jouit pas de la propriété de Baire, puisque, si H jouissait de cette propriété, il en serait de même de l'ensemble $H - \underset{x,y}{E}[x = -1]$ qui coïncide évidemment avec J_1, et comme nous savons, l'ensemble J_1 ne jouit pas de la propriété de Baire.

Donc, *si* $2^{\aleph_0} = \aleph_1$, *il existe une fonction d'une variable réelle qui jouit de la propriété de Baire et dont l'image géométrique ne jouit pas de la propriété de Baire.*

Voici encore une remarque due à M. Kuratowski.

En conservant les notations utilisées plus haut, posons

$$f(x,y) = 1 \text{ si } (x,y) \in Q, \text{ et } f(x,y) = 0 \text{ si } (x,y) \text{ non } \in Q.$$

La fonction de deux variables réelles $f(x,y)$ ne jouit pas de la propriété de Baire, puisque l'ensemble Q ne jouit pas de la propriété de Baire. Or, la fonction $f(x,y)$ dépend évidemment seulement de y, et si l'on pose, pour x et y réels, $f(x,y) = \vartheta(y)$, la fonction $\vartheta(y)$ d'une variable réelle y jouit de la propriété de Baire (puisque $\underset{y}{E}[\vartheta(y) \neq 0] = K$ et K est un ensemble toujours de première catégorie).

Donc, *si* $2^{\aleph_0} = \aleph_1$, *une fonction d'une variable réelle qui jouit de la propriété de Baire, considérée comme une fonction de deux variables réelles, peut ne jouir pas de la propriété de Baire.*

Ou encore, posons, pour x et y réels: $F(x,y) = y$ — ce sera évidemment une fonction continue de deux variables réelles x, y. La fonction $g(x,y) = F(x, \vartheta(y))$ est, comme on voit sans peine, la fonction caractéristique de l'ensemble (plan) Q, donc une fonction qui ne jouit pas de la propriété de Baire. Donc:

Si $2^{\aleph_0} = \aleph_1$, *il existe une fonction de deux variables réelles qui ne jouit pas de la propriété de Baire et qui est une fonction continue (de deux variables réelles) de fonctions (d'une variable réelle) jouissant de la propriété de Baire.*

(Pour les fonctions d'une variable réelle, ainsi que pour les fonctions de deux variables réelles de fonctions de deux variables réelles, un tel cas est, comme on sait, impossible).

On Linearly Measurable Plane Sets of Points of Upper Density $^1/_2$.

By

J. Gillis (Cambridge, England).

§ 1. The general theory of linear measure and measurability of plane sets of points is due to Carathéodory[1]), Gross[2]) and Esterman[3]); but Besicovitch[4]), in a paper to which I shall refer as (*B*), was the first to investigate the geometrical properties of linearly measurable plane sets. Later, Besicovitch and Walker[5]), proved a further important result, and, although their actual theorem is irrelevant to my present purpose, I shall have to make frequent use of the arguments they use to establish some of their auxiliary theorems. When doing so, I shall refer to their paper as (*B* and *W*). I proved to state some definitions and the relevant known theorems.

§ 2. Let A be a plane set of points and p an arbitrarily chosen positive number; let $U(p, A)$ denote a finite or denumerable set of convex areas $\{U_k(p, A)\}$ such that:

(I) every point of A is interior to at least one of the areas $U_k(p, A)$, and

(II) the diameter d_k of $U_k(p, A)$ is less than p for each k;

then the lower bound of $\sum_U d_k$ is denoted by L_p. As p decreases,

[1]) Über das lineare Maß von Punktmengen. *Nachrichten der K. Gesellschaft der Wissenschaften zu Göttingen*, Math.-Phys. Klasse, (1914).

[2]) *Monatshefte für Math. und Physik* (1918).

[3]) *Abhandlungen aus dem Math. Sem. der Hamb. Univ.* (1925).

[4]) *Math. Annalen* (1927), pp. 422—464.

[5]) *Proc. of the London Math. Soc*, Ser. 2, Vol. 32, Part 2, pp. 142—153.

L_p is non-increasing and, therefore, $\lim_{p\to 0} L_p = L^* A$ exists. We call $L^* A$ the *exterior linear measure* of A.

If A is such that, for each set W of finite exterior linear measure,

$$L^* W = L^*[A \times W] + L^*[W - A \times W],$$

then A is said to be *linearly measurable*, or, as we shall say for simplicity, *measurable*, and we write $L^* A = LA$.

If now A is a measurable set and $c(a, r)$ is the circle with centre a and radius r, we write [6])

$$\limsup_{r\to 0} \frac{L[A \times c(a, r)]}{2r} = D^*(a, A) = \textit{upper density of } A \textit{ at } a,$$

and

$$\liminf_{r\to 0} \frac{L[A \times c(a, r)]}{2r} = D_*(a, A) = \textit{lower density of } A \textit{ at } a.$$

Theorem. *At almost all points a of a measurable set A,*

$$0 \leqslant D_*(a, A) \leqslant 1 \quad \textit{and} \quad \tfrac{1}{2} \leqslant D^*(a, A) \leqslant 1,$$

and these results are the best possible.

If, at a point a, $D^*(a, A) = D_*(a, A)$, then we call their common value the *density* of A at a and denote it by $D(a, A)$.

A point a of A at which the density exists and is equal to 1 is called a *regular* point of A; any other point is called an *irregular* point. If almost all points of A are regular points, then A is said to be a *regular set*; if almost all points are irregular then A is said to be an *irregular set*. I now invoke a number of theorems [6]) which I do not propose to prove.

Theorem. *The density of a measurable set is zero at all points outside the set, except for a set of points of linear measure zero.*

From this we can deduce the following theorems.

Theorem of permanence of densities. *Let $A_1, A_2, \ldots$ be a finite or denumerable set of measurable sets such that $L(A_1 + A_2 + \ldots)$ is finite. Then at almost every point a of the set $\sum_n A_n$ the upper and lower densities are respectively equal to those of the set A_n which contains a.*

[6]) These definitions and the succeeding ones as well as all the theorems quoted in this section are taken almost verbatim from (B) where these theorems are proved.

Theorem of Decomposition. *The set of all regular points of a measurable set A is a regular set, and the set of all irregular points is an irregular set.*

§ 3. It follows from the Theorem of Decomposition that the study of measurable sets can be reduced to the separate studies of regular and irregular sets. The properties of regular sets have been investigated by Besicovitch [7]) who established that at almost every point of such a set there exists a tangent, and that almost all points of the set are contained in a finite or denumerable set of rectifiable curves which can be constructed so as to have total length arbitrarily near to the measure of our original set. Regular sets are thus generalizations of rectifiable curves and have no strikingly new properties. Irregular sets, on the other hand, have properties fundamentally different from those of rectifiable curves, and it is with such sets that this paper will deal.

§ 4. I shall deal, in particular, with sets of upper density $\frac{1}{2}$ (i. e. sets at almost all points of which the upper density is equal to $\frac{1}{2}$); we shall see that these sets have a very simple structure. It is my belief that such sets have projection of zero measure on almost all directions, though I have been unable to prove this. The following theorem will be proved (subject to a slight modification of the definition of LA):

Given a linearly measurable plane set A, we can write $A = G + R$, where

(I) $LR = 0$, *and*

(II) *corresponding to each point x of G there is a set of directions $P(x)$, of measure greater than or equal to $\frac{\pi}{2}$ (the measure of the set of all directions being taken as 2π) such that, if θ is any direction belonging to $P(x)$, then*

$$\liminf_{r\to 0} \frac{\text{measure of projection of } [A \times c(x, r)] \text{ on } \theta}{L[A \times c(x, r)]} = 0.$$

The chief interest of this result it that is brings to light a further fundamental difference between regular sets and those of upper density $\frac{1}{2}$.

[7]) (B) pp. 438—451.

§ 5. If, in our definition of linear measure and measurability, we had restricted the convex areas to be circles, we should have obtained a new definition of measure. If we denote it by L_c, then it is clear that $L_c A \geqslant LA$. It has been shown [8]) that

(I) If A is a regular set, then $L_c A = LA$,

(II) If A is an irregular set, then $LA \leqslant L_c A \leqslant \frac{2}{\sqrt{3}} A$

and these results are the best possible, in the obvious sense of the term. It is convenient, when dealing with irregular sets, to use L_c measure rather than L measure (the latter may be referred to as Carathéodory measure). The results obtained in this paper hold, mutatis mutandis, for L measure, but the use of L_c measure introduces a certain simplification of detail, and it is for L_c measure that I shall prove my result. I shall begin with an analysis of the structure of sets of upper density $\frac{1}{2}$, but for this I need to make use of a known property of such sets. I shall not prove it in detail but shall content myself with stating the main stages in the proof and referring the reader to the original paper [9]), where a full proof is to be found.

§ 6. Suppose that we are given a linearly measurable plane set A of upper density $\frac{1}{2}$, and arbitrary positive numbers ε, η, γ. Let l be a positive number and α a point of A such that the mean density of A in the circle $c(\alpha, r)$ is less than $\frac{1}{2} + \varepsilon$ for all $r < l$. Let A_l denote the set of all such points α. Clearly $L_c A_l \to L_c A$ as $l \to 0$, and hence, by fixing l_0 sufficiently small, we can have $L_c A_{l_0} > L_c A - \gamma$.

Write

$$B = A_{l_0}, \quad \text{then} \quad L_c(A - B) < \gamma. \tag{1}$$

By a lemma proved in (B) p. 42 for Carathéodory measure, but the proof of which applies without change to L_c measure, we can find δ so that, for any finite or denumerable set of circles $C(\delta)$, each of diameter less than δ,

$$\sum_{C(\delta)} 2r > L_c\{B \times C(\delta)\} - \eta \tag{2}$$

and we can take $\delta < l_0$.

[8]) (B) pp. 458—464.

[9]) See $(B$ and $W)$, pp. 145—148, for a complete exposition of the matter of § 6.

We deduce, by the argument used in the proof Vitali's Theorem, that there exists a set O of non-overlapping circles, each of diameter less than or equal to δ, such that

$$\left| \sum_{O} 2r - L_c B \right| < \eta \tag{3}$$

and O covers almost all points of B.

Write now

$$O = C_1 + C_2 + C_3$$

where C_1 is the set of circles of O in each of which the mean density of B is less than $1 - \gamma$, C_2 the set for which the mean density lies in the closed interval $[1 - \gamma, 1 + \gamma]$, and C_3 that for which the mean density is greater than $1 + \gamma$.

It is shown that

$$L_c\{B \times (c_1 + c_3)\} < \frac{3\eta}{\gamma} - \eta$$

and hence that

$$L_c\{B \times C_2\} > L_c B - \frac{3\eta}{\gamma} + \eta.$$

Next it is shown that in every circle $c(0, r)$ of C_2 all the included points of B lie in the narrow ring between the circles $c(0, r)$ and $c(0, \overline{1 - \sigma r})$, where $\sigma = 4\varepsilon + 2\gamma$.

§ 7. I proceed to apply these results. For our purpose it will be convenient to take $\gamma = \varepsilon = \sqrt{\eta}$.

We have

$$L\{B \times (C_1 + C_3)\} < \frac{3\eta}{\gamma} - \eta < 3\gamma \tag{4}$$

$$L(B \times C_2) > L_c B - 3\gamma \tag{5}$$

$$\sigma = 6\gamma. \tag{6}$$

Summing up the results quoted in the proceding paragraph, using the inequalities (1), (4) and (5) and equation (6), and replacing γ by $\gamma/6$, we can assert that:

Given a measurable set A of upper density $\frac{1}{2}$ and an arbitrary positive number γ, we can find a set C_2 of non-overlapping circles and can write $A = \Gamma + D$, where

(I) $L_c \Gamma < \gamma$,

(II) *The points of D are included in C_2, and in any circle $c(0, r)$ of C_2 the included points of D lie in the ring between the circles $c(0, r)$, $c(0, \overline{1-\gamma}\, r)$,*

(III) *In each circle of C_2 the mean density of D lies in the closed interval $[1-\gamma/6, 1+\gamma/6]$ and, a fortiori, in the interval $(1-\gamma, 1+\gamma)$.*

§ 8. The circles of C_2 are denumerable; $S_1, S_2, \ldots, S_n, \ldots$ (say). The set $D \times S_n$ is again measurable and of upper density $\frac{1}{2}$ (by the theorem of permanence of densities). Therefore the result of the last paragraph can be applied to $D \times S_n$, and here it will be convenient to use $\gamma/2^{n+1}$ instead of γ. The set of $D \times S_n$ which corresponds to Γ will have measure less than $\gamma/2^{n+1}$ and can, for the moment, be „disregarded". We may treat each $D \times S_n$ in this way. The total measure of the set of points „disregarded" in this process will be less than $\Sigma\, \gamma/2^{n+1} = \gamma/2$.

The remainder of $D \times C_2$ will again be contained in a denumerable set of circles, and the points of $D \times S_n$ belonging to one of these circles will lie in a ring of relative width $\gamma/2^{n+1}$. This process can be continued indefinitely. Suppose that, at the rth time of application, we are faced with the points of D in the annuli $\mathfrak{A}_1, \mathfrak{A}_2, \ldots$. Apply the process again using $\gamma/2^{r+m}$, in place of the γ in the enunciation of the preceding paragraph, when dealing with $\mathfrak{A}_m$.

Thus, the measure of the set of „disregarded point" arising from the rth application of the analysis is less than $\gamma/2^{r-1}$, and the total measure of the set of all disregarded points less than 2γ.

Let

$$Q = \text{the „disregarded set"}$$

and

$$C = A - Q.$$

$L_c Q < 2\gamma$ and the following describes the structure of C:

We have a succesion $\Lambda_1, \Lambda_2, \ldots, \Lambda_n, \ldots$ of finite or denumerable sets of non-overlapping annuli such that, for every n, each member of Λ_{n+1} is interior to some member of Λ_n, the ratio of the width of each annulus of Λ_n to the radius of its outer circle is less than, or equal to, $\gamma/2^n$, and $\Lambda_1 \times \Lambda_2 \times \Lambda_3 \times \ldots \times \Lambda_n \times \ldots = C$.

In particular, each point x of C is given by $x = \prod_{n=1}^{\infty} \lambda_n$, where λ_n is an annulus of Λ_n, and $\lambda_n \supset \lambda_{n+1}$.

§ 9. Now choose an arbitrary positive integer K and choose some fixed direction as the initial direction. Starting from the latter, divide each circle of each Λ_n into K equal sectors. If the mean density of C in a sector is greater than $\frac{1}{60}$, then we call this sector *compact*; if it is greater than $1+\frac{1}{60}$, then we call the sector *dense*.

Let Δ be the set of points of C which, in an infinity of each sequence of circles defining them, are contained in a dense sector. Let P be a point of Δ. Then, corresponding to each of the dense sectors defining P, we can find a circle of which P is the centre and in which the mean density of C is greater than $\frac{1}{2}+\frac{1}{120}$; and the radii of these circles tend to zero. Thus

$$D^*(P, C) \geqslant \tfrac{1}{2} + \tfrac{1}{120}.$$

But C is a set of upper density $\frac{1}{2}$. Hence

$$L_c \Delta = 0.$$

Now let D_n denote the set of dense sectors of Λ_n.

Let $\Delta_i =$ the set of points of C which are contained in no D_n for $n \geqslant i$. Then

$$\Delta_i \subset \Delta_{i+1} \text{ for each } i, \text{ and } \Delta + \sum_{i=1}^{\infty} \Delta_i = C$$

i. e.

$$L_c \left[\sum_{i=1}^{\infty} \Delta_i\right] = L_c C$$

and hence

$$L_c \Delta_i > L_c C - \gamma \quad \text{for} \quad i \geqslant N = N(\gamma).$$

We may regard Λ_N as the starting point for the construction of C, and we shall lose no generality by calling it Λ_1. Then in Λ_1, and in all succeeding Λ_n, suppress entirely any circle which contains a dense sector. By what I have shown, the measure of the set of points in these sectors will be less than γ.

Therefore, if $\delta_1, \delta_2, \ldots, \delta_n, \ldots$ denote the suppressed circles and $r_1, r_2, \ldots, r_n, \ldots$ their respective radii,

$$\gamma < \sum_n \left(1 + \tfrac{1}{60}\right) \frac{2 r_n}{K} \pi$$

i. e.

$$\sum_n 2 r_n < \frac{K\gamma}{\pi\left(1 + \frac{1}{60}\right)};$$

hence the measure of the set of all points „lost“ in this way is less than $\frac{K\gamma(1+\gamma)}{\pi(1+\frac{1}{60})}$ (since the mean density of C in each circle is less than $1+\gamma$), and

$$\frac{K\gamma(H\gamma)}{\pi(1+\frac{1}{60})} < \tfrac{1}{3}K\gamma, \text{ for sufficiently small } \gamma.$$

Henceforth, *I* shall denote by C the set so modified.

§ **10.** Now, in each remaing circle of the sequence $\{\Lambda_n\}$, there is no dense sector.

Let $l(\lambda_n) =$ the number of compact sectors in the circle λ_n of Λ_n, of radius r_n.

The total measure of C in the non-compact sectors is

$$\leqslant \tfrac{1}{60}\cdot\frac{K-l(\lambda_n)}{K}\cdot 2\pi r_n;$$

therefore the total measure of C in the compact sectors is

$$\geqslant 2r_n\left(1-\frac{\gamma}{2^n}\right) - \tfrac{1}{60}\cdot\frac{K-l(\lambda_n)}{K}\cdot 2\pi r_n;$$

hence there exists a sector in which the measure of C is

$$\geqslant \frac{2r_n}{l(\lambda_n)}\left(1-\frac{\gamma}{2^n}\right) - \tfrac{1}{60}\,\frac{K-l(\lambda_n)}{Kl(\lambda_n)}\cdot 2\pi r_n.$$

But, by hypothesis, λ_n contains no dense sector, and therefore

$$1+\tfrac{1}{60} \geqslant \frac{K}{\pi l(\lambda_n)}\left(1-\frac{\gamma}{2^n}\right) - \frac{K-l(\lambda_n)}{60\,l(\lambda_n)}$$

$$= \frac{K}{l(\lambda_n)}\left[\frac{1-\frac{\gamma}{2^n}-\frac{\pi}{60}}{\pi}\right] + \frac{1}{60};$$

thus $\frac{l(\lambda_n)}{K} > \frac{9}{10\pi}$, if γ is chosen sufficiently small i. e.

(7) $$l(\lambda_n) > \frac{K}{4}.$$

Now let $l = \operatorname{Min}\{l(\lambda_n)\}$.

By (7), $l > \frac{K}{4}$, *and each circle has at least l compact sectors.*

§ **11.** We can enumerate the sectors of each circle by the positive integers $1, 2, \ldots, K$. These represent, not particular sectors, but sectors of directions.

If $\lambda_1 \supset \lambda_2 \supset \ldots \supset \lambda_n \supset \ldots \supset x$, and $x \supset \lambda_i \times \alpha_i$, $(i = 1, 2, \ldots)$, where the α_i's are numerals from $1 \ldots K$, then x may be said to have the „expansion“ $\alpha_1\alpha_2\ldots$

It is, a priori, conceivable that there should be $K-l$ numerals from $1\ldots K$ which appear in the expansion of no x of the set. But, if we pick any $K-l+1$ of the numerals, the set of x in the expansion of which none of these appear will be shown to have measure zero.

Let this set be X_0. In every circle defining X_0, at least one of the excluded sectors is a compact one, i. e. in it the mean density of C is greater than $\frac{1}{60}$. [For convenience, I shall write α for $\frac{1}{60}$]. Let $C_1, C_2, \ldots, C_n, \ldots$ be a sequence of the circles and $\beta_1, \beta_2, \ldots, \beta_n, \ldots$ be the indices of excluded compact sectors. X_0 has no point of $\beta_1 \times C_1$. Hence

$$\frac{L_c[C\times\beta_1\times C_1]}{L_c[C\times C_1]} \geqslant \frac{d\cdot\frac{2\pi r}{K}}{2r(1+\gamma)} = \frac{\pi d}{(1+\gamma)K},$$

i. e. the measure of the part of X_0 included in C_1 is

$$\leqslant\left[1-\frac{\pi d}{K(1+\gamma)}\right]\cdot L_c[C+C_1].$$

This holds for every member of Λ_1 containing points of X_0. Hence

$$L_c[X_0\times\Lambda_1] \leqslant \left[1-\frac{\pi d}{(1+\gamma)K}\right]\cdot L_c[C\times\Lambda_1].$$

Similarly, in Λ_2, we must omit another compact sector of each circle, i. e.

$$L_c[X_0\times\Lambda_2] \leqslant \left[1-\frac{\pi d}{(1+\gamma)K}\right]\cdot L_c[X_0\times\Lambda_1]$$
$$\leqslant \left[1-\frac{\pi d}{(1+\gamma)K}\right]^2\cdot L_c(C\times\Lambda_1].$$

This can obviously be continued indefinitely.

$$L_c X_0 = L_0[X_0 \times \Lambda_n], \text{ since } X_0 \subset C = \prod_{n=1}^{\infty} \Lambda_n, \text{ is}$$

$$\leqslant \left[1 - \frac{\pi d}{(1+\gamma)K}\right]^n \cdot L_c C, \text{ for all } n.$$

Hence

$$L_c X_0 = 0.$$

We can easily extend the above reasoning to show that the set X of x, in the expansion of which a given set of $K-l+1$ numerals occur only a finite number of times, has measure zero. For the set X_n of x, in which none of the $K-l+1$ numerals comes after the n th place, has measure zero. This can be seen by applying the above argument, starting from Λ_n; i. e. since

$$X = \sum_{n=1}^{\infty} X_n \quad \text{we receive} \quad L_c X = 0.$$

§ 12. Let now $H = C - X$. $L_c H = L_c C > 0$.

We can subdivide H into a finite number of sub-sets each of which has the property that the same l numerals occur infinitely many times in the expansion of each x of it, i. e.

$$H = H_1 + H_2 + \ldots + H_r.$$

Since $L_c H > 0$, not all H_i can be of measure zero. Suppose that $L_c H_1 > 0$.

Let $x \subset H_1$, where $\lambda_1 \supset \lambda_2 \supset \ldots \supset \lambda_n \supset \ldots \supset x$, $\lambda_n \subset \Lambda_n$, and $x = \prod_{n=1}^{\infty} \lambda_n$.

Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_l$, are numerals which occur infinitely often in the expansion of each x of H_1. If a direction θ belongs to α_i, then the projection of $\lambda_n \times \alpha_i \times H_1$ on θ has measure less than $r_n - r_n(1-\sigma_n)\cos\frac{2\pi}{K}$, where $r_n =$ radius of λ_n and $\sigma_n =$ relative width of the annulus.

This is less than

$$(8) \qquad r_n\left[1 - \left(1 - \frac{2\pi^2}{K^2}\right)\left(1 - \frac{\gamma}{2^{n-2}}\right)\right] < \frac{3\pi^2}{K^2} \cdot r_n,$$

for sufficiently big n.

Now the diameter of $\lambda_n \times \alpha_i$

$$(8^*) \qquad = \frac{2\pi r_n}{K} + O\left(\frac{r_n}{K^3}\right),$$

$$(9) \qquad L_c[H_1 \times \lambda_n \times \alpha_i] \geqslant \frac{2\pi r_n}{60 K}$$

$$(10) \qquad \text{and} \quad m\left(\sum{}' \alpha_i\right) > \frac{\pi}{2}.$$

From (8), (9), and (10) we see that, for a set of θ of measure greater than $\pi/2$, we have

$$(11) \qquad \frac{\text{measure of projection on } \theta \text{ of } H_1 \times \lambda_n \times \alpha_i}{L_c[H_1 \times \lambda_n \times \alpha_i]} = O\left(\frac{1}{K}\right).$$

Similar results hold for $H_2, H_3, \ldots, H_r$.

Also, by the theorem of permanence of density, if $d(U_x)$ denote the diameter of any sufficiently small neighbourhood U_x of a point x of H_i,

$$(12) \qquad L_c[U_x \times (A - H_i)] = o[d(U_x)].$$

I now write

$$V[A, U_x, \theta] \quad \text{for} \quad \frac{\text{measure of projection on } \theta \text{ of } [A \times U_x]}{L_c[A \times U_x]}.$$

From (11) and (12) it follows that, if x belongs to H,

$$(13) \qquad \liminf_{d(U_x)\to 0} V[A, U_x, \theta] = O\left(\frac{1}{K}\right).$$

Also, by what was shown in §§ 3, 4, 6

$$(14) \qquad L_e(A - H) \leqslant 2\gamma + \gamma + \tfrac{1}{3}K\gamma < \tfrac{1}{2}K\gamma,$$

for sufficiently big K. It follows that:

Given a measurable set A of upper density $\frac{1}{2}$ and positive numbers K, ϵ_1, we can write $A = H + E_1$, where

(I) $L_c E_1 < \epsilon_1$

and

(II) *corresponding to each point x of H there exists a set of directions θ of measure greater than or equal to $\frac{\pi}{2}$ such that*

$$\liminf_{d(U_x)\to 0} V[A, U_x, \theta] = O\left(\frac{1}{K}\right).$$

For we need only put $\gamma = \frac{2\epsilon_1}{K}$, and the result follows.

§ 13. A dissection of the set A such as that described in the final result of § 7 will be called a (K, ϵ_1) dissection. The property (II) of the set H can conveniently be described as the property $\{P, K\}$. Suppose now that we are given an arbitrary positive number ϱ. Corresponding to each positive integer n carry out an $\left(n, \frac{\varrho}{2^n}\right)$ dissection. Thus, corresponding to each n, we have $A = G_n + R_n$, where

(I) $L_c R_n < \frac{\varrho}{2^n}$,

and

(II) *G_n has the property $\{P, n\}$.*

Now write

$$R = \sum_{n=1}^{\infty} R_n, \qquad G = \prod_{n=1}^{\infty} G_n, \qquad A = G + R.$$

I note that $L_c R < \varrho$.

Moreover, if x is a point of G then, for each n, there is a set $P_n(x)$ of directions θ such that

$$\liminf_{d(U_x)\to 0} V[A, U_x, \theta] < \frac{g}{n} \tag{15}$$

where g is a constant, depending only on A, and independent of x, n. Indeed, it is clear from (8) and (8*) that we may even take $g = \frac{3\pi}{2}$. Also

$$\mathrm{m}\, P_n(x) > \frac{\pi}{2}. \tag{16}$$

If we now extend $P_n(x)$ so as to include all θ for which (15) holds, then (16) is, a fortiori, true, and

$$P_n(x) \supset P_{n+1}(x) \quad \text{for each } n;$$

hence $\lim_{n\to\infty} P_n(x) = \prod_{n=1}^{\infty} P_n(x) = P(x)$, in particular, exists, and

$$\mathrm{m}\, P(x) \geqslant \frac{\pi}{2}.$$

Clearly, if θ is contained in $P(x)$, then

$$\liminf_{d(U_x)\to 0} V[A, U_x, \theta] = 0.$$

Thus given A, ϱ we can write $A = G + R$ where

(I) $L_c R < \varrho$

and

(II) *if x is a point of G there is a set $P(x)$ of directions θ such that*

$$\liminf_{d(U_x)\to 0} V[A, U_x, \theta] = 0, \quad \textit{for all } \theta \textit{ of } P(x).$$

Now the property (II) of G is independent of ϱ which can be taken arbitrarily small without affecting the truth of (II). *Hence the above theorem must be true with $L_c R = 0$.*

The validity of this result with U_x restricted to be a circle with x as centre is immediately and trivially deducible from the theorem obtained. This completes the theorem.

§ 14. It is obvious from the argument that, by using less crude inequalities than those actually employed, I could have improved on the number $\frac{\pi}{2}$ which occurs in this theorem. This would, however, have involved some slight complications of arithmetic and, especially in view of the opinions expressed in § 4, I do not think that the improvement in the result would be either useful or necessary.

It will also be noticed that I have assumed the measurability of most of the sets which have arisen out of my analysis. This was to avoid obscuring the theme by an excess of detail, but, in all cases, measurability can be proved quite trivially.