d'ensembles linéaires parfaits, il existe un ensemble linéaire non dénombrable qui est transformé par toute fonction de la famille Φ en un ensemble qui est de 1re catégorie sur tout ensemble parfait de la famille Φ_1.

En admettant l'hypothèse du continu, on obtient du théorème $\mathrm{II}^{\mathrm{bis}}$ tout de suite ce

Corollaire $\mathrm{II}^{\mathrm{bis}}$: *Si $2^{\aleph_0} = \aleph_1$, il existe un ensemble linéaire non dénombrable qui est transformé par toute fonction mesurable d'une variable réelle en un ensemble qui est de 1re catégorie sur tout ensemble parfait* [1]).

[1]) Cf. W. Sierpiński *C. R. Soc. Sc. Varsovie* XXII (1929), p. 58.

On tangents to general sets of points.

By

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In this note we consider the problem of the set of points at which the tangent to a given set exists.

We deal with two definitions of the tangent: one is independent of the notion of measure; the other is based upon this notion. In either case we prove the

Theorem. *The set of points at which the tangent to a given set exists is always of finite or enumerably infinite linear measure* [1]).

First definition of tangent. *Given a plane set E and a limit point M of the set, either belonging to the set or not, we say that a line MI is the tangent to the set at the point M if to any two lines MI', MI'', however near MI and on different sides of MI, there corresponds a positive number r such that all the points of the set E which belong to the circle $c(M, r)$ (of centre M and radius r) lie in the acute angles between MI' and MI''.*

We shall first prove our theorem using this definition.

Denote by G the set of all the points of the plane at which the tangent to the set E exists. Let h be a fixed line and $\theta < \frac{\pi}{2}$. Denote by $G(\theta)$ the subset of the points of G at which the tangent to E makes with h an angle $< \frac{\theta}{2}$.

Let $G(\theta, r)$, $r > 0$, be the subset of $G(\theta)$ consisting of the points M

[1]) A set E is said to be of enumerably infinite linear measure if it can be represented as the sum of an enumerably infinite system of sets each of finite linear measure.

for which every segment of length $< r$ joining M to a point of E makes with h an angle $< \frac{\theta}{2}$. We obviously have

$$G(\theta) = \lim_{r \to 0} G(\theta, r)$$

so that

$$G(\theta) = \sum_{n=0}^{\infty} G\left(\theta, \frac{r}{2^n}\right) \tag{1}$$

Take now a pair of coordinate axes in the plane of the set E so that the directions of the bisectors of the first quadrant and of the angle θ coincide, and divide the whole plane into squares of side $\frac{r}{2}$ by parallels to the coordinate axes. Taking an arbitrary point M_1 of $G(\theta, r)$ we conclude that the segment joining M_1 to an arbitrary point of E lying in the same square as M_1 will make with h an angle $< \frac{\theta}{2}$. And *a fortiori* a segment $M_1 M_2$ joining two points of $G(\theta, r)$ that lie in the same square, will make with h an angle $\leqq \frac{\theta}{2}$.

Taking an arbitrary square containing points of $G(\theta, r)$ we define the function $y = f(x)$ by the coordinates of the points of $G(\theta, r)$ belonging to this square. Denoting by H the set of abscissae of these points we define $f(x)$ by the continuity condition at limit points of H and linear interpolation at exterior points of H. The function $y = f(x)$ is an increasing function of x in the square and represents a continuous curve. Thus all the points of $G(\theta, r)$ lie on a finite number of monotone curves (equal to the number of squares containing points of $G(\theta, r)$), from which it follows that $G(\theta, r)$ has a finite linear measure. Then by (1) $G(\theta)$ has a finite or an enumerably infinite linear measure, which proves the theorem.

Passing to the second definition of the tangent we shall first give the definition of ν-dimensional sets and ν-dimensional measure [1]. Let E be a plane set of points. Denote by $A(\delta)$ a finite or enumerably infinite set of open convex areas of diameters less than $< \delta$ containing all the points of E. Denote by l the diameter of the variable element of the set $A(\delta)$ and consider the sum of numbers l^ν corresponding to all areas of $A(\delta)$. We write this sum in the form

$$\sum_{A(\delta)} l^\nu.$$

Define now the lower bound of this sum $\sum\limits_{A(\delta)} l^\nu$ for all possible sets $A(\delta)$, then the ν-dimensional measure $m_\nu E$ of E is defined by the equation

$$m_\nu E = \lim_{\delta \to 0} l \cdot bd \sum_{A(\delta)} l^\nu.$$

To every set E corresponds a number ν_0 such that

$$m_\nu E = 0 \quad \text{for all} \quad \nu > \nu_0$$
$$m_\nu E = \infty \quad \text{for all} \quad \nu < \nu_0.$$

The set E is called a ν_0-dimensional set or simpler a ν_0-set and the number ν_0 is called the dimensional number of the set E.

Of course we assume the set E to be measurable with respect to this measure in Caratheodory's sense.

The ν-dimensional density is defined in the obvious way.

Remark.

An immediate application of the theorem which has just been proved to the case of a Jordan arc leads to this interesting result.

If the dimensional number of the set of points represented by a Jordan arc is greater than 1 (it may be equal to any number from 1 to 2) then the arc has a tangent only at exceptional points of the arc.

Second definition of tangent. *Given a set E of finite and positive ν-dimensional measure and a point M (either belonging to the set E or not) at which the lower ν-dimensional density of E is positive, a line MI is called the tangent to the set E at the point M if for any two lines MI', MI'' however near MI and on different sides of MI, the part of the set E included in the obtuse angles between MI' and MI'' has zero ν-dimensional density at the point M.*

We shall now prove our theorem for the case of this definition of the tangent. Denote by G the set of points at which the tangent

[1] Hausdorff. Dimension und äusseres Mass. *Math. Annalen* 79 (1918), pp. 157—179.

A. S. Besicovith. On linear sets of fractional dimensions. *Math. Annalen* 101 (1929), pp. 161—193.

o the set E exists. Let $G(\alpha)$ be the subset of G of points at which the lower ν-dimensional density is greater than a positive number α. We have

(2) $$G = \lim_{\alpha \to 0} G(\alpha).$$

Consider now the set $G(\alpha, \lambda)$ of all the points M of $G(\alpha)$ at which the following condition is satisfied

$$m_\nu\{E \times c(M, l)\} > \alpha (2l)^\nu \quad \text{for all} \quad l \leqq \lambda.$$

We have

(3) $$G(\alpha) = \lim G(\alpha, \lambda).$$

Let now h be a fixed line and $\theta < \frac{\pi}{2}$. Denote by $G(\alpha, \lambda, \theta)$ the set of the points of $G(\alpha, \lambda)$ at which the tangent makes with h an angle $< \frac{\theta}{2}$. Then the set $G(\alpha, \lambda)$ may be represented in the form

(4) $$G(\alpha, \lambda) = \sum_\theta G(\alpha, \lambda, \theta)$$

where the sum is extended over a finite set of overlapping angles θ (bisected by different lines h) covering the whole angle from 0 to 2π. Take now an angle θ_1, $\theta < \theta_1 < \frac{\pi}{2}$, with the same bisector as θ and a positive number ε satisfying the condition

(5) $$2^\nu \varepsilon < \alpha \left(\sin \frac{\theta_1 - \theta}{2}\right)^\nu.$$

Denote by

$$\{c(M, l) \text{ outside } \theta\}$$

the system of radii of $c(M, l)$ making an angle $\geqq \frac{\theta}{2}$ with h and let $G(\alpha, \lambda, \theta, \lambda_1)$, $0 < \lambda_1 < \lambda$, be the set of points M of $G(\alpha, \lambda, \theta)$ for which the following condition is satisfied

(6) $$m_\nu[E \times \{c(M, l) \text{ outside } \theta\}] < \varepsilon (2l)^\nu \quad \text{for} \quad l < \lambda_1$$

We have

(7) $$G(\alpha, \lambda, \theta) = \lim_{\lambda_1 \to 0} G(\alpha, \lambda, \theta, \lambda_1).$$

Divide the plane of the set E into squares of the side $\frac{\lambda_1}{4}$, and

let M_1, M_2 be any pair of points of $G(\alpha, \lambda, \theta, \lambda_1)$ belonging to the same square. Then we shall prove that

the direction $M_1 M_2$ makes with h an angle $< \frac{\theta_1}{2}$.

For, if not, the distance from M_2 to the sides of the angle θ, with the vertex at M_1 and the bisector parallel to h, is greater than $M_1 M_2 \sin \frac{\theta_1 - \theta}{2}$ and thus the circle $c\left(M_2, M_1 M_2 \sin \frac{\theta_1 - \theta}{2}\right)$ consists of points whose joins to M_1 make angles greater than or equal to $\frac{\theta}{2}$ with h.

We have

$$m_\nu\left\{E \times c\left(M_2, M_1 M_2 \sin \frac{\theta_1 - \theta}{2}\right)\right\} > \alpha \left(2 M_1 M_2 \sin \frac{\theta_1 - \theta}{2}\right)^\nu$$

and as the circle $c(M_1, 2M_1M_2)$ contains the circle $c\left(M_2, M_1 M_2 \sin \frac{\theta_1 - \theta}{2}\right)$ we conclude that

(8) $$m_\nu\left[E \times \left\{c(M_1, 2 M_1 M_2) \text{ outside } \theta\right\}\right] > \alpha \left(2 M_1 M_2 \sin \frac{\theta_1 - \theta}{2}\right)^\nu$$

on the other hand by (6)

(9) $$m_\nu[E \times \{c(M_1, 2 M_1 M_2) \text{ outside } \theta\}] < \varepsilon (4 M_1 M_2)^\nu.$$

We conclude from (5) that (8) and (9) are contradictory and thus our statement is proved. Then in the same way as in the case of the first definition of the tangent we prove that the set $G(\alpha, \lambda, \theta, \lambda_1)$ lies on a finite number of rectifiable curves, i. e. that it is of finite linear measure. Then from (7), (4), (3), (2) we conclude consecutively that each of the sets

$$G(\alpha, \lambda, \theta), \quad G(\alpha, \lambda), \quad G(\alpha), \quad G$$

is of at most enumerably infinite linear measure, which proves the theorem.

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