfrak{J}$	Rings of type (β_1).
$\mathfrak{P}\neq\mathfrak{P}^*=\mathfrak{S}\neq\mathfrak{N}\neq\mathfrak{J}$	Rings of type (ω), other than those of type (β_1) but including those of type (β_2^*,β_1).
$\mathfrak{P}\neq\mathfrak{P}^*\neq\mathfrak{S}=\mathfrak{N}=\mathfrak{J}$	Rings of type (β_3).
$\mathfrak{P}\neq\mathfrak{P}^*\neq\mathfrak{S}=\mathfrak{N}\neq\mathfrak{J}$	Rings of type (β_2) including those of type (β_1,β_1).
$\mathfrak{P}\neq\mathfrak{P}^*\neq\mathfrak{S}\neq\mathfrak{N}\neq\mathfrak{J}$	General rings without unit, including those of types (β_2^*,β_2), (β_3^*,β_3) and (ω,ω) other than those of type (β_1,β_1).

We know from R Th. 25 that the relation $\mathfrak{P}=\mathfrak{P}^*$ is characteristic for rings with unit and implies $\mathfrak{P}^*=\mathfrak{S}$; from Ths. 3.6 and 8.1 that the relations $\mathfrak{P}\neq\mathfrak{P}^*=\mathfrak{S}$ are characteristic for rings of type (ω); from Th. 2.3 (2) that the relation $\mathfrak{S}=\mathfrak{N}$ is characteristic for totally multiplicative rings; from Ths. 5.2 and 6.1 that the relation $\mathfrak{N}=\mathfrak{J}$ is characteristic for the finite rings and the rings of type (β_3); and from R Th. 24 that the relations $\mathfrak{J}\neq\mathfrak{S}$, $\mathfrak{N}\neq\mathfrak{S}$ both imply $\mathfrak{J}\neq\mathfrak{N}$. We are thus left with at most nine possible combinations of equalities and inequalities between the classes of ideal — the eight listed in the table, and the system of relations $\mathfrak{P}\neq\mathfrak{P}^*=\mathfrak{S}=\mathfrak{N}=\mathfrak{J}$ which does not appear there. This ninth system cannot actually occur since the relation $\mathfrak{N}=\mathfrak{J}$ characterizes rings in which either $\mathfrak{P}=\mathfrak{P}^*=\mathfrak{S}$ — the finite case — or $\mathfrak{P}\neq\mathfrak{P}^*\neq\mathfrak{S}$ — the case of type (β_3). Setting aside the cases where $\mathfrak{N}=\mathfrak{J}$ as already settled, we see that in the six remaining cases those where $\mathfrak{S}=\mathfrak{N}$ exhaust the infinite totally multiplicative rings of types other than (β_3). Of these, the only ones with unit are those of type (α), the only ones with $\mathfrak{P}\neq\mathfrak{P}^*=\mathfrak{S}$ are those of type (β_1), and the only ones with $\mathfrak{P}\neq\mathfrak{P}^*\neq\mathfrak{S}$ are those of type (β_2), as we see by reference to Ths. 2.6, 3.6 and 6.4. We are thus left with the three cases where $\mathfrak{S}\neq\mathfrak{N}$ and hence $\mathfrak{N}\neq\mathfrak{J}$. These cases exhaust the Boolean rings which are not totally multiplicative. Among these rings the ones with unit satisfy also the relations $\mathfrak{P}=\mathfrak{P}^*=\mathfrak{S}$, the ones of type (ω) the relations $\mathfrak{P}\neq\mathfrak{P}^*=\mathfrak{S}$, and the remaining ones the relations $\mathfrak{P}\neq\mathfrak{P}^*\neq\mathfrak{S}$. The distribution of the types (β_3^*), (β_3^*), (β_2^*,β_1), (β_2^*,β_2), (β_3^*,β_3) and (ω,ω) among these three cases is settled by noting that no rings of the last four types just listed have units and that, in accordance with Th. 8.3, only type (β_2^*,β_1) is included in type (ω). The general distribution of the type (ω,ω) between the totally multiplicative cases and the other cases is settled by Ths. 3.7 and 8.3.

It will be observed that each row of the table contains at least one type of Boolean ring for which an explicit construction has been provided in § 2, 3, 4, 5, or 6. Hence each of the eight systems of relations is actually realized. It will be observed further that the three entries with $\mathfrak{S}\neq\mathfrak{N}$ are of comparable degress of generality in view of $\mathfrak{R}$ Th. 1 and Th. 3.8, while the remaining entries characterize quite special Boolean rings.

With regard to the class $\mathfrak{E}$ of prime ideals, we note the following facts:

Theorem 10.2. *The relation $\mathfrak{E}\subset\mathfrak{N}$ implies the relations $\mathfrak{E}\subset\mathfrak{P}^*$ and $\mathfrak{N}=\mathfrak{J}$, thus characterizing the finite Boolean rings and the Boolean rings of type (β_3). The assertion that $\mathfrak{E}\Delta\mathfrak{E}\mathfrak{N}$ is a one-element class characterizes the Boolean rings of types (β_3^*) and (β_3^*,β_3).*

From $\mathfrak{R}$ Th. 38, we know that $\mathfrak{E}\subset\mathfrak{N}$ implies $\mathfrak{E}\subset\mathfrak{P}^*$. $\mathfrak{R}$ Th. 66 shows that $\mathfrak{E}\subset\mathfrak{N}$ implies $\mathfrak{N}=\mathfrak{J}$: for, the ideal $\mathfrak{e}$ is semiprincipal and hence normal; and every other ideal, being the product of its prime ideal divisors, is normal in accordance with $\mathfrak{R}$ Th. 29. The case where $\mathfrak{E}\Delta\mathfrak{E}\mathfrak{N}$ is a one-element class — that is, where there exists exactly one non-normal prime ideal — is settled by Th. 7.2.

By virtue of Th. 9.4 we can easily specialize the table of Th. 10.1 for the case of countable rings.

Theorem 10.3. *The possible reductions of the inclusion relations $\mathfrak{P}\subset\mathfrak{P}^*\subset\mathfrak{S}\subset\mathfrak{N}\subset\mathfrak{J}$ to equalities in the case of a countable Boolean ring are summarized in the following table:*

$\mathfrak{P}=\mathfrak{P}^*=\mathfrak{S}=\mathfrak{N}=\mathfrak{J}$	Finite rings.
$\mathfrak{P}=\mathfrak{P}^*=\mathfrak{S}\neq\mathfrak{N}\neq\mathfrak{J}$	General rings with unit, including those of type (β_3^*).
$\mathfrak{P}\neq\mathfrak{P}^*\neq\mathfrak{S}=\mathfrak{N}=\mathfrak{J}$	Rings of type (β_3).
$\mathfrak{P}\neq\mathfrak{P}^*\neq\mathfrak{S}\neq\mathfrak{N}\neq\mathfrak{J}$	General rings without unit, including those of type (β_3^*,β_3).

§ 11. Some Special Ideals. In this section we shall consider ideals constructed from the classes $\mathfrak{P}^*\Delta\mathfrak{P}$, $\mathfrak{S}\Delta\mathfrak{P}^*$, $\mathfrak{N}\Delta\mathfrak{S}$, $\mathfrak{J}\Delta\mathfrak{N}$. In particular we associate with each class the least ideal containing all its members. The associated ideal is $\mathfrak{o}$ in case the class is void; otherwise it is the sum of all ideals in the class. The ideal obtained in this way from $\mathfrak{P}^*\Delta\mathfrak{P}$ or from $\mathfrak{S}\Delta\mathfrak{P}^*$ is easily seen to be either $\mathfrak{o}$ or $\mathfrak{e}$, the first case corresponding to the relations $\mathfrak{P}=\mathfrak{P}^*$, $\mathfrak{P}^*=\mathfrak{S}$ respectively, the second to the relations $\mathfrak{P}\neq\mathfrak{P}^*$, $\mathfrak{P}^*\neq\mathfrak{S}$ respectively. Thus the classes $\mathfrak{P}^*\Delta\mathfrak{P}$, $\mathfrak{S}\Delta\mathfrak{P}^*$ do not lead to results which it is necessary to examine more carefully. On the other hand, the facts established in Ths. 7.2 and 7.4 show that the situation is different with respect to the classes $\mathfrak{N}\Delta\mathfrak{S}$, $\mathfrak{J}\Delta\mathfrak{N}$.

Using the existence and divisibility properties of prime ideals, we find the following results:

Theorem 11.1. *In a Boolean ring A, let $\mathfrak{q}$ and $\mathfrak{r}$ be the least ideals containing all ideals in the classes $\mathfrak{N}\Delta\mathfrak{S}$ and $\mathfrak{J}\Delta\mathfrak{N}$ respectively. Then $\mathfrak{q}\subset\mathfrak{r}$; and each of them is equal to $\mathfrak{o}$, is equal to $\mathfrak{e}$, or is a non-normal prime ideal. The possible combinations and the types of ring which they characterize are exhibited in the following table, in which the entry * signifies that the ideal in question is a non-normal prime ideal:*

$\mathfrak{q}$	$\mathfrak{r}$	A
$\mathfrak{o}$	$\mathfrak{o}$	Finite or of type (β_3).
$\mathfrak{o}$	$\mathfrak{e}$	Ring of types (α), (β_1), (β_2).
*	*	Ring of types (β_3^*), (β_3^*,β_3).
*	$\mathfrak{e}$	Ring of types (β_2^*), (β_2^*,β_1), (β_2^*,β_2).
$\mathfrak{e}$	$\mathfrak{e}$	General ring.

We first prove that $\mathfrak{q}\subset\mathfrak{r}$. If $\mathfrak{N}\Delta\mathfrak{S}$ is void, then $\mathfrak{q}=\mathfrak{o}$ and the relation $\mathfrak{q}\subset\mathfrak{r}$ is trivial. If $\mathfrak{N}\Delta\mathfrak{S}$ is not void, then there exists a non-simple normal ideal $\mathfrak{a}$ and $\mathfrak{q}$ is the sum of all such ideals. Now if $\mathfrak{a}$ is any such ideal, so is $\mathfrak{a}'$; and the ideal $\mathfrak{a}\vee\mathfrak{a}'$ is not normal, by $\mathfrak{R}$ Th. 24. Hence we have $\mathfrak{a}\subset\mathfrak{a}\vee\mathfrak{a}'\subset\mathfrak{r}$ whenever $\mathfrak{a}\,\epsilon\,\mathfrak{N}\Delta\mathfrak{S}$. The desired relation $\mathfrak{q}\subset\mathfrak{r}$ then follows. We may observe that, if $\mathfrak{N}\Delta\mathfrak{S}$ is not void, then $\mathfrak{q}\neq\mathfrak{o}$. It is also easily seen that, if $\mathfrak{J}\Delta\mathfrak{N}$ is void, $\mathfrak{r}=\mathfrak{o}$ and that otherwise $\mathfrak{r}\neq\mathfrak{o}$. Thus the relations $\mathfrak{q}=\mathfrak{o}$, $\mathfrak{r}=\mathfrak{o}$ are equivalent respectively to the relations $\mathfrak{N}=\mathfrak{S}$, $\mathfrak{J}=\mathfrak{N}$.

Next let us prove that, when $\mathfrak{r}\neq\mathfrak{o}$, $\mathfrak{r}\neq\mathfrak{e}$, then $\mathfrak{r}$ is a non-normal prime ideal. Since $\mathfrak{r}\neq\mathfrak{o}$, there exists a non-normal ideal $\mathfrak{a}$; and $\mathfrak{R}$ Th. 24 shows that $\mathfrak{a}\vee\mathfrak{a}'$ is not normal. Since $\mathfrak{r}\neq\mathfrak{e}$, there exists an element a not in $\mathfrak{r}$. For any such element a, the ideal $\mathfrak{a}(a)\vee\mathfrak{a}$ is not contained in $\mathfrak{r}$ and so must be normal. Since it is also not contained in $\mathfrak{q}\subset\mathfrak{r}$, it must even be simple. Using the indicated properties of the ideals $\mathfrak{a}\vee\mathfrak{a}'$ and $\mathfrak{a}(a)\vee\mathfrak{a}$, we have

$$\mathfrak{e}=(\mathfrak{a}(a)\vee\mathfrak{a})\vee(\mathfrak{a}(a)\vee\mathfrak{a})'=\mathfrak{a}(a)\vee\mathfrak{a}\vee\mathfrak{a}'(a)\mathfrak{a}'=\mathfrak{a}(a)\vee(\mathfrak{a}\vee\mathfrak{a}')\subset\mathfrak{a}(a)\vee\mathfrak{r},$$

$$\mathfrak{a}\vee\mathfrak{a}'\subset\mathfrak{r},\qquad \mathfrak{o}=\mathfrak{a}'\mathfrak{a}''=(\mathfrak{a}\vee\mathfrak{a}')'\supset\mathfrak{r}',\qquad \mathfrak{r}'=\mathfrak{o}.$$

Thus an ideal containing $\mathfrak{r}$ and any element not in $\mathfrak{r}$ must coincide with $\mathfrak{e}$, so that $\mathfrak{r}$ is divisorless and hence prime; and the relation $\mathfrak{r}'=\mathfrak{o}$ shows that $\mathfrak{r}$ is not normal. In a similar way, we show that, if $\mathfrak{q}\neq\mathfrak{o}$ and $\mathfrak{q}\neq\mathfrak{e}$, then $\mathfrak{q}$ is a non-normal prime ideal. Let $\mathfrak{a}$ be a non-simple normal ideal and a an element not in $\mathfrak{q}$. Then $\mathfrak{a}'$ is a non-simple normal ideal, and $\mathfrak{a}\vee\mathfrak{a}'\subset\mathfrak{q}$. Also $\mathfrak{a}(a)\vee\mathfrak{a}$ is a normal ideal by Th. 1.2; and, since it is not contained in $\mathfrak{q}$, it must be simple. By reasoning exactly like that used above, we have $\mathfrak{e}\subset\mathfrak{a}(a)\vee\mathfrak{q}$, $\mathfrak{q}'=\mathfrak{o}$. It follows, as before, that $\mathfrak{q}$ is a non-normal prime ideal.

The possible combinations are now reduced to the five listed in the table, together with the one where $\mathfrak{q}=\mathfrak{o}$ and $\mathfrak{r}$ is a non-normal prime ideal. We shall see presently that this particular combination cannot arise. From the preceding results we know that $\mathfrak{r}=\mathfrak{o}$ if and only if $\mathfrak{J}=\mathfrak{N}$ and that $\mathfrak{r}=\mathfrak{o}$ implies $\mathfrak{q}=\mathfrak{o}$. We also know from Ths. 5.2 and 6.1 that the relation $\mathfrak{J}=\mathfrak{N}$ characterizes the finite rings and the rings of type (β_3). If $\mathfrak{r}$ is prime, we must have $\mathfrak{q}=\mathfrak{r}$ or $\mathfrak{q}=\mathfrak{o}$ by virtue of the relation $\mathfrak{q}\subset\mathfrak{r}$. In either case we see that the sum of all non-simple ideals in A is the ideal $\mathfrak{s}=\mathfrak{q}\vee\mathfrak{r}=\mathfrak{r}$. The relation $\mathfrak{r}\neq\mathfrak{o}$ implies in particular that $\mathfrak{J}\neq\mathfrak{S}$. Since $\mathfrak{s}$ is prime, Th. 7.2 shows that A is a ring of type (β_3^*) or (β_3^*,β_3). In rings of these types we have $\mathfrak{N}\neq\mathfrak{S}$ and hence $\mathfrak{q}\neq\mathfrak{o}$. On the other hand, if A is a ring of either of these types, the sum $\mathfrak{s}$ of all non-simple ideals is a prime ideal by Th. 7.2. Obviously $\mathfrak{s}=\mathfrak{q}\vee\mathfrak{r}=\mathfrak{r}$, so that $\mathfrak{r}$ is prime. Thus we see that the rings of types (β_3^*) and (β_3^*,β_3) are characterized by the statement that $\mathfrak{r}$ is prime; and that, when $\mathfrak{r}$ is prime, $\mathfrak{q}$ is also prime and equal to $\mathfrak{r}$. We now have to treat the cases where $\mathfrak{r}=\mathfrak{e}$. We have $\mathfrak{q}=\mathfrak{o}$, $\mathfrak{r}=\mathfrak{e}$ if and only if the relations $\mathfrak{N}=\mathfrak{S}$, $\mathfrak{J}\neq\mathfrak{N}$ are valid. These relations characterize the totally multiplicative rings of types (α), (β_1), and (β_2) as we see by reference to Th. 10.1 or to earlier

theorems. If $\mathfrak{q}$ is prime and $\mathfrak{r}=\mathfrak{e}$, we have $\mathfrak{N}\neq\mathfrak{S}$ and Th. 7.4 shows that the ring A is of one of the types (β_2^*), (β_2^*,β_1), (β_2^*,β_2). Conversely, if A is of one of these types, then $\mathfrak{q}$ is prime and $\mathfrak{r}=\mathfrak{e}$, by virtue of the same theorem. Thus the only case remaining is that where $\mathfrak{q}=\mathfrak{r}=\mathfrak{e}$. This case occurs in all rings not of the types listed under the other cases, and is therefore to be regarded as the general case.

§ 12. Optimum Character of § 1. We shall now proceed to show that the general results stated in § 1 are the best possible, in a certain sense. It is of course obvious that in any ring for which relations of equality hold between some of the classes $\mathfrak{P}$, $\mathfrak{P}^*$, $\mathfrak{S}$, $\mathfrak{N}$, $\mathfrak{J}$, as listed in Th. 10.1, the results presented in the tables of § 1 can be correspondingly reduced. Thus, for example, if $\mathfrak{N}=\mathfrak{S}$, the tables of Th. 1.2 can be simplified by writing $\mathfrak{S}$ for $\mathfrak{N}$ throughout; and it would also be permissible to strike out altogether the rows and columns labeled "$\mathfrak{N}$". What we propose to show is that, apart from the simplifications arising in this way or in connection with certain special types of ring, no reductions of the results of § 1 are possible. For convenience in comparison, we shall take up the theorems of § 1 in serial order, stating for each of them a similarly numbered theorem discussing its optimum character.

Theorem 12.1. *Under the condition $\mathfrak{a}\subset\mathfrak{b}$, the ideals $\mathfrak{a}$ and $\mathfrak{b}$ in a Boolean ring A can be assigned arbitrarily to the classes $\mathfrak{P}$, $\mathfrak{P}^*$, $\mathfrak{S}$, $\mathfrak{N}$, $\mathfrak{J}$, with only the following exceptions:*

(1) *the assignment is subject to the general restrictions given in Th. 1.1;*

(2) *the assignment is subject to the limitations imposed by equalities between the classes $\mathfrak{P}$, $\mathfrak{P}^*$, $\mathfrak{S}$, $\mathfrak{N}$, $\mathfrak{J}$;*

(3) *in a ring of type (β_3^*) or (β_3^*,β_3), the relation $\mathfrak{a}\subset\mathfrak{b}$ is impossible when $\mathfrak{a}$ is not normal and $\mathfrak{b}$ is normal but not simple.*

In (3) we have a new algebraic characterization of the two types concerned.

We shall begin with a consideration of the exception (3). Suppose we wish to find a non-normal ideal $\mathfrak{a}$ and a non-simple normal ideal $\mathfrak{b}$ such that $\mathfrak{a}\subset\mathfrak{b}$. Obviously, we cannot do so if $\mathfrak{N}=\mathfrak{S}$, so that we must assume the relation $\mathfrak{N}\neq\mathfrak{S}$ at the start. Then we can select a non-simple normal ideal $\mathfrak{c}_1$, observing that $\mathfrak{c}_1\vee\mathfrak{c}_1'$ is not normal, $\mathfrak{c}_1'$ normal but not simple. If we can find an ideal $\mathfrak{c}_2$ such that $\mathfrak{c}_1\vee\mathfrak{c}_1'$

is not contained in $\mathfrak{c}_2 \vee \mathfrak{c}_2'$, we can prove that one of the ideals $(\mathfrak{c}_2 \vee \mathfrak{c}_2')\mathfrak{c}_1$, $(\mathfrak{c}_2 \vee \mathfrak{c}_2')\mathfrak{c}_1'$ is not normal. With proper choice of notation, we may arrange that the first of them should have this property; and we can then complete our construction by putting $\mathfrak{a}=(\mathfrak{c}_2 \vee \mathfrak{c}_2')\mathfrak{c}_1$, $\mathfrak{b}=\mathfrak{c}_1$. The necessary proof runs as follows. The orthocomplement of $(\mathfrak{c}_2 \vee \mathfrak{c}_2')\mathfrak{c}_1=((\mathfrak{c}_2 \vee \mathfrak{c}_2')\mathfrak{c}_1)\mathfrak{c}_1$ relative to $\mathfrak{c}_1$ is given by

$$((\mathfrak{c}_2 \vee \mathfrak{c}_2')\mathfrak{c}_1)'\mathfrak{c}_1=(\mathfrak{c}_2 \vee \mathfrak{c}_2')'\mathfrak{c}_1=\mathfrak{c}_1'\mathfrak{c}_2''\mathfrak{c}_1=\mathfrak{o}.$$

Thus if $(\mathfrak{c}_2 \vee \mathfrak{c}_1')\mathfrak{c}_1$ were normal, we would have

$$\mathfrak{c}_1 \subset ((\mathfrak{c}_2 \vee \mathfrak{c}_2')\mathfrak{c}_1)''=(\mathfrak{c}_2 \vee \mathfrak{c}_2')\mathfrak{c}_1 \subset \mathfrak{c}_2 \vee \mathfrak{c}_2'.$$

Similarly, if $(\mathfrak{c}_2 \vee \mathfrak{c}_2')\mathfrak{c}_1'$ were normal, we would have $\mathfrak{c}_1' \subset \mathfrak{c}_2 \vee \mathfrak{c}_2'$. Hence if both ideals were normal we would have $\mathfrak{c}_1 \vee \mathfrak{c}_1' \subset \mathfrak{c}_2 \vee \mathfrak{c}_2'$ against hypothesis. We now have to consider the case where the relation $\mathfrak{c}_1 \vee \mathfrak{c}_1' \subset \mathfrak{c}_2 \vee \mathfrak{c}_2'$ holds whatever the ideal $\mathfrak{c}_2$. Here we first suppose that $\mathfrak{c}=\mathfrak{c}_1 \vee \mathfrak{c}_1'$ is not prime. Then there exists an element c such that $c \epsilon \mathfrak{c}$, $\mathfrak{a}(c) \vee \mathfrak{c} \neq \mathfrak{e}$. By Th. 1.2 the ideal $\mathfrak{a}(c) \vee \mathfrak{c}_1$ is normal; but the relations $(\mathfrak{a}(c) \vee \mathfrak{c}_1) \vee (\mathfrak{a}(c) \vee \mathfrak{c}_1)'=\mathfrak{a}(c) \vee \mathfrak{c}_1 \vee \mathfrak{a}'(c)\mathfrak{c}_1'=\mathfrak{a}(c) \vee \mathfrak{c} \neq \mathfrak{e}$ show that it is not simple. If we put $\mathfrak{c}_2=\mathfrak{a}(c) \vee \mathfrak{c}_1$ we therefore have a non-simple normal ideal such that $\mathfrak{c}_2 \vee \mathfrak{c}_2'$ is not contained in $\mathfrak{c}_2 \vee \mathfrak{c}_1'$. Consequently the construction carried out above can be repeated with the rôles of $\mathfrak{c}_1$ and $\mathfrak{c}_2$ interchanged. We are thus left to consider the case where $\mathfrak{c}_1 \vee \mathfrak{c}_1'$ is a prime ideal contained in $\mathfrak{c}_2 \vee \mathfrak{c}_2'$ whatever the ideal $\mathfrak{c}_2$. According to Th. 7.1 this is precisely the case where A is of type (β_3^*) or (β_3^*, β_3): for we have $\mathfrak{N} \neq \mathfrak{S}$ and hence $\mathfrak{J} \neq \mathfrak{S}$; and we can take $\mathfrak{p}=\mathfrak{c}_1 \vee \mathfrak{c}_1'$ in Th. 7.1 (1). We complete our discussion by showing that in this case the desired construction is impossible. Let $\mathfrak{b}$ be a non-simple normal ideal and $\mathfrak{a}$ an ideal contained in $\mathfrak{b}$. Since $\mathfrak{p} \subset \mathfrak{b} \vee \mathfrak{b}' \neq \mathfrak{e}$ we have $\mathfrak{p}=\mathfrak{b} \vee \mathfrak{b}'$; and since $\mathfrak{a} \subset \mathfrak{b}$ we have $\mathfrak{a}'' \subset \mathfrak{b}''=\mathfrak{b} \subset \mathfrak{p}$. Thus the relations $\mathfrak{p} \subset \mathfrak{a} \vee \mathfrak{a}' \subset \mathfrak{a}'' \vee \mathfrak{a}' \subset \mathfrak{p}$ hold when $\mathfrak{a}' \subset \mathfrak{p}$; and the relations $\mathfrak{a} \vee \mathfrak{a}' \neq \mathfrak{p}$, $\mathfrak{a} \vee \mathfrak{a}' \supset \mathfrak{p}$ imply $\mathfrak{a} \vee \mathfrak{a}'=\mathfrak{e}$ when $\mathfrak{a}'$ is not contained in $\mathfrak{p}$. In either case we have $\mathfrak{a} \vee \mathfrak{a}'=\mathfrak{a}'' \vee \mathfrak{a}'$ and hence $\mathfrak{a}=\mathfrak{a}\mathfrak{a}''=(\mathfrak{a} \vee \mathfrak{a}')\mathfrak{a}''=(\mathfrak{a}'' \vee \mathfrak{a}')\mathfrak{a}''=\mathfrak{a}''$, so that $\mathfrak{a}$ is normal.

Of the twenty-one examples left for us to give after the exceptions (1) and (3) are taken into account, fourteen are trivial. The five examples where $\mathfrak{a}$ and $\mathfrak{b}$ belong to the same class are found by taking $\mathfrak{a}=\mathfrak{b}$ and choosing $\mathfrak{b}$ as a non-normal ideal, a non-simple normal ideal, and so on. The four remaining examples where $\mathfrak{b}$ is semiprincipal but not principal are obtained by taking $\mathfrak{b}$ as the ideal $\mathfrak{e}$ in a ring without unit and choosing $\mathfrak{a}$ as a non-normal ideal, a non-simple normal ideal, and so on. The three remaining examples where $\mathfrak{a}$ is principal are obtained by choosing $\mathfrak{b}$ as a non-normal ideal, a non-simple normal ideal, and so on, and putting $\mathfrak{a}=\mathfrak{a}(a)$ where a is an element of $\mathfrak{b}$. Of the two remaining examples where $\mathfrak{b}$ is principal, one is easily constructed. If $\mathfrak{c}$ is a non-simple normal ideal, there exists an element b such that $\mathfrak{a}(b)\mathfrak{c}$ is not principal, by R Th. 26. Since $\mathfrak{a}(b)\mathfrak{c} \subset \mathfrak{a}(b)$, we know from Th. 1.1 that $\mathfrak{a}(b)\mathfrak{c}$ cannot even be simple. On the other hand, $\mathfrak{a}(b)\mathfrak{c}$ is normal by Th. 1.2. Thus on putting $\mathfrak{a}=\mathfrak{a}(b)\mathfrak{c}$ and $\mathfrak{b}=\mathfrak{a}(b)$ we obtain an example where $\mathfrak{a}$ is normal but not simple and $\mathfrak{b}$ is principal. Of the three remaining examples where $\mathfrak{b}$ is not normal, one is easily constructed: for, if $\mathfrak{a}$ is a non-simple normal ideal, then $\mathfrak{b}=\mathfrak{a} \vee \mathfrak{a}'$ is not normal and contains $\mathfrak{a}$, so that we have an instance where $\mathfrak{a}$ is normal but not simple and $\mathfrak{b}$ is not normal. We note that none of the preceding fourteen examples breaks down except in the presence of an equality between the classes $\mathfrak{P}$, $\mathfrak{P}^*$, $\mathfrak{S}$, $\mathfrak{N}$, $\mathfrak{J}$ which automatically excludes it from consideration. We are thus left with seven further examples to construct.

We begin with the two remaining examples where $\mathfrak{b}$ is not normal. We first obtain an example where $\mathfrak{a}$ is simple but not semiprincipal, and $\mathfrak{b}$ is not normal. Clearly no such example exists unless $\mathfrak{P}^* \neq \mathfrak{S}$ and $\mathfrak{N} \neq \mathfrak{J}$. We therefore assume these relations. Let $\mathfrak{c}_1$ be non-normal, $\mathfrak{c}_2$ simple but not semiprincipal. Then $\mathfrak{c}_2'$ is also simple but not semiprincipal. If $\mathfrak{c}_1 \vee \mathfrak{c}_2$ and $\mathfrak{c}_1 \vee \mathfrak{c}_2'$ were both normal, Th. 1.2 would show that $\mathfrak{c}_1=(\mathfrak{c}_1 \vee \mathfrak{c}_2)(\mathfrak{c}_1 \vee \mathfrak{c}_2')$ is also normal against hypothesis. Hence one of these ideals is not normal; and we may arrange our notation so that the first is not normal. The desired example is then obtained by putting $\mathfrak{a}=\mathfrak{c}_2$, $\mathfrak{b}=\mathfrak{c}_1 \vee \mathfrak{c}_2$. Next we assume $\mathfrak{P} \neq \mathfrak{P}^*$, $\mathfrak{N} \neq \mathfrak{J}$, and give an example where $\mathfrak{a}$ is semiprincipal but not principal and $\mathfrak{b}$ is not normal. We choose $\mathfrak{c}$ as a non-normal ideal and a as an element in $\mathfrak{c}''$ but not in $\mathfrak{c}$. The ideal $\mathfrak{a}'(a)$ is semiprincipal but not principal, by R Th. 25. The ideal $\mathfrak{c} \vee \mathfrak{a}'(a)$ cannot be normal: for, if it were, we should have

$$\mathfrak{c}'' \vee \mathfrak{a}'(a) \supset \mathfrak{c} \vee \mathfrak{a}'(a)=(\mathfrak{c} \vee \mathfrak{a}'(a))''=(\mathfrak{c}'\mathfrak{a}(a))' \supset \mathfrak{c}'' \vee \mathfrak{a}'(a),\ \mathfrak{c} \vee \mathfrak{a}'(a)=\mathfrak{c}'' \vee \mathfrak{a}'(a),$$

and hence $\mathfrak{c} \supset \mathfrak{a}(a)\mathfrak{c}=\mathfrak{a}(a)(\mathfrak{c} \vee \mathfrak{a}'(a))=\mathfrak{a}(a)(\mathfrak{c}'' \vee \mathfrak{a}'(a))=\mathfrak{a}(a)\mathfrak{c}''=\mathfrak{a}(a)$, contrary to hypothesis. Thus we obtain the desired example by putting $\mathfrak{a}=\mathfrak{a}'(a)$, $\mathfrak{b}=\mathfrak{c} \vee \mathfrak{a}'(a)$.

We pass next to the two remaining examples where $\mathfrak{b}$ is normal but not simple. First, we assume $\mathfrak{P}^*\neq\mathfrak{S}$, $\mathfrak{S}\neq\mathfrak{N}$, and give an example where $\mathfrak{a}$ is simple but not semiprincipal and $\mathfrak{b}$ is normal but not simple. We choose $\mathfrak{c}_1$ in $\mathfrak{S}\Delta\mathfrak{P}^*$ and $\mathfrak{c}_2$ in $\mathfrak{N}\Delta\mathfrak{S}$. The ideals $\mathfrak{c}_1\vee\mathfrak{c}_2$ and $\mathfrak{c}_1'\vee\mathfrak{c}_2$ cannot both be simple; for, if they were, the ideal $\mathfrak{c}_2=(\mathfrak{c}_1\vee\mathfrak{c}_2)(\mathfrak{c}_1'\vee\mathfrak{c}_2)$ would be simple by Th. 1.2, against hypothesis. On the other hand, both are normal by Th. 1.2. Since both $\mathfrak{c}_1$ and $\mathfrak{c}_1'$ are simple but not semiprincipal, we may adjust our notation so that $\mathfrak{c}_1\vee\mathfrak{c}_2$ is normal but not simple. We then obtain the desired example by putting $\mathfrak{a}=\mathfrak{c}_1$ and $\mathfrak{b}=\mathfrak{c}_1\vee\mathfrak{c}_2$. Next we assume that $\mathfrak{P}\neq\mathfrak{P}^*$, $\mathfrak{S}\neq\mathfrak{N}$, and construct an example of a semiprincipal, but not principal, ideal contained in a non-simple normal ideal. We choose $\mathfrak{c}$ as a non-simple normal ideal and a as an element which is not in the ideal $\mathfrak{c}\vee\mathfrak{c}'\neq\mathfrak{e}$. Then the ideal $\mathfrak{c}\vee\mathfrak{a}'(a)$ is normal by Th. 1.2, but cannot be simple since, if it were, we should have

$$\mathfrak{a}(a)=\mathfrak{a}(a)\mathfrak{e}=\mathfrak{a}(a)((\mathfrak{c}\vee\mathfrak{a}'(a))\vee(\mathfrak{c}\vee\mathfrak{a}'(a))')=$$
$$=\mathfrak{a}(a)(\mathfrak{c}\vee\mathfrak{a}'(a)\vee\mathfrak{c}'\mathfrak{a}(a))=\mathfrak{a}(a)(\mathfrak{c}\vee\mathfrak{c}')\subset\mathfrak{c}\vee\mathfrak{c}',$$

against hypothesis. Hence we obtain the desired example by putting $\mathfrak{a}=\mathfrak{a}'(a)$, $\mathfrak{b}=\mathfrak{c}\vee\mathfrak{a}'(a)$.

We consider next the two remaining examples where $\mathfrak{b}$ is simple but not semiprincipal. First we assume that $\mathfrak{N}\neq\mathfrak{J}$, $\mathfrak{P}^*\neq\mathfrak{S}$ and give an example where $\mathfrak{a}$ is non-normal and $\mathfrak{b}$ is simple but not semiprincipal. If $\mathfrak{c}_1$ is a non-simple ideal and $\mathfrak{c}_2$ is a non-semiprincipal simple ideal, the argument used in the discussion of the exception (3) can be applied to show that $(\mathfrak{c}_1\vee\mathfrak{c}_1')\mathfrak{c}_2$ and $(\mathfrak{c}_1\vee\mathfrak{c}_1')\mathfrak{c}_2'$ are not both normal: for if they were we would have $\mathfrak{c}_1\vee\mathfrak{c}_1'\supset\mathfrak{c}_2\vee\mathfrak{c}_2'=\mathfrak{e}$ against hypothesis. Since $\mathfrak{c}_2'$ is simple but not semiprincipal we may suppose our notation so chosen that $(\mathfrak{c}_1\vee\mathfrak{c}_1')\mathfrak{c}_2$ is not normal. We then obtain the desired example by putting $\mathfrak{a}=(\mathfrak{c}_1\vee\mathfrak{c}_1')\mathfrak{c}_2$, $\mathfrak{b}=\mathfrak{c}_2$. Next, we assume that $\mathfrak{N}\neq\mathfrak{S}$, $\mathfrak{S}\neq\mathfrak{P}^*$ and give an example where $\mathfrak{a}$ is normal but not simple and $\mathfrak{b}$ is simple but not semiprincipal. We choose $\mathfrak{c}_1$ as a non-simple normal ideal and $\mathfrak{c}_2$ as a non-semiprincipal simple ideal. Then $\mathfrak{c}_1'$ is also simple but not semiprincipal. The ideals $\mathfrak{c}_1\mathfrak{c}_2$, $\mathfrak{c}_1\mathfrak{c}_2'$ are both normal by Th. 1.2; but they cannot both be simple since, if they were, we would have $\mathfrak{c}_1=\mathfrak{c}_1\mathfrak{e}=\mathfrak{c}_1(\mathfrak{c}_2\vee\mathfrak{c}_2')=\mathfrak{c}_1\mathfrak{c}_2\vee\mathfrak{c}_1\mathfrak{c}_2'$ so that $\mathfrak{c}_1$ would be simple contrary to hypothesis. We may suppose our notation adjusted so that $\mathfrak{c}_1\mathfrak{c}_2$ is not simple, replacing $\mathfrak{c}_2$ by $\mathfrak{c}_2'$ if necessary. We then obtain the desired example by putting $\mathfrak{a}=\mathfrak{c}_1\mathfrak{c}_2$, $\mathfrak{b}=\mathfrak{c}_2$.

Finally, we come to the one remaining case where $\mathfrak{b}$ is principal. We assume that $\mathfrak{J}\neq\mathfrak{N}$ and give an example where $\mathfrak{a}$ is non-normal and $\mathfrak{b}$ is principal. If $\mathfrak{c}$ is a non-normal ideal, then $\mathfrak{c}\vee\mathfrak{c}'\neq\mathfrak{e}$ since $\mathfrak{c}$ is certainly not simple. If c is an element not in $\mathfrak{c}\vee\mathfrak{c}'$, then the ideal $(\mathfrak{c}\vee\mathfrak{c}')\mathfrak{a}(c)$ cannot be normal: for, if it were, we would have $\mathfrak{c}\vee\mathfrak{c}'\supset(\mathfrak{c}\vee\mathfrak{c}')\mathfrak{a}(c)=\mathfrak{a}(c)$, contrary to hypothesis, by virtue of the relations $(\mathfrak{c}\vee\mathfrak{c}')\mathfrak{a}(c)=((\mathfrak{c}\vee\mathfrak{c}')\mathfrak{a}(c))''\mathfrak{a}(c)=(\mathfrak{c}\vee\mathfrak{c}')''\mathfrak{a}(c)=\mathfrak{e}\mathfrak{a}(c)$. Hence we obtain the desired example by putting $\mathfrak{a}=(\mathfrak{c}\vee\mathfrak{c}')\mathfrak{a}(c)$ and $\mathfrak{b}=\mathfrak{a}(c)$.

Theorem 12.2. *The tables of Th.* 1.2 *give best possible results with the following exceptions:*

(1) *the tables are to be reduced whenever there is any relation of equality between the classes* $\mathfrak{P}$, $\mathfrak{P}^*$, $\mathfrak{S}$, $\mathfrak{N}$, $\mathfrak{J}$;

(2) *in any ring of type* (β_3^*) *or type* (β_3^*, β_3), $\mathfrak{a}\mathfrak{b}$ *is normal whenever* $\mathfrak{a}$ *is; and* $\mathfrak{a}'$ *is simple whenever* $\mathfrak{a}$ *is not normal.*

In (2) *we have further algebraic characterizations of the two types* (β_3^*), (β_3^*, β_3).

The elementary properties of the three classes $\mathfrak{P}$, $\mathfrak{P}^*$, $\mathfrak{S}$ show that the first three entries of the first table can be improved only by assuming some equality between these classes. If $\mathfrak{a}$ is normal but not simple, then so is $\mathfrak{a}'$ since $\mathfrak{a}'\vee\mathfrak{a}''=\mathfrak{a}\vee\mathfrak{a}'\neq\mathfrak{e}$. Hence the last two entries can be improved, in view of the inclusion $\mathfrak{N}\subset\mathfrak{J}$, only by assuming an equality between the classes $\mathfrak{N}$ and $\mathfrak{S}$. In a ring of type (β_3^*) or type (β_3^*, β_3), the relation $\mathfrak{a}\subset\mathfrak{a}''$, in which $\mathfrak{a}''$ is known to be normal, shows in accordance with Th. 12.1 (3) that, if $\mathfrak{a}$ is not normal, then $\mathfrak{a}''$, and hence also $\mathfrak{a}'$, is simple. On the other hand a ring with $\mathfrak{N}\neq\mathfrak{S}$ in which $\mathfrak{a}\,\epsilon\,\mathfrak{J}\Delta\mathfrak{N}$ implies $\mathfrak{a}'\,\epsilon\,\mathfrak{S}$ is necessarily of type (β_3^*) or type (β_3^*, β_3). This we prove by showing that in such a ring there is exactly one non-normal prime ideal and then applying Th. 7.2. By Th. 11.1, the sum $\mathfrak{q}$ of all non-simple normal ideals is either prime or equal to $\mathfrak{e}$. Hence if the ring contains two distinct non-normal prime ideals $\mathfrak{p}_1$, $\mathfrak{p}_2$, their product cannot contain $\mathfrak{q}$. Hence one of them, let us say $\mathfrak{p}_1$, does not contain $\mathfrak{q}$; and there must exist a non-simple normal ideal $\mathfrak{a}$ which is not contained in $\mathfrak{p}_1$. The ideal $\mathfrak{a}\mathfrak{p}_1$ is contained in $\mathfrak{a}$ so that $(\mathfrak{a}\mathfrak{p}_1)'\supset\mathfrak{a}'$. On the other hand the relation $(\mathfrak{a}\mathfrak{p}_1)'\mathfrak{a}\mathfrak{p}_1=\mathfrak{o}$ implies $(\mathfrak{a}\mathfrak{p}_1)'\mathfrak{a}\subset\mathfrak{p}_1'=\mathfrak{o}$ and hence $(\mathfrak{a}\mathfrak{p}_1)'\subset\mathfrak{a}'$. We thus find that $(\mathfrak{a}\mathfrak{p}_1)'=\mathfrak{a}'$, $(\mathfrak{a}\mathfrak{p}_1)''=\mathfrak{a}''=\mathfrak{a}\neq\mathfrak{a}\mathfrak{p}_1$. Thus the ideal $\mathfrak{a}\mathfrak{p}_1$ is not normal and the ideal $(\mathfrak{a}\mathfrak{p}_1)'=\mathfrak{a}'$ is normal but not simple.

Since this result contradicts our hypothesis, we conclude that there exists at most one non-normal prime ideal. The relation $\mathfrak{N} \neq \mathfrak{S}$ together with Th. 10.2 shows that at least one non-normal prime ideal exists. This completes the discussion.

To show that the entries in the third table are the best possible we apply Th. 12.1. Since $\mathfrak{a}\mathfrak{b}=\mathfrak{b}\mathfrak{a}$ we may confine our attention to the entries on or below the principal diagonal. We take $\mathfrak{a} \subset \mathfrak{b}$ so that $\mathfrak{a}\mathfrak{b}=\mathfrak{a}$ and assign $\mathfrak{a}$ and $\mathfrak{b}$ to such classes as are called for by the various entries in the table. Thus, unless we encounter one of the special exceptions noted under Th. 12.1, we obtain precisely the entries given in the second table and conclude that all these entries, except possibly the three entries "$\mathfrak{P}$" which have to be made in the first column because of the condition $\mathfrak{a} \subset \mathfrak{b}$, are the best possible. Even in the general case where $\mathfrak{a}$ is not assumed to be contained in $\mathfrak{b}$, an entry "$\mathfrak{P}$" is obviously the best possible. Hence the general table gives best possible results unless there are relations of equality between some of the class $\mathfrak{P}$, $\mathfrak{P}^*$, $\mathfrak{S}$, $\mathfrak{N}$, $\mathfrak{J}$ or, possibly, unless the ring considered is of type (β_3^*) or type (β_3^*, β_3). That an improvement can be made in the latter case is shown as follows: if $\mathfrak{a}$ is normal and $\mathfrak{b}$ arbitrary, then $\mathfrak{a}\mathfrak{b} \subset \mathfrak{a}$ implies that $\mathfrak{a}\mathfrak{b}$ must be normal by Th. 12.1. (3).

We proceed similarly in the case of the second table. Since $\mathfrak{a} \vee \mathfrak{b} = \mathfrak{b} \vee \mathfrak{a}$, we may confine our attention to entries on or above the principal diagonal. If we take $\mathfrak{a} \subset \mathfrak{b}$, we have $\mathfrak{a} \vee \mathfrak{b} = \mathfrak{b}$. On assigning $\mathfrak{a}$ and $\mathfrak{b}$ to the various classes called for by the various entries in the table, we obtain all entries as given in Th. 1.2 except the entry $\mathfrak{J}$ for the case where $\mathfrak{a}$ and $\mathfrak{b}$ are normal; and the only possibility of improvement occurs when there is a relation of equality between some of the classes $\mathfrak{P}$, $\mathfrak{P}^*$, $\mathfrak{S}$, $\mathfrak{N}$, $\mathfrak{J}$. Now if $\mathfrak{a}$ is normal but not simple, $\mathfrak{a}'$ has the same properties and $\mathfrak{a} \vee \mathfrak{a}'$ is not normal since $(\mathfrak{a} \vee \mathfrak{a}')'' = (\mathfrak{a}'\mathfrak{a}'')' = \mathfrak{o}' = \mathfrak{e} \neq \mathfrak{a} \vee \mathfrak{a}'$. Thus the exceptional entry is also the best possible. It should be observed that the exceptional case (3) of Th. 12.1 does not cause trouble, since we do not work below the diagonal.

In Ths. 12.1 and 12.2 we arranged the proofs so that we could ascertain the precise effect of each possible equality between any two of the classes $\mathfrak{P}$, $\mathfrak{P}^*$, $\mathfrak{S}$, $\mathfrak{N}$, $\mathfrak{J}$. In the succeeding theorems our analysis will not be quite so detailed.

Theorem 12.3. *The tables of Th.* 1.3 *give best possible results with the following exceptions:*

(1) *the tables are to be reduced whenever there is any relation of equality between the classes* $\mathfrak{P}$, $\mathfrak{P}^*$, $\mathfrak{S}$, $\mathfrak{N}$, $\mathfrak{J}$;

(2) *in a ring of type* (β_3^*), *the product* $\mathfrak{a}\mathfrak{b}$ *of a principal ideal* $\mathfrak{a}$ *with an arbitrary ideal* $\mathfrak{b}$ *is semiprincipal relative to* $\mathfrak{b}$;

(3) *in a ring of type* (β_2^*), *the product* $\mathfrak{a}\mathfrak{b}$ *of a principal ideal* $\mathfrak{a}$ *with a normal ideal* $\mathfrak{b}$ *is semiprincipal relative to* $\mathfrak{b}$; *but the product* $\mathfrak{a}\mathfrak{b}$ *of a principal ideal* $\mathfrak{a}$ *with a non-normal ideal* $\mathfrak{b}$ *may be non-semiprincipal relative to* $\mathfrak{b}$.

In (2) *and* (3) *we have further algebraic characterizations of the rings of the respective types* (β_3^*) *and* (β_1^*).

We observe that, because of the equivalence of the relations $\mathfrak{a} \subset \mathfrak{b}$, $\mathfrak{a} \vee \mathfrak{b} = \mathfrak{b}$, $\mathfrak{a}\mathfrak{b} = \mathfrak{a}$, the first table is a special case of the two others. Hence an entry in the second or third table which is the same as the corresponding entry in the first can be improved only in case the latter can be improved. Thus we see that it is sufficient for us to consider the first table, the last two entries in the first row of the second, and the second and third entries in the first column of the third.

The entries "$\mathfrak{P}$" in the first table are obviously the best possible. Since $\mathfrak{a}$ is principal relative to $\mathfrak{b}$ if and only if it is principal, Ths. 1.1 and 12.1 show that no further entries "$\mathfrak{P}$" can be made in the first table. Thus the entries "$\mathfrak{P}^*$" in the first table are the best possible also. The entries "$\mathfrak{P}^*$" in the second and third places of the first column of the third table are also the best possible, since, on taking $\mathfrak{a}$ as a non-principal ideal which is either semiprincipal or simple and $\mathfrak{b}$ as a principal ideal contained in $\mathfrak{a}$, we see that $\mathfrak{a}$ is not principal in $\mathfrak{a} \vee \mathfrak{b} = \mathfrak{a}$. In view of the inclusion relations $\mathfrak{P} \subset \mathfrak{P}^* \subset \mathfrak{S} \subset \mathfrak{N} \subset \mathfrak{J}$, no improvement in the last row of the first table can be made without improving the first entry. That this entry is the best possible we prove as follows. By Th. 12.1, we choose $\mathfrak{b}$ as a principal ideal and $\mathfrak{a}$ as a non-normal ideal contained in $\mathfrak{b}$. If $\mathfrak{a}$ were normal relative to $\mathfrak{b}$, we would have $\mathfrak{a} = \mathfrak{a}''\mathfrak{b}$ and would conclude that $\mathfrak{a}$, as the product of the normal ideals $\mathfrak{a}''$ and $\mathfrak{b}$, is normal. We show similarly that the entries in the fourth row are the best possible, by examination of the first. Taking $\mathfrak{b}$ as a principal ideal and $\mathfrak{a}$ as a non-simple normal ideal contained in $\mathfrak{b}$, we see immediately that $\mathfrak{a}$ cannot be simple relative to $\mathfrak{b}$: for, if it were, we would have $\mathfrak{a}\vee\mathfrak{a}'=\mathfrak{a}\vee\mathfrak{a}'\mathfrak{b}\vee\mathfrak{a}'\mathfrak{b}'=\mathfrak{b}\vee\mathfrak{a}'\mathfrak{b}'=\mathfrak{b}\vee\mathfrak{b}'=\mathfrak{e}$ since $\mathfrak{a}\subset\mathfrak{b}$ implies $\mathfrak{a}'\supset\mathfrak{b}'$. We show similarly that the last four entries in the third row are the best possible, by treating the first of them. We take $\mathfrak{b}$ as a non-principal semiprincipal ideal and $\mathfrak{a}$ as a non-semiprincipal simple ideal contained in $\mathfrak{b}$. Then $\mathfrak{a}'$ is simple but not semiprincipal and $\mathfrak{b}'$ is principal. If $\mathfrak{a}$ were semiprincipal relative to $\mathfrak{b}$, then $\mathfrak{a}'\mathfrak{b}$ would be principal since $\mathfrak{a}$ is not; and the fact that $\mathfrak{b}'$ is principal would show that $\mathfrak{a}'\mathfrak{b}'$ and $\mathfrak{a}'=\mathfrak{a}'\mathfrak{b}\vee\mathfrak{a}'\mathfrak{b}'$ are principal. We thus reach a contradiction, since $\mathfrak{a}'$ is not principal. Finally we show that the last two entries in the second row are the best possible, by treating the first of them. We take $\mathfrak{b}$ as a non-simple normal ideal and $\mathfrak{a}$ as a non-principal semiprincipal ideal contained in $\mathfrak{b}$. If $\mathfrak{a}$ were semiprincipal relative to $\mathfrak{b}$, then $\mathfrak{a}'\mathfrak{b}$ would be principal; and it would follow that $\mathfrak{b}=\mathfrak{b}(\mathfrak{a}\vee\mathfrak{a}')=\mathfrak{a}\vee\mathfrak{a}'\mathfrak{b}$, as the sum of semiprincipal ideals, is semiprincipal. We are thus left to consider the exceptional entries in the second table.

We discuss next the construction of an example of a principal ideal $\mathfrak{a}=\mathfrak{a}(a)$ and a non-simple normal ideal $\mathfrak{b}$ such that $\mathfrak{a}\mathfrak{b}$ is not semiprincipal relative to $\mathfrak{b}$. Assuming $\mathfrak{R}\neq\mathfrak{S}$, we select a non-simple normal ideal $\mathfrak{b}$. First, let us suppose that this can be done in such a way that $\mathfrak{b}\vee\mathfrak{b}'$ is not prime. Then there exists an element a not in $\mathfrak{b}\vee\mathfrak{b}'$ such that $\mathfrak{a}(a)\vee\mathfrak{b}\vee\mathfrak{b}'\neq\mathfrak{e}$. It follows at once that $\mathfrak{a}'(a)\vee\mathfrak{b}\vee\mathfrak{b}'\neq\mathfrak{e}$: for otherwise we would have

$$\mathfrak{a}(a)\vee\mathfrak{b}\vee\mathfrak{b}'=(\mathfrak{a}(a)\vee\mathfrak{b}\vee\mathfrak{b}')\mathfrak{e}=(\mathfrak{a}(a)\vee\mathfrak{b}\vee\mathfrak{b}')(\mathfrak{a}'(a)\vee\mathfrak{b}\vee\mathfrak{b}')=\mathfrak{b}\vee\mathfrak{b}',$$

contrary to our choice of a. If $\mathfrak{a}(a)\mathfrak{b}$ were principal, then the ideal $\mathfrak{a}'(a)\vee\mathfrak{b}\vee\mathfrak{b}'=\mathfrak{a}'(a)\vee\mathfrak{b}'\vee\mathfrak{a}(a)\mathfrak{b}$ would be normal by Th. 1.2; and we would find that $\mathfrak{a}'(a)\vee\mathfrak{b}\vee\mathfrak{b}'=(\mathfrak{a}'(a)\vee\mathfrak{b}\vee\mathfrak{b}')''=(\mathfrak{a}(a)\mathfrak{b}'\mathfrak{b}'')'=\mathfrak{o}'=\mathfrak{e}$. If $\mathfrak{a}'(a)\mathfrak{b}$ were principal we would have $\mathfrak{a}(a)\vee\mathfrak{b}\vee\mathfrak{b}'=\mathfrak{e}$ in the same way. Thus we can obtain the desired example by putting $\mathfrak{a}=\mathfrak{a}(a)$ and using the ideal $\mathfrak{b}$ originally selected. If now we find two non-simple normal ideals $\mathfrak{b}_1$ and $\mathfrak{b}_2$ such that $\mathfrak{b}_1\vee\mathfrak{b}_1'$ and $\mathfrak{b}_2\vee\mathfrak{b}_2'$ are distinct but both prime, we can construct a non-simple normal ideal $\mathfrak{b}$ such that $\mathfrak{b}\vee\mathfrak{b}'$ is not prime. Since $\mathfrak{b}_1$, $\mathfrak{b}_1'$, $\mathfrak{b}_2$, $\mathfrak{b}_2'$ are all non-simple normal ideals, since $\mathfrak{p}_1=\mathfrak{b}_1\vee\mathfrak{b}_1'$ cannot contain both $\mathfrak{b}_2$ and $\mathfrak{b}_2'$, and since $\mathfrak{p}_2=\mathfrak{b}_2\vee\mathfrak{b}_2'$ cannot contain both $\mathfrak{b}_1$ and $\mathfrak{b}_1'$, we can choose our notation so that $\mathfrak{p}_1\vee\mathfrak{b}_2=\mathfrak{e}$, $\mathfrak{p}_2\vee\mathfrak{b}_1=\mathfrak{e}$. The ideal $\mathfrak{b}=\mathfrak{b}_1\mathfrak{b}_2$ is normal. We can show that $(\mathfrak{b}_1\mathfrak{b}_2)'$ is contained in $\mathfrak{p}_1\mathfrak{p}_2$. By symmetry, it is sufficient to prove that $(\mathfrak{b}_1\mathfrak{b}_2)'\subset\mathfrak{p}_1$ Let c be an element of $(\mathfrak{b}_1\mathfrak{b}_2)'$. Then c, as an element of $\mathfrak{p}_1\vee\mathfrak{b}_2=\mathfrak{e}$, can be expressed in the form $c=a\vee b$ where $a\,\epsilon\,\mathfrak{p}_1$, $b\,\epsilon\,\mathfrak{b}_2$. Since $b<c\,\epsilon\,(\mathfrak{b}_1\mathfrak{b}_2)'$, we have $b\,\epsilon\,(\mathfrak{b}_1\mathfrak{b}_2)'$; and since $\mathfrak{a}(b)\subset\mathfrak{b}_2$, we have $\mathfrak{a}(b)\mathfrak{b}_1=\mathfrak{a}(b)\mathfrak{b}_1\mathfrak{b}_2=\mathfrak{o}$, $\mathfrak{a}(b)\subset\mathfrak{b}_1'\subset\mathfrak{p}_1$. We see therefore that $b\,\epsilon\,\mathfrak{p}_1$, $c\,\epsilon\,\mathfrak{p}_1$, and $(\mathfrak{b}_1\mathfrak{b}_2)'\subset\mathfrak{p}_1$. The ideal $\mathfrak{b}=\mathfrak{b}_1\mathfrak{b}_2$ thus satisfies the relation $\mathfrak{b}\vee\mathfrak{b}'\subset\mathfrak{p}_1\mathfrak{p}_2$ so that $\mathfrak{b}$ is not simple and $\mathfrak{b}\vee\mathfrak{b}'$ is not prime. We have thus settled all cases except that where there exists a prime ideal $\mathfrak{p}$ such that $\mathfrak{b}\vee\mathfrak{b}'=\mathfrak{p}$ for every non-simple normal ideal $\mathfrak{b}$. Even in this case we can still obtain the desired example if the given ring has no unit. Let a be an element not in $\mathfrak{p}$. Then the ideal $\mathfrak{a}'(a)$ is not principal and is contained in $\mathfrak{p}$. Hence if $\mathfrak{b}$ is a non-simple normal ideal, the ideals $\mathfrak{a}'(a)\mathfrak{b}$ and $\mathfrak{a}'(a)\mathfrak{b}'$ cannot both be principal: for, if they were, $\mathfrak{a}'(a)=\mathfrak{a}'(a)\mathfrak{p}=\mathfrak{a}'(a)\mathfrak{b}\vee\mathfrak{a}'(a)\mathfrak{b}'$ would be principal. We may suppose that $\mathfrak{a}'(a)\mathfrak{b}$ is not principal. If $\mathfrak{a}(a)\mathfrak{b}$ were principal it would be principal relative to $\mathfrak{a}(a)$. Thus we would find that its orthocomplement $\mathfrak{a}(a)\mathfrak{b}'$ relative to the ring $\mathfrak{a}(a)$ with unit is principal. We could then conclude that $\mathfrak{a}(a)\mathfrak{p}=\mathfrak{a}(a)\mathfrak{b}\vee\mathfrak{a}(a)\mathfrak{b}'$ is principal. On the other hand, since $\mathfrak{p}'=(\mathfrak{b}\vee\mathfrak{b}')'=\mathfrak{b}'\mathfrak{b}''=\mathfrak{o}$, $\mathfrak{p}$ is not normal and Th. 1.6 (2) shows that $\mathfrak{a}(a)\mathfrak{p}$ is not normal. Hence we see that neither $\mathfrak{a}'(a)\mathfrak{b}$ nor $\mathfrak{a}(a)\mathfrak{b}$ is principal. We can therefore obtain the desired example by putting $\mathfrak{a}=\mathfrak{a}(a)$ and using the ideal $\mathfrak{b}$. Hence the only possibility of improving the fourth entry in the first row of the second table occurs in the case of a ring with unit in which there is a prime ideal $\mathfrak{p}$ such that $\mathfrak{p}=\mathfrak{b}\vee\mathfrak{b}'$ for every non-simple normal ideal $\mathfrak{b}$. By reference to Ths. 7.1-7.4 and 11.1, we see that such a ring must be of type (β_3^*) or type (β_2^*).

We conclude by showing that simplifications of the second table occur for rings of these types. Every non-simple ideal in a ring of type (β_3^*) satisfies the relation $\mathfrak{p}=\mathfrak{b}\vee\mathfrak{b}'$, where $\mathfrak{p}$ is the prime ideal of Th. 7.1 (1). If a is an arbitrary element, then one of the elements a and a' is in $\mathfrak{p}$; hence either $\mathfrak{a}(a)\mathfrak{p}$ or $\mathfrak{a}(a')\mathfrak{p}=\mathfrak{a}'(a)\mathfrak{p}$ is principal in $\mathfrak{p}$ and $\mathfrak{a}(a)\mathfrak{p}$ is semiprincipal relative to $\mathfrak{p}$. In particular, if a is not in $\mathfrak{p}$, then $\mathfrak{a}(a)\mathfrak{p}$ is semiprincipal but not principal relative to $\mathfrak{p}$ by Th. 1.8. Considered as an ideal in $\mathfrak{p}$, the ideal $\mathfrak{b}$ is simple. Hence the ideal $\mathfrak{a}(a)\mathfrak{b}=\mathfrak{a}(a)\mathfrak{p}\mathfrak{b}$ is semiprincipal relative to $\mathfrak{b}$ by an application of Th. 1.3 within the ring $\mathfrak{p}$. Every non-simple normal ideal $\mathfrak{b}$ in a ring of type (β_2^*) satisfies the relation $\mathfrak{p}=\mathfrak{b}\vee\mathfrak{b}'$, where $\mathfrak{p}$ is the prime ideal of Th. 7.3 (1). Hence the argument used above shows that $\mathfrak{a}(a)\mathfrak{b}$ is semiprincipal relative to $\mathfrak{b}$ for every a and every such $\mathfrak{b}$. If we apply the preceding work to the ring $\mathfrak{p}$, for which $\mathfrak{P}\neq\mathfrak{P}^*\neq\mathfrak{S}=\mathfrak{R}$, in both cases, we find that a and $\mathfrak{b}$ can be chosen so that $\mathfrak{a}(a)\mathfrak{b}$ is not principal: for every non-principal semiprincipal ideal in $\mathfrak{p}$ is representable in the form $\mathfrak{a}(a)\mathfrak{p}$, $a \epsilon \mathfrak{p}$, by proper choice of a' in $\mathfrak{p}$. In a ring of type (β_2^*), there exist a principal ideal $\mathfrak{a}(a)$ and a non-normal ideal $\mathfrak{b}$ such that $\mathfrak{a}(a)\mathfrak{b}$ is not semiprincipal relative to $\mathfrak{b}$. By Th. 7.2, there is a non-normal prime ideal $\mathfrak{q}$ distinct from $\mathfrak{p}$. If we take a in $\mathfrak{q}$ but not in $\mathfrak{p}$, we obtain the desired example by putting $\mathfrak{a}=\mathfrak{a}(a)$, $\mathfrak{b}=\mathfrak{p}\mathfrak{q}$. We then know that $\mathfrak{a}(a)\mathfrak{b}$ and $\mathfrak{a}(a')\mathfrak{b}=\mathfrak{a}'(a)\mathfrak{b}$ are simple relative to $\mathfrak{b}$. Considering $\mathfrak{a}(a)$ and $\mathfrak{b}$ as ideals in $\mathfrak{q}$, we know that $\mathfrak{a}(a)$ is not contained in $\mathfrak{b}=\mathfrak{p}\mathfrak{q}$ and that $\mathfrak{b}$ is a non-normal prime ideal relative to $\mathfrak{q}$ by Th. 1.7. Hence $\mathfrak{a}(a)\mathfrak{b}$ cannot be principal, by virtue of Th. 1.8. Since a' is in $\mathfrak{p}$ but not in $\mathfrak{q}$ a similar argument shows that $\mathfrak{a}(a')\mathfrak{b}$ cannot be principal. Hence $\mathfrak{a}(a)\mathfrak{b}$ is not semiprincipal relative to $\mathfrak{b}$. It follows that $\mathfrak{b}$ is not normal, a fact which can be verified directly also.

Theorem 12.4. *The table of Th.* 1.4 *gives best possible results except when reductions are made possible by equalities between some of the classes* $\mathfrak{P}, \mathfrak{P}^*, \mathfrak{S}, \mathfrak{N}, \mathfrak{F}$.

The entries "$\mathfrak{P}$" in the table are obviously the best possible. If we take $\mathfrak{b}$ as a non-principal ideal in any of the four classes $\mathfrak{P}^*, \mathfrak{S}, \mathfrak{N}, \mathfrak{J}$ and put $\mathfrak{a}=\mathfrak{b}$, we see that $\mathfrak{a}$ is semiprincipal but not principal relative to $\mathfrak{b}$ and belongs in A to the same class as $\mathfrak{b}$. Hence the entries in the second row of the table are the best possible; and so also the last three entries in the last column are the best possible. Since $\mathfrak{a}$ is normal (simple) relative to $\mathfrak{b}$ if it is normal (simple) in A, by Th. 1.3, it is evident that the first four entries in the fourth and fifth rows are the best possible. Since the class of simple ideals relative to $\mathfrak{b}$ contains the class of semiprincipal ideals relative to $\mathfrak{b}$ no improvement in the third and fourth entries of the third row can be made without improvement of the corresponding entries in the second row; but the latter are already known to be the best possible. To complete our discussion we have only to study the second entry in the third row. Using Th. 12.1, we choose $\mathfrak{b}$ as a non-principal semiprincipal ideal and $\mathfrak{a}$ as a non-semiprincipal simple ideal contained in $\mathfrak{b}$. Then $\mathfrak{a}$ is simple but not principal relative to $\mathfrak{b}$. In order to show that $\mathfrak{a}$ is not semiprincipal relative to $\mathfrak{b}$, we prove that $\mathfrak{a}'\mathfrak{b}$ is not principal: if it were we would have $\mathfrak{a}'=\mathfrak{a}'\mathfrak{b}\vee\mathfrak{a}'\mathfrak{b}'$, where $\mathfrak{a}'\mathfrak{b}'$ is principal since $\mathfrak{a}'$ is simple and $\mathfrak{b}'$ principal, and hence $\mathfrak{a}'$ would be principal against hypothesis. Thus $\mathfrak{a}$ and $\mathfrak{b}$ furnish the desired example.

Theorem 12.5. *Under the condition* $\mathfrak{a}\subset\mathfrak{b}$, *the ideal* $\mathfrak{b}$ *can be assigned to an arbitrary class, and the ideal* $\mathfrak{a}$ *to an arbitrary class relative to* $\mathfrak{b}$, *with only the following exceptions*:

(1) *the assignment is subject to the general restrictions given in Th.* 1.5;

(2) *the assignment is subject to the limitations imposed by equalities between the classes* $\mathfrak{P}, \mathfrak{P}^*, \mathfrak{S}, \mathfrak{N}, \mathfrak{J}$ *in the given ring*;

(3) *in a ring of type* (β_3^*) *or type* (β_3^*, β_3) *the choice of* $\mathfrak{a}$ *to be not simple relative to* $\mathfrak{b}$ *and of* $\mathfrak{b}$ *to be not normal is impossible*;

(4) *in a ring of type* (β_2^*), *type* (β_2^*, β_1) *or type* (β_2^*, β_2), *the choice of* $\mathfrak{a}$ *to be normal but not simple relative to* $\mathfrak{b}$ *and of* $\mathfrak{b}$ *to be normal but not simple is impossible.*

(5) *in a ring of type* (ω, ω) *the choice of* $\mathfrak{a}$ *to be simple but not semiprincipal relative to* $\mathfrak{b}$ *and of* $\mathfrak{b}$ *to be simple but not semiprincipal is impossible.*

In (3) *and* (4), *we have further algebraic characterizations of the various types* (β_3^*), (β_3^*, β_3), (β_2^*), (β_2^*, β_1) *and* (β_2^*, β_2).

Any relation of equality between the classes of ideals in the given ring will have an effect on the classes of ideals $\mathfrak{a}$ considered relative to a containing ideal $\mathfrak{b}$. It is our purpose to show that apart from this and the various other exceptions noted above, $\mathfrak{b}$ can be assigned to any of the five classes at pleasure and, independently, $\mathfrak{a}$ to any of the five classes relative to $\mathfrak{b}$.

We begin with the exceptional cases mentioned in (3), (4), (5). Assuming that $\mathfrak{J}\neq\mathfrak{N}$ and hence that $\mathfrak{J}\neq\mathfrak{S}$, we construct an example where $\mathfrak{b}$ is not normal and $\mathfrak{a}$ is not normal relative to $\mathfrak{b}$. In a ring with $\mathfrak{J}\neq\mathfrak{S}$, there exist ideals $\mathfrak{c}$ such that $\mathfrak{c}'=\mathfrak{o}$, $\mathfrak{c}\neq\mathfrak{e}=\mathfrak{c}''$: if $\mathfrak{a}$ is a non-simple ideal then $\mathfrak{c}=\mathfrak{a}\vee\mathfrak{a}'$ has these properties. Let $\mathfrak{b}$ and $\mathfrak{c}$ be two such non-normal ideals with $\mathfrak{b}\neq\mathfrak{c}$. By proper choice of notation we may suppose that $\mathfrak{b}\mathfrak{c}\neq\mathfrak{b}$. Considering $\mathfrak{a}=(\mathfrak{b}\mathfrak{c})\mathfrak{b}=\mathfrak{b}\mathfrak{c}$ an as ideal in $\mathfrak{b}$ we have $(\mathfrak{b}\mathfrak{c})''\mathfrak{b}=\mathfrak{b}\mathfrak{c}''=\mathfrak{b}$ so that $\mathfrak{a}$ is not normal relative to the non-normal ideal $\mathfrak{b}$. Thus we have the desired example unless there is only one non-normal ideal $\mathfrak{b}$ with $\mathfrak{b}'=\mathfrak{o}$. This ideal $\mathfrak{b}$ must be prime: for $\mathfrak{a}\supset\mathfrak{b}$ implies $\mathfrak{a}'\subset\mathfrak{b}'=\mathfrak{o}$, $\mathfrak{a}'=\mathfrak{o}$ so that either $\mathfrak{a}$ is non-normal and $\mathfrak{a}=\mathfrak{b}$ or $\mathfrak{a}$ is normal and $\mathfrak{a}=\mathfrak{a}''=\mathfrak{e}$. Now if $\mathfrak{a}$ is an arbitrary ideal then either $\mathfrak{a}\vee\mathfrak{a}'=\mathfrak{e}$ or $\mathfrak{a}\vee\mathfrak{a}'$ is a non-normal ideal with $(\mathfrak{a}\vee\mathfrak{a}')'=\mathfrak{o}$ so that $\mathfrak{a}\vee\mathfrak{a}'=\mathfrak{b}$. Hence $\mathfrak{b}$ has the properties demanded under (1) of Th. 7.1; and our exceptional case proves to fall under the case where the given ring is of type (β_3^*) or type (β_3^*, β_3). That all such rings are actually exceptional is easily proved. In fact we can prove that in such a ring the relation $\mathfrak{a}\subset\mathfrak{b}$ implies that $\mathfrak{a}$ is simple relative to $\mathfrak{b}$ whenever $\mathfrak{b}$ is not simple. If $\mathfrak{p}$ is the prime ideal of Th. 7.1 (1) and if $\mathfrak{b}$ is not simple we have $\mathfrak{b}\vee\mathfrak{b}'=\mathfrak{p}\subset\mathfrak{a}\vee\mathfrak{a}'$. Hence we obtain the relations $\mathfrak{a}\vee\mathfrak{a}'\mathfrak{b}=\mathfrak{a}\mathfrak{b}\vee\mathfrak{a}'\mathfrak{b}=(\mathfrak{a}\vee\mathfrak{a}')\mathfrak{b}\supset\mathfrak{p}\mathfrak{b}=\mathfrak{b}$, so that $\mathfrak{a}\vee\mathfrak{a}'\mathfrak{b}=\mathfrak{b}$ and $\mathfrak{a}$ is simple relative to $\mathfrak{b}$.

We next assume that $\mathfrak{J}\neq\mathfrak{N}$ and hence that $\mathfrak{J}\neq\mathfrak{S}$, and construct an example where $\mathfrak{a}$ is normal but not simple relative to $\mathfrak{b}$ and $\mathfrak{b}$ is not normal. We choose a normal ideal $\mathfrak{c}_1$ and a non-simple ideal $\mathfrak{c}_2$. Let us suppose that we can so choose them that $\mathfrak{c}_2\vee\mathfrak{c}_2'$ is not contained in $\mathfrak{c}_1\vee\mathfrak{c}_1'$. We then obtain the desired example by putting $\mathfrak{a}=\mathfrak{c}_1(\mathfrak{c}_2\vee\mathfrak{c}_2')$, $\mathfrak{b}=\mathfrak{c}_2\vee\mathfrak{c}_2'$: for $\mathfrak{a}$ is normal relative to $\mathfrak{b}$ by virtue of the fact that $\mathfrak{c}_1$ is normal; but $\mathfrak{a}$ is not simple relative to $\mathfrak{b}$ because of the relations $\mathfrak{a}\vee\mathfrak{a}'\mathfrak{b}=\mathfrak{c}_1(\mathfrak{c}_2\vee\mathfrak{c}_2')\vee\mathfrak{c}_1'(\mathfrak{c}_2\vee\mathfrak{c}_2')=(\mathfrak{c}_1\vee\mathfrak{c}_1')(\mathfrak{c}_2\vee\mathfrak{c}_2')\neq\mathfrak{c}_2\vee\mathfrak{c}_2'=\mathfrak{b}$. We now examine the conditions under which such a choice of $\mathfrak{c}_1$ and $\mathfrak{c}_2$ is impossible. Clearly, if every normal ideal is simple we shall have $\mathfrak{c}_1\vee\mathfrak{c}_1'=\mathfrak{e}\supset\mathfrak{c}_2\vee\mathfrak{c}_2'$. In this case we see that the given ring is totally multiplicative by Th. 2.3 (2). Hence any ideal $\mathfrak{b}$ is totally multiplicative, and an ideal $\mathfrak{a}$ normal relative to $\mathfrak{b}$ is necessarily simple relative to $\mathfrak{b}$, again by Th. 2.3 (3). Now let us assume that $\mathfrak{N}\neq\mathfrak{S}$. If we cannot effect the previous construction using non-simple normal ideals $\mathfrak{c}_1$ and $\mathfrak{c}_2$ it can only be because for all such ideals $\mathfrak{c}_1\vee\mathfrak{c}_1'=\mathfrak{c}_2\vee\mathfrak{c}_2'$. Even in this situation we can effect the previous construction using a non-simple normal ideal $\mathfrak{c}_1$ and a non-normal ideal $\mathfrak{c}_2$ unless $\mathfrak{c}_2\vee\mathfrak{c}_2'\subset\mathfrak{c}_1\vee\mathfrak{c}_1'$ for every non-normal ideal $\mathfrak{c}_2$ and every non-simple normal ideal $\mathfrak{c}_1$. Thus the only possible exceptional case, as we see by reference to Th. 11.1, is that where, first, the sum of all non-simple normal ideals is a prime ideal $\mathfrak{q}$, given here by $\mathfrak{q}=\mathfrak{c}_1\vee\mathfrak{c}_1'=\mathfrak{c}_2\vee\mathfrak{c}_2'\neq\mathfrak{e}$ when $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are such ideals, and, second, the sum of all non-normal ideals, being contained in $\mathfrak{q}$, coincides with it. By Th. 11.1 this case occurs only for rings of type (β_3^*) or type (β_3^*,β_3). From the preceding work we know that such rings actually constitute exceptional cases here.

We consider next the construction of an example where $\mathfrak{b}$ is normal but not simple and $\mathfrak{a}$ is normal but not simple relative to $\mathfrak{b}$. We assume that $\mathfrak{N}\neq\mathfrak{S}$, since no such example could be obtained otherwise. As in the preceding paragraph, we consider the case where non-simple normal ideals $\mathfrak{c}_1$ and $\mathfrak{c}_2$ can be obtained such that $\mathfrak{c}_1\vee\mathfrak{c}_1'$ does not contain $\mathfrak{c}_2\vee\mathfrak{c}_2'$. The ideals $\mathfrak{c}_1\mathfrak{c}_2$, $\mathfrak{c}_1'\mathfrak{c}_2'$ are normal in the given ring and also normal relative to $\mathfrak{c}_2$ and $\mathfrak{c}_2'$ respectively. However, $\mathfrak{c}_1\mathfrak{c}_2$ and $\mathfrak{c}_1'\mathfrak{c}_2'$ cannot both be simple relative to $\mathfrak{c}_2$ and $\mathfrak{c}_2'$ respectively: for, if they were, we should have $\mathfrak{c}_1\mathfrak{c}_2\vee\mathfrak{c}_1'\mathfrak{c}_2=\mathfrak{c}_2$, $\mathfrak{c}_1'\mathfrak{c}_2'\vee\mathfrak{c}_1\mathfrak{c}_2'=\mathfrak{c}_2'$ and hence $\mathfrak{c}_1\vee\mathfrak{c}_1'\supset(\mathfrak{c}_1\vee\mathfrak{c}_1')(\mathfrak{c}_2\vee\mathfrak{c}_2')=\mathfrak{c}_2'\vee\mathfrak{c}_2'$, contrary to hypothesis. We then obtain the desired example by putting $\mathfrak{a}=\mathfrak{c}_1\mathfrak{c}_2$, $\mathfrak{b}=\mathfrak{c}_2$ or $\mathfrak{a}=\mathfrak{c}_1'\mathfrak{c}_2'$, $\mathfrak{b}=\mathfrak{c}_2'$. We see therefore that our construction breaks down only in the case where the sum of all non-simple normal ideals is a prime ideal $\mathfrak{q}$. According to Th. 11.1 this situation occurs only for rings of one of the types (β_3^*), (β_3^*,β_3), (β_2^*), (β_2^*,β_1), (β_2^*,β_2). We know already that the first two of these types provide actual exceptional cases. We can easily show that the three remaining types do likewise. Let $\mathfrak{p}$ be the prime ideal of Th. 7.3 (1). Then if $\mathfrak{b}$ is normal but not simple, we have $\mathfrak{p}\subset\mathfrak{b}''\vee\mathfrak{b}'=\mathfrak{b}\vee\mathfrak{b}'\neq\mathfrak{e}$ and hence $\mathfrak{p}=\mathfrak{b}\vee\mathfrak{b}'$. Thus if $\mathfrak{a}$ is an arbitrary ideal normal relative to $\mathfrak{b}$, we have $\mathfrak{a}=\mathfrak{a}''\mathfrak{b}$, $\mathfrak{a}\vee\mathfrak{a}'\mathfrak{b}=(\mathfrak{a}''\vee\mathfrak{a}')\mathfrak{b}\supset\mathfrak{p}\mathfrak{b}=(\mathfrak{b}\vee\mathfrak{b}')\mathfrak{b}=\mathfrak{b}$ and hence $\mathfrak{a}\vee\mathfrak{a}'\mathfrak{b}=\mathfrak{b}$, so that $\mathfrak{a}$ is simple relative to $\mathfrak{b}$. This completes the discussion.

The exception (5) has already been established in Th. 8.2.

Of the nineteen examples which we still have to give, seventeen offer no difficulty. The five cases where $\mathfrak{a}$ is principal relative to $\mathfrak{b}$ are treated by assigning $\mathfrak{b}$ to any of the five classes $\mathfrak{P}$, $\mathfrak{P}^*$, $\mathfrak{N}$, $\mathfrak{S}$, $\mathfrak{J}$ at pleasure and taking $\mathfrak{a}$ as the principal ideal generated by an element of $\mathfrak{b}$. The four remaining cases where $\mathfrak{a}$ is semiprincipal relative to $\mathfrak{b}$ but not principal are treated by assigning $\mathfrak{b}$ to any of the four classes $\mathfrak{P}^*$, $\mathfrak{S}$, $\mathfrak{N}$, $\mathfrak{J}$ at pleasure and putting $\mathfrak{a}=\mathfrak{b}$. The four remaining cases where $\mathfrak{a}$ is not normal relative to $\mathfrak{b}$ are treated by assigning $\mathfrak{b}$ to one of the four classes $\mathfrak{P}$, $\mathfrak{P}^*$, $\mathfrak{S}$, $\mathfrak{N}$, at pleasure and choosing $\mathfrak{a}$ as a non-normal ideal contained in $\mathfrak{b}$: for according to Th. 1.4, $\mathfrak{a}$ cannot be normal relative to $\mathfrak{b}$ under these circumstances unless it is normal in the given ring. The three remaining cases where $\mathfrak{a}$ is normal but not simple relative to $\mathfrak{b}$ are treated by assigning $\mathfrak{b}$ to any of the classes $\mathfrak{P}$, $\mathfrak{P}^*$, $\mathfrak{S}$ at pleasure and choosing $\mathfrak{a}$ as a non-simple normal ideal contained in $\mathfrak{b}$: for $\mathfrak{a}$ is then normal relative to $\mathfrak{b}$ by Th. 1.3 but, by virtue of Th. 1.4, cannot be simple relative to $\mathfrak{b}$. In a ring for which $\mathfrak{P}^*\neq\mathfrak{S}$, we take $\mathfrak{a}$ as a non-semiprincipal simple ideal and put $\mathfrak{b}=\mathfrak{e}$, since $\mathfrak{e}$ is semiprincipal but not principal, in order to obtain an example where $\mathfrak{a}$ is simple but not semiprincipal relative to $\mathfrak{b}$ and $\mathfrak{b}$ is semiprincipal but not principal.

There are still two cases to be studied. First we consider the construction of an ideal $\mathfrak{a}$ which is simple but not semiprincipal relative to a non-simple normal ideal $\mathfrak{b}$. We choose a non-simple normal ideal $\mathfrak{c}_1$ and a non-semiprincipal simple ideal $\mathfrak{c}_2$ such that $\mathfrak{c}_1\subset\mathfrak{c}_2$. Since $\mathfrak{c}_2'$ is simple, $\mathfrak{c}_1\vee\mathfrak{c}_2'$ is normal. On the other hand $\mathfrak{c}_1\vee\mathfrak{c}_2'$ is not simple: for, if it were, we would have

$$\mathfrak{e}=(\mathfrak{c}_1\vee\mathfrak{c}_2')\vee(\mathfrak{c}_1\vee\mathfrak{c}_2')'=\mathfrak{c}_1\vee\mathfrak{c}_2'\vee\mathfrak{c}_1'\mathfrak{c}_2\subset\mathfrak{c}_1\vee\mathfrak{c}_1'$$

since $\mathfrak{c}_1\subset\mathfrak{c}_2$ implies $\mathfrak{c}_1'\supset\mathfrak{c}_2'$; and we would thus have $\mathfrak{e}=\mathfrak{c}_1\vee\mathfrak{c}_2'$ against hypothesis. Now $\mathfrak{c}_2'=\mathfrak{c}_2'(\mathfrak{c}_2'\vee\mathfrak{c}_1)$ is simple relative to $\mathfrak{c}_2'\vee\mathfrak{c}_1$; but neither $\mathfrak{c}_2'$ nor $\mathfrak{c}_2=\mathfrak{c}_2(\mathfrak{c}_2'\vee\mathfrak{c}_1)=\mathfrak{c}_2''(\mathfrak{c}_2'\vee\mathfrak{c}_1)$ can be principal. Hence we obtain the desired example by putting $\mathfrak{a}=\mathfrak{c}_2'$ and $\mathfrak{b}=\mathfrak{c}_1\vee\mathfrak{c}_2'$.

Finally we consider the construction of an ideal $\mathfrak{a}$ which is simple but not semiprincipal relative to a non-normal ideal $\mathfrak{b}$. We choose a non-normal ideal $\mathfrak{c}_1$ and a non-semiprincipal simple ideal $\mathfrak{c}_2$ such that $\mathfrak{c}_1\subset\mathfrak{c}_2$. The ideal $\mathfrak{c}_1\vee\mathfrak{c}_2'$ is not normal: for, if it were, $\mathfrak{c}_1=\mathfrak{c}_2(\mathfrak{c}_1\vee\mathfrak{c}_2')$ would be normal against hypothesis. Now the ideal $\mathfrak{c}_2'=\mathfrak{c}_2'(\mathfrak{c}_1\vee\mathfrak{c}_2')$ is simple relative to $\mathfrak{c}_1\vee\mathfrak{c}_2'$ but neither $\mathfrak{c}_2'$ nor $\mathfrak{c}_1=\mathfrak{c}_2(\mathfrak{c}_1\vee\mathfrak{c}_2')=\mathfrak{c}_2''(\mathfrak{c}_1\vee\mathfrak{c}_2')$ can be principal. Hence we obtain the desired example by putting $\mathfrak{a}=\mathfrak{c}_2'$, $\mathfrak{b}=\mathfrak{c}_1\vee\mathfrak{c}_2'$. This completes the proof of the theorem.

It is evident that Ths. 1.6, 1.7, and 1.8 give complete information on the topics discussed. The only question left open is that of the existence of a prime ideal of given class either containing or not containing an ideal $\mathfrak{a}$ of given class. In this connection we have:

Theorem 12.6. *Let $\mathfrak{a}$ be an ideal in a Boolean ring A with $\mathfrak{a}\neq\mathfrak{e}$. In order that $\mathfrak{a}$ have a normal prime divisor it is necessary and sufficient that $\mathfrak{a}'$ contain an atomic element; and in order that every prime ideal divisor of $\mathfrak{a}$ be normal it is necessary that $\mathfrak{a}=\mathfrak{s}'$ where $\mathfrak{s}$ is an atomic system. In order that there exist a normal prime ideal which is not a divisor of $\mathfrak{a}$ it is necessary and sufficient that $\mathfrak{a}$ contain an atomic element; and in order that every prime ideal which is not a divisor of $\mathfrak{a}$ be normal it is necessary that $\mathfrak{s}\subset\mathfrak{a}\subset\mathfrak{s}''$ where $\mathfrak{s}$ is an atomic system.*

By R Th. 38 we know that an ideal $\mathfrak{p}$ is prime and normal if and only if $\mathfrak{p}=\mathfrak{a}'(a)$ where a is an atomic element. If $\mathfrak{a}\subset\mathfrak{p}=\mathfrak{a}'(a)$, then $\mathfrak{a}'\supset\mathfrak{a}''(a)=\mathfrak{a}(a)$ so that $a\,\epsilon\,\mathfrak{a}'$; and, if $a\,\epsilon\,\mathfrak{a}'$, then $\mathfrak{a}(a)\subset\mathfrak{a}'$, $\mathfrak{p}=\mathfrak{a}'(a)\supset\mathfrak{a}''\supset\mathfrak{a}$ so that $\mathfrak{p}\supset\mathfrak{a}$. If $\mathfrak{p}$ does not contain $\mathfrak{a}$, then $\mathfrak{a}\vee\mathfrak{p}=\mathfrak{e}$ so that $\mathfrak{a}(a)=\mathfrak{a}(a)(\mathfrak{a}\vee\mathfrak{p})=\mathfrak{a}(a)\mathfrak{a}\vee\mathfrak{a}(a)\mathfrak{a}'(a)=\mathfrak{a}(a)\mathfrak{a}\subset\mathfrak{a}$ so that $a\,\epsilon\,\mathfrak{a}$; if $a\,\epsilon\,\mathfrak{a}$ then $\mathfrak{p}=\mathfrak{a}'(a)$ does not contain a or $\mathfrak{a}$. If $\mathfrak{p}\supset\mathfrak{a}$, where $\mathfrak{p}$ is prime, implies that $\mathfrak{p}$ is normal, then $\mathfrak{a}$, as the product of its prime ideal divisors, is normal; and $\mathfrak{a}'$ contains an atomic system $\mathfrak{s}$. Since

$b\,\epsilon\,\mathfrak{s}'$ implies $b\,\epsilon\,\mathfrak{a}'(a)$ for every a in $\mathfrak{s}$, we see that $\mathfrak{s}'\subset\mathfrak{a}$. On the other hand $\mathfrak{s}\subset\mathfrak{a}'$ implies $\mathfrak{s}'\supset\mathfrak{a}''=\mathfrak{a}$. Hence we have $\mathfrak{a}=\mathfrak{s}'$. If every prime ideal not containing $\mathfrak{a}$ is normal, then $\mathfrak{a}$ contains an atomic system $\mathfrak{s}$ such that every prime ideal not containing $\mathfrak{a}$ is given by $\mathfrak{p}=\mathfrak{a}'(a)$, $a\,\epsilon\,\mathfrak{s}$. We shall prove that $\mathfrak{a}\mathfrak{s}'=\mathfrak{o}$. If $b\,\epsilon\,\mathfrak{a}\mathfrak{s}'$, then $ab=0$ for every a in $\mathfrak{s}$ and hence $b\,\epsilon\,\mathfrak{a}'(a)$ for $a\,\epsilon\,\mathfrak{s}$; and also $b\,\epsilon\,\mathfrak{p}$ whenever $\mathfrak{p}$ is a prime ideal containing $\mathfrak{a}$. Thus b is contained in every prime ideal and must be the element 0. Hence $\mathfrak{a}\mathfrak{s}'=\mathfrak{o}$ at we wished to prove. The relations $\mathfrak{s}\subset\mathfrak{a}$ and $\mathfrak{a}\mathfrak{s}'=\mathfrak{o}$ show that $\mathfrak{s}\subset\mathfrak{a}\subset\mathfrak{s}''$ [16]).

[16]) Through the courtesy of the Editors, it is possible for me to add the following bibliographical indications: Jaśkowski on a meeting of the Polish Math. Soc. (cf. Annales Soc. Pol. de Math. 12, p. 122) has considered totally additive Boolean rings with countable bases and without atomic elements, and gave the result quoted in footnote 14; Kuratowski, as cited by Tarski, Fund. Math. 6 (1924), pp. 94-95, has found Th. 5.1; Mazurkiewicz, Mon. Math. u. Phys. 41 (1934), pp. 343-352, has also considered deductive systems; Mostowski, Fund. Math. 29 (1937), pp. 34-53, studies countable Boolean rings; results of Tarski closely related to Ths. 2.2, 2.4, 4.1, 5.2, 6.2, 9.1 and 9.2 are to be found in Fund. Math. 24 (1934), p. 180, 26 (1936), p. 285 and pp. 287-288, in Comptes Rendus du I Congrès des Mathématiciens Polonais 1927 (Supplément des Ann. Soc. Pol. de Math. 1929, pp. 29-33), in Ann. Soc. Pol. de Math. 1936, pp. 186 and 190, and Comptes-Rendus Soc. Sc. Varsovie 30 (1937).