Some properties of arbitrary functions [1]).

By

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1. Introduction.

The object of this paper is to derive various properties of arbitrary functions of two and of three real variables concerning limits of functional values as a straight line or a plane is approached. As the point of departure the following theorem [2]) of Professor Blumberg is used:

If $f(x,y)$ is an arbitrary real function defined in a plane π; s a straight line in π; and d_1, d_2, two directions of approach to s on the same side of it, then, for every point P of s with the possible exception of $\aleph_0$ points, I_{Pd_1} overlaps or abuts I_{Pd_2}, where I_{Pd_1} is the interval whose end points are the limits inferior and superior of f as (x,y) approaches P along the direction d_1, and I_{Pd_2} is defined similarly.

Theorems 1 and 2 are generalizations of this theorem in different directions. First the effect upon the exceptional set of the neglect of various sets of points will be shown. Then the two given directions will be freed of their fixed positions to give a rather striking property of functions of two real variables. Finally an extension will be made to functions of three variables.

2. Approach with Neglect of Certain Sets.

Let $f(x,y)$ be an arbitrary function of two real variables, s a given straight line in the xy plane, and d a direction in the xy plane.

[1]) The author is intebted to Professor Blumberg for very helpful suggestions and criticisms in the preparation of this article.

[2]) Fundamenta Mathematicae, vol 16 (1930). p. 77.

Since we shall deal with linear sets of various types, we shall use the letter τ to represent the variable type of set. Denote by l^τ_{Pd} (u^τ_{Pd}) the lower (upper) bound of the values of $f(x,y)$ as the variable point (x,y) approaches a point P of s along the direction d with the admissible neglect of a linear set of points of type τ. With the neglect of each different set of type τ the value of l^τ_{Pd} may differ. The set of values for l^τ_{Pd}, for P and d fixed, corresponding to the set of all possible sets of type τ which may be neglected, has an upper bound. In the theorem which follows the set of type τ neglected as each point P is approached may be a set for which l^τ_{Pd} is arbitrarily near this upper bound. The absence of a superscript indicates that no points are to be neglected. Let I^τ_{Pd} be the interval $(l^\tau_{Pd}, u^\tau_{Pd})$. The sum of a denumerable number of sets of type τ will be called a set of type τ_σ. We shall assume that τ is such that the sum of a finite number of sets of type τ is a set of type τ, that a set consisting of a finite number of points is a set of type τ, that two sets of type τ are not sufficient to constitute the linear continuum, and that τ is invariant under projection.

Theorem 1. a) *If $f(x,y)$ is an arbitrary function of two real variables, s a given straight line in the xy plane, and d_1, d_2 two given directions of approach to s on the same side, then $I^\tau_{Pd_1}$ overlaps or abuts I_{Pd_2} at every point P of s except possibly at points belonging to a set of type τ_σ; and* b) *furthermore, if T is any set of type τ_σ on s, there exists a function $f(x,y)$ such that at every point P of T, $I^\tau_{Pd_1}$ neither overlaps nor abuts I_{Pd_2}, and at every point P of s which does not belong to T, $I^\tau_{Pd_1}$ contains I_{Pd_2}.*

Proof. Let r be a given number and E_r the set of points P of s at which $l^\tau_{Pd_1} > r$. There is an interval J_{Pn} in direction d_1 with P as one end point and of length $< \frac{1}{n}$, in which $f > r$ except possibly at points of a set T_{Pn} of type τ contained in J_{Pn}. Let J'_{Pn} be the projection of J_{Pn} on s in direction d_2 and T'_{Pn} the projection of T_{Pn}. The points of E_r which are end points of the intervals J'_{Pn} but are interior to no J'_{Pn}, for n fixed and P varying over E_r, form at most a denumerable set D_{rn}, which is a set of type τ_σ.

Select from the set of intervals J'_{Pn}, for n fixed and P varying over E_r, a denumerable subset containing the same interior points, and let T'_n be the sum of the T'_{Pn} in the denumerable subset selected.

Let $T_r = \sum_{n=1}^{\infty}(T'_n + D_{rn})$. The set T_r is a set of type τ_σ. At every point P of $E_r - T_r$, $u_{Pd_2} \geqslant r$.

Let the value of r vary over all rational numbers r_ν, and let $T = \sum_{\nu=1}^{\infty} T_{r_\nu}$. The set T is also a set of type τ_σ. If P is any point of s which does not belong to the set T, then $u_{Pd_2} \geqslant l^{\tau}_{Pd_1}$, for if P is a point at which $u_{Pd_2} < l^{\tau}_{Pd_1}$, there is a rational number r_ν such that $l^{\tau}_{Pd_1} > r_\nu > u_{Pd_2}$, and therefore P belongs to the set T_{r_ν}.

Similarly we can prove that, except possibly at points of a set of type τ_σ, $l_{Pd_2} \leqslant u^{\tau}_{Pd_1}$.

An example will suffice to establish Part b).

Suppose s to be a given line in the xy plane and T a set of points on s of type τ_σ, i. e., $T = \sum_{n=1}^{\infty} T_n$ where T_n is a set of type τ. Let d_1 and d_2 be two given directions of approach to s on the same side. At all points (x,y) on lines through points of T_n in direction d_2, define $f(x,y)$ to be $\frac{1}{2^n}$ and at all other points of the plane to be 0. In approaching a point of T_n in the direction d_1 neglect all points at which the line of direction d_1 crosses lines of direction d_2 through points of $\sum_{i=1}^{n} T_i$. Then at points of T_n, $l^{\tau}_{Pd_1} = 0$, $u^{\tau}_{Pd_1} \leqslant \frac{1}{2^{n+1}}$ and $l_{Pd_2} = u_{Pd_2} = \frac{1}{2^n}$. At all points of s which do not belong to T, $l^{\tau}_{Pd_1} = 0$ and $l_{Pd_2} = u_{Pd_2} = 0$.

This theorem is of particular interest for sets of type τ which are such that τ_σ is equivalent to τ, as, for instance, denumerable sets, exhaustible sets, and sets of measure zero.

3. Variation of Directions.

Theorem 2. a) *If $f(x,y)$ is an arbitrary function of two variables, and s a given straight line in the xy plane, then at every point P of s except possibly at points of a denumerable set, $I_{P\alpha}$ overlaps or abuts $I_{P\beta}$ for all pairs of directions α and β on the same side of s; and* b) *furthermore, if D is any denumerable set of points on s, there exists a function $f(x,y)$ such that at every point P of s which does not belong to D, $I_{P\alpha}$ overlaps or abuts $I_{P\beta}$ for every pair of directions α and β of approach to s on the same side, and at every point P of D, $I_{P\alpha}$ neither overlaps nor abuts $I_{P\beta}$.*

Proof. All directions involved in this proof are assumed to be on one side of s.

Let $\varkappa$, λ be two given directions and r a given number. Let E_r be the set of points P of s such that along at least one direction between $\varkappa$ and λ the lower bound of values of f at P is $> r$. Let $d_{P\varkappa\lambda}$ be a chosen direction between $\varkappa$ and λ at a point P of E_r along which $l_{Pd} > r$, and $\{d_{P\varkappa\lambda}\}$ the set of directions associated one with each point P of E_r. Let δ be a given direction not contained between $\varkappa$ and λ. At each point P there is an interval J_{Pn} of length $< \frac{1}{n}$ and in the direction $d_{P\varkappa\lambda}$ associated with P, at every point of which $f > r$. Let J'_{Pn} be the interval on s which is the projection of J_{Pn} in the direction δ. The points of E_r which are end points of the J'_{Pn} but are interior to no J'_{Pn}, for n fixed and P varying over E_r, form a denumerable set D_{rn}. If $D_r = \sum_{n=1}^{\infty} D_{rn}$, then at a point of $E_r - D_r$, $u_{P\delta} \geqslant r$. Moreover, if d is any direction on the opposite side of δ from $\varkappa$ and λ, it follows that $u_{Pd'} \geqslant r$.

Let r vary over all rational numbers r_ν and let $\Delta = \sum_{\nu=1}^{\infty} D_{r_\nu}$. Also let $\varkappa$, λ, and δ each vary over a dense denumerable set of directions, independently except for the restriction that δ never be included between $\varkappa$ and λ. Let D be the sum of the denumerable number of sets Δ thus obtained. The set D is denumerable.

We are now prepared to show that if P is any point at which there exists a pair of directions α and β such that $l_{P\alpha} > u_{P\beta}$, the point P belongs to D. For from the dense set of directions we can select a $\varkappa$ and λ including α in the angle between them but not including β, and a δ between β and the nearer of the two directions $\varkappa$ and λ; and we can select a rational number r_ν such that $l_{P\alpha} > r_\nu > u_{P\beta}$. The point P belongs to the set E_{r_ν} associated with $\varkappa$ and λ, but, since between $\varkappa$ and λ there is a $d_{P\varkappa\lambda}$, which may or may not coincide with α, along which $l_{Pd} > r_\nu$ while $u_{P\beta} < r_\nu$, the point P belongs to the denumerable set D_{r_ν} associated with the $\varkappa$, λ, δ chosen.

Similarly we can prove that, for all pairs of directions α and β, $l_{P\beta} \leqslant u_{P\alpha}$, except possibly at points of a denumerable set.

Part b) will be established by means of an example.

Let s be a given line in the xy plane and D a denumerable set of points on s. At each point of D on one side of s take a circle tangent to s in such a way that no two circles overlap or are tangent to each other. At all points outside of this set of circles define $f(x,y)$ to be 0. In each circle suppose all possible chords to be drawn and extended to intersect a line t parallel to s and one unit distant on the same side of s as the circles. If this point of intersection is at a distance ξ from the point at which t crosses the x axis (y axis if t is parallel to the x axis) define $f(x,y)$ to be equal to ξ at every point of the chord. Then at a point of s which does not belong to D, for any pair of directions α and β of approach to s, 0 will be among the set of limits of functional values in each direction, thus insuring the overlapping or abutting of $I_{P\alpha}$ and $I_{P\beta}$, and at a point of D, for any pair of directions α and β, $I_{P\alpha}$ and $I_{P\beta}$ will each consist of a single point and will not coincide.

If a slight restriction is imposed on one of the directions of approach the result in Theorem 1 a) is valid with the directions freed of their fixed positions. It is only necessary to remark that at each point there may be a pencil of type τ_σ to which one of the directions can not belong, where a pencil of type τ_σ is the pencil of lines obtained by joining each point of a linear set of type τ_σ to a point not on the line.

In these theorems rectilinearity is not an essential property. By an analysis situs transformation, for example, the straight lines may be converted into curves, and results for curves analogous to those for straight lines are thus made readily accessible. It is also possible to use connected sets instead of straight lines.

4. Functions of Three Variables.

All planes parallel to a given plane in a 3-space will be said to have the same planar direction, that of the given plane. Greek letters will be used to designate planar directions. Let $f(x,y,z)$ be a function of three variables, q a plane in the 3-space of the independent variables, and P a point in the plane q. By the lower bound l_{P_π} at the point P of $f(x,y,z)$ in the planar direction π on one side of q is meant the upper bound of the lower bounds of values of f at points of semicircles with P as center located in a plane of direction π through P. The upper bound u_{P_π} is defined similarly, and the interval (l_{P_π}, u_{P_π}) is denoted by I_{P_π}. The lower bound l_{P_d} at the point P of f in the linear direction d is the lower bound of values of f as the point (x,y,z) approaches P along the linear direction d. Likewise the upper bound u_{P_d} is defined, and the interval (l_{P_d}, u_{P_d}) denoted by I_{P_d}.

Theorem 3. *If $f(x,y,z)$ is an arbitrary function of three variables, and q a given plane, then, at every point P of q except possibly at points of an exhaustible set, I_{P_π} overlaps or abuts I_{P_d} for any planar direction π and any linear direction d on the same side of q.*

Proof. All lines and planes are assumed to be on one side of q.

If d is contained in π, every I_{P_d} is contained in I_{P_π}.

A sheaf H_{P_k} of planes is defined to consist of all planes through P perpendicular to the lines contained in a cone of vertex P with axis in direction k.

Let E_r be the set of points P of q at which there exists at least one plane contained in the sheaf H_{P_k} along which the lower bound of values of f as P is approached is $>r$. Let π be the planar direction of a chosen plane contained in H_{P_k} at a point P of E_r along which $l_{P_\pi}>r$, and $\{\pi\}$ the set of directions associated one with each point P of E_r. Each point P is the center of a semicircle S_{P_n} of radius $<\frac{1}{n}$ lying in the plane associated with P, at every point of which $f(x,y,z)>r$.

Let C_b be a cone of linear directions not contained in the sheaf H_{P_k}, the axis of which has the linear direction b and the angle of which may be taken arbitrarily small. Project each semicircle S_{P_n} in the directions C_b on the plane q and let the product of the projections be S'_{P_n}. The S'_{P_n} will be convex regions in the plane q. The points P of E_r which are interior to no S'_{P_n}, for n fixed and P varying over E_r, form a nowhere dense set N_{rn}. If $N_r=\sum_{n=1}^{\infty} N_{rn}$, then at a point of E_r-N_r, $u_{P_d}\geqslant r$ for all d contained in C_b.

Let r vary over all rational numbers r_ν and let $N=\sum_{\nu=1}^{\infty} N_{r_\nu}$. The set N is exhaustible. Let b and k each vary over a dense denumerable set of linear directions in the 3-space. Let M be the sum of the denumerable number of sets N thus obtained. The set M is exhaustible.

We are now prepared to show that if P is any point at which there exists a planar direction π and a linear direction d such that $l_{P_\pi} > u_{P_d}$, the point P belongs to M. For a sheaf H_{P_k} containing a plane of direction π and a cone C_b containing a line of direction d but containing no line contained in H_{P_k} can be selected; and a rational number r_ν can be selected such that $l_{P_\pi} > r_\nu > u_{P_d}$ The point P belongs to the set E_{r_ν} associated with k, but since in H_{P_k} there is a plane of direction π', which may or may not coincide with π, along which $l_{P_{\pi'}} > r_\nu$ while $u_{P_d} < r_\nu$, the point P belongs to the N_{r_ν} associated with the k and b chosen.

Similarly it can be proved that for all pairs consisting of a planar direction π and a linear direction d, $u_{P_\pi} \geqslant l_{P_d}$ except possibly at points of an exhaustible set.

Über die zusammenziehende und Lipschitzsche Transformationen [1]).

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Es sei A eine beliebige in einem metrischen Raume $\mathfrak{C}$ enthaltene Menge, und $\varrho(x, y)$ bezeichne die Entfernung zweier Punkte x und y von $\mathfrak{C}$.

Eine Transformation [2]) f, die die Menge A auf eine Untermenge von $\mathfrak{C}$ abbildet und die Bedingung

$$(1) \qquad \varrho(f(x), f(y)) \leqslant \varrho(x, y)$$

erfüllt, heisse eine *zusammenziehende Transformation*, oder kurz eine *Zusammenziehung* [3]).

Das ist also eine Verallgemeinerung der isometrischen Trans-

[1]) Die Ergebnisse dieser Arbeit wurden grösstenteils schon in den Jahren 1926—1930 gefunden und in meiner Magister-Dissertation (Warschau, 1930) zusammengefasst. Jetzt werden sie etwas weitergeführt und gleichzeitig vereinfacht. Vgl. auch: A. Lindenbaum et A. Tarski, *Communication sur les recherches de la Théorie des Ensembles*, Comptes Rendus de la Société des Sciences de Varsovie XIX (1926). p. 327—328.

[2]) Transformation = Abbildung. (Im allgemeinen nicht unbedingt schlichte, und zwar nicht unbedingt eindeutige Funktion. Vgl. § 22).

[3]) Der Begriff der Transformation, die die inverse Bedingung $(\varrho(f(x), f(y)) \geqslant \varrho(x, y))$ erfüllt, wurde schon von E. Schmidt als „asphinktische Transformation" eingeführt. Vgl. E. Schmidt, *Über die Definition des Begriffes der Länge krummer Linien*, Math. Ann. 55 (1902). Die Transformation mit der Bedingung (1) wurde letztens von A. Kolmogoroff gebraucht und „dehnungslose Abbildung" genannt. Vgl. A. Kolmogoroff, *Beiträge zur Masstheorie*, Math. Ann. 107 (1933). Vgl. auch Fussnote [4]).