Concerning a problem of K. Borsuk.

By

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The following problem was recently proposed by M. K. Borsuk[1]): Is every (compact) subcontinuum C of E_n, which cuts E_n and for which there exists for every $\epsilon > 0$ an ϵ-transformation[2]) of C into a set C' such that $C \cdot C' = 0$, an $(n-1)$-dimensional manifold?

Although, as pointed out below, the answer to this question is negative, even (if $n > 2$) for the case where C is a Jordan continuum[3]) (unless restrictions be put upon the Brouwer numbers $p^r(C)$ where $r > 0$), we are able to obtain several positive results through either of two devices, viz., by further restrictions on the nature of C, or by modification of the type of transformation.

[1]) *Fund. Math.*, 20 (1933), p. 285, Problem 54. Since this paper was presented for publication, M. Borsuk has communicated to me that it was his intention to formulate this problem only for *lokal zusammenziehbare* continua, the words „lokal zusammenziehbare" having been omitted through an oversight (for definition see *Fund. Math.*, vol. 19, p. 236). Thus, M. Borsuk's problem should really be stated as „Ist jedes lokal zusammenziehbares Teilkontinuum C... eine $(n-1)$-dimensionale Mannigfaltigkeit?" However, as M. Borsuk points out in his communication, since all Betti numbers of a finite dimensional, *lokal zusammenziehbar* compact space are finite (for proof see M. Borsuk's note „Zur kombinatorischen Eigenschaften der Retrakte", *Fund. Math.*, vol. 21), my Theorem 3 below furnishes an *affirmative* answer to M. Borsuk's problem (amended as just stated) for the case $n = 3$.

[2]) That is, a continuous mapping f of C such that if $P \subset C$, then $\varrho[P, f(P)] < \epsilon$.

[3]) By Jordan continuum (= continuous curve, = Peano continuum) we mean a locally connected, compact continuum.

It follows easily from the conditions stated in the problem that C is a common boundary of two and only two domains of E_n, and if C is a Jordan continuum then these domains are uniformly locally connected[4]). Thus, when C is a Jordan continuum and $n = 2$, C is a simple closed curve; and if $n = 3$ and $p^1(C)$ is finite, C is a closed 2-dimensional manifold.

We also consider the problem: In E_n let C be a compact continuum which cuts E_n and which may be deformed continuously without meeting itself[5]); is C an $(n-1)$-manifold? We show that C is a Jordan continuum whose complement is two uniformly locally connected domains having C as a common boundary, and for $n = 2$, 3 is a closed $(n-1)$-manifold. In every case, whether C cuts E_n or not, its points are regularly accessible[6]) from its complement.

Theorem 1. *In E_n, let C be a compact continuum such that for every $\epsilon > 0$ there exists an ϵ-transformation of C into a set C' such that $C \cdot C' = 0$. Then the complement of C is either one domain whose boundary is C, or two domains of which C is the common boundary.*

Proof. Let D be a domain complementary to C. Denoting the boundary of D by B, suppose P is a point of C not in B, and let Q be an arbitrary fixed point of D. Then B separates P and Q in E_n. Let $f(C) = C'$ be an ϵ-transformation of C such that $C \cdot C' = 0$, and with ϵ small enough that $f(P)$ is not in D and B does not meet $P + Q$ during the rectilinear deformation[7]) of B into $f(B)$. By a theorem of Pontrjagin[8]) $f(B)$ separates P and Q in E_n. Now

[4]) A domain D is uniformly locally connected if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if P and Q are points of D whose distance aparts is $< \delta$, then P and Q may be joined by an arc of D of diameter $< \epsilon$.

[5]) The precise definition is given below.

[6]) That is, given a point P of C and an $\epsilon > 0$ there exists a $\delta > 0$ such that if Q is a point of $(E_n - C) \cdot S(P, \delta)$ then P and Q may be joined by an arc which lies wholly in $S(P, \epsilon)$ and meets C only in P.

[7]) That is, a deformation $F(B, t)$, $0 \leqq t \leqq 1$, such that if $P \subset B$ the point P moves along the straight line joining P and $f(P)$ in such a way that $\varrho[P, F(P, t)] / \varrho[P, f(P)] = t$.

[8]) L. Pontrjagin, *Zum Alexanderschen Dualitätssatz*, Gött. Nach., Math.-Phys. Kl., 1927, pp. 315—322, Satz IV. I have made considerable use, both in the present paper and in others, of this fundamental and important theorem of Pontrjagin, so that I feel I should point out what seems to me a defect in its proof and a method for rectifying it. I refer to the last sentence of Pontrjagin's proof, wherein it is asserted that from $\tilde{\gamma}^{n-r-1} \cdot K^{r+1} \neq 0$ it follows that

C' is a continuum, and consequently lies either wholly in D or wholly in $E_n - \bar{D}$. The former case has been ruled out, since $f(P)$ is not in D. In the latter case, $\bar{D} + C$ is a continuum joining P and Q and not meeting $f(B)$, so that this case is impossible. Consequently $B \equiv C$.

There are at most two domains complementary to C. For suppose there are three domains $D_i\,(i=1,2,3)$; let P_i denote a point of D_i. From the three points P_i we form three sets of pairs of points, (P_j, P_k). Let f denote a transformation of C into a set C' such that $C \cdot C' = 0$ and such that C' separates each of the pairs $(P_j P_k)$ (See preceding paragraph) As above, C' must lie wholly in one domain complementary to C, say in D_1. Then $\bar{D}_2 + \bar{D}_3 + C$ is a continuum that does not meet C' and yet contains both P_2 and P_3, thus violating the fact that C' separates P_2 and P_3. Consequently C has at most two complementary domains. This completes the proof.

Before considering the case where C is a Jordan continuum, we note the reason for stipulating this condition. In E_2, let C consist of the following set of points: 1) All points on the curve $y = \sin 1/x$ where $0 < x \leqq 1/\pi$, 2) all points $(0, y)$ for $-1 \leqq y \leqq +1$, and 3) an arc which joins the points $(0, -1)$ and $(1/\pi, 0)$ without otherwise containing any points of 1) + 2) and lies, except for these points, wholly in the fourth quadrant of the plane. For any $\epsilon > 0$, there exists, in either of the domains complementary to C,

$\gamma^{n-r-1} \cdot C \neq 0$. This overlooks the fact that γ^{n-r-1} and C may be mutually exclusive and yet after deformation $\tilde{\gamma}^{n-r-1} \cdot K^{r+1} \neq 0$, as may be shown by simple examples. I should like to suggest substituting the following argument for the last paragraph of Pontrjagin's proof:

Suppose $\Gamma^r \sim 0$ in $E_n - \Phi$; there exists, then, a complex $K^{r+1} \subset E_n - \Phi$ which is bounded by Γ^r. Let P^n be a polyhedral neighborhood of F excluding Γ^r and so small that during the deformation Δ, P^n does not meet Γ^r; also such that for every point $x \subset \Delta(P^n)$, the relation $\varrho(x, \Phi) < \epsilon$ holds, where $\varrho(K^{r+1}, \Phi) = \epsilon$. Clearly $\Delta(P^n) \supset \Phi$.

The cycle Γ^r links P^n, since it links F. Therefore by Satz II there exists a cycle $\tilde{\gamma}^{n-r-1}$ of P^n which is linked with Γ^r. The deformation Δ carries γ^{n-r-1} into a (possibly singular) cycle $\tilde{\gamma}^{n-r-1}$ which is linked with $\Gamma^r = \Delta(\Gamma^r)$. Consequently $\tilde{\gamma}^{n-r-1} \cdot K^{r+1} \neq 0$. However, we have

$$\tilde{\gamma}^{n-r-1} \subset \Delta(P^n) \subset E_n - K^{r+1},$$

and thus the supposition that $\Gamma^r \sim 0$ in $E_n - \Phi$ leads to a contradiction.

a homeomorph $C' = f(C)$ which constitutes an ϵ-transformation of C into C'.

An analogous example may be given in E_3, using a surface which has one of its cross-sections similar to the example in the preceding paragraph, and in which the analogue of the continuum of condensation, M, described in 2) above is such that for every $\epsilon > 0$ there is a rectangular parallelopiped with two of its dimensions $< \epsilon$ enclosing M.

It is to be noted from the above examples that 1) to require that C' be a homeomorph of C is not in itself a requirement strong enough to make C an $(n-1)$-manifold, and 2) in each case C is not a Jordan continuum.

Theorem 2. *Under the same hypothesis as in Theorem 1, with the additional condition that C be a Jordan continuum which cuts E_n, the domains complementary to C are uniformly locally connected.*

For the proof of Theorem 2 we require the following lemma:

Lemma. *Let C be any closed point set in E_n, A a point of C, ϵ an arbitrary positive number, and Γ^r a cycle that links C in $S(A, \epsilon)$* [9]*. Then for any positive number $\epsilon' < \epsilon$ such that $\Gamma^r \subset S(A, \epsilon')$, there exists a positive number η such that if $f(C)$ is any η-transformation of C, then Γ^r links $f(C)$ in $S(A, \epsilon')$.*

Proof of Lemma. Let η be such that $0 < \eta < \frac{1}{2}(\epsilon - \epsilon')$, as well as such that a rectilinear deformation or C into $f(C)$ does not meet Γ^r.

Let Δ denote a rectilinear deformation such as that just mentioned. Let $F = F(A, \epsilon)$. Using a method of proof employed by Pontrjagin [10], the deformation Δ may be extended to an η-deformation Δ' of $F + C$ which agrees on C with Δ and carries no point of F into $S(A, \epsilon')$. By the deformation theorem of Pontrjagin [8] the set $F' + C'$ (where F' and C' denote the sets into which Δ' carries F and C respectively) is linked by Γ^r. But then Γ^r links C' in $S(A, \epsilon')$, since if it bounded a complex in $S(A, \epsilon')$ that did not meet C', such a complex would not meet $F' + C'$.

Proof of Theorem 2. Denoting the domains complementary to C by D_1 and D_2, suppose that D_1 is not uniformly locally connected. Then there exist an $\epsilon > 0$ and a point A of C such that

[9]) That is, Γ^r does not bound a complex in $S(A, \epsilon)$ that does not meet C.

[10]) Loc. cit., p. 321 (proof of Hilfssatz).

in every $S(A,\delta)$, for $0<\delta<\epsilon$, there exist two points of D_1 that cannot be joined by a subcontinuum of $D_1\cdot S(A,\epsilon)$.

Since C is a Jordan continuum there is a $\delta>0$ such that $C\cdot S(A,\delta)\subset M$, where M is ihe component of $C\cdot S(A,\epsilon)$ determined by A. Let P and Q denote points of $D_1\cdot S(A,\delta/2)$ that cannot be joined by a subcontinuum of $D_1\cdot S(A,\epsilon)$. Also, let R be a point of $D_2\cdot S(A,\delta/2)$.

Obviously C separates each of the point pairs (P,Q), (P,R), (Q,R) in $S(A,\epsilon)$. Let H be a subcontinuum of M that contains all points of $M\cdot S(A,\delta)$[11]. Also, let ϵ' be a number such that $\epsilon>\epsilon'>\delta$ and $S(A,\epsilon')$ contains H. By the above Lemma there exists $\eta>0$ such that any η-transformation of C separates each of the above point-pairs in $S(A,\epsilon')$. Let f denote such an η-transformation with the added stipulations that a) $C\cdot f(C)=0$ (permissible by hypothesis), b) $\eta<\delta/2$, and c) $f(H)\subset S(A,\epsilon')$. We note that

$$f(C)\cdot S(A,\delta/2)\subset f(H)\cdot S(A,\delta/2).$$

There are two cases to be considered. 1) Suppose $f(C)=C'\subset D_2$. On the straight line interval PQ let a_1 be the first point of M in the order from P to Q, and a_2 the first point of M in the reverse order. Obviously a_1 and a_2 are points of H, and consequently Pa_1+H+Qa_2 is a subcontinuum of $S(A,\epsilon')$ joining P and Q and containing no point of C'. But this contradicts the fact that C' separates P and Q in $S(A,\epsilon')$. 2) Suppose $C'\subset D_1$. Let b be the first point of M on the straight line interval RP in the order from R to P. On the respective straight line intervals PR and QR, let a_1 and a_2 be the first points of M in the order named. Then on Pa_1-a_1 there exists a point c_1 of $f(H)$, else Pa_1+H+Rb is a subcontinuum of $S(A,\epsilon')$ joining P and R and not meeting C'. Similarly, on Qa_2-a_2 there is a point c_2 of $f(H)$. But then $Pc_1+f(H)+Qc_2$ is a subcontinuum of $S(A,\epsilon')$ that contains no point of C, contradicting the fact that C separates P and Q in $S(A,\epsilon)$.

In either case, then, we have a contradiction, and consequently D_1 is uniformly locally connected. Similarly D_2 is uniformly locally connected.

[11]) See R. L. Wilder, *On connectde and regular point sets*, Bull. Amer. Math. Soc., vol. 34 (1928), pp. 649—655, Th. 5.

For the case $n=2$ we may interpolate the following corollary:

Corollary 1. *Under the hypothesis of Theorem 2, with $n=2$, C is a simple closed curve*[12]).

For $n>2$, even under the hypothesis of Theorem 2, C is not in general an $(n-1)$-manifold. For it is easy to set up, in E_3, a surface C satisfying the conditions of Theorem 2, consisting of a „sphere with infinitely many handles" where the handles converge to a point P. It is quite likely, however, that with the proper restrictions on the connectivity of C or its complementary domains, the surface C will necessarily be an $(n-1)$-manifold. Thus, in E_3 we have.

Theorem 3. *In E_3, let C be a compact Jordan continuum such that $p^1(C)$ is finite, C cuts E_3, and for every $\epsilon>0$ C is ϵ-transformable into a set C' such that $C\cdot C'=0$. Then C is a closed 2-dimensional manifold.*

This theorem is an immediate consequence of my recently established analogue in E_3 of the Schoenflies plane converse theorem[13]) together with the result of Theorem 2 above and well-known duality relations[14]).

If the reader will recall the examples indicated above, he will notice that in the case of each, the ϵ-transformation which we have asserted to exist cannot be extended to any sort of continuous deformation of C into the transformed set, without that the set C meet itself (or pass through itself) during the course of the deformation. It would be interesting to see what conclusions are possible in case we require that for a given set C deformations exist during the course of which C does not meet itself. For this purpose we find it necessary to prove first a general theorem on the local lin-

[12]) See R. L. Moore, *A characterization of Jordan regions by properties having no reference to their boundaries*, Proc. Nat. Acad. Sci., 4 (1918), pp. 364—370.

[13]) To appear soon in Mathematische Annalen in my paper *On the properties of domains and their boundaries in E_n*. See Bull. Amer. Math. Soc., 36 (1930), p. 219, abstract N° 196.

[14]) F. Frankl. Wien Akad. d. Wiss. Math. Naturw. Kl., Sitz., Abt. IIa, 136 (1927), pp. 689—699; P. Alexandroff, Annals of Math., 30 (1928), p. 101—187; S. Lefschetz, Annals of Math., 29 (1928), pp. 232—254, and earlier papers referred to in the last two citations. For a definition of the numbers (Brouwer) $p^r(C)$ see L. E. J. Brouwer, Math. Ann., 72 (1912), pp. 422—425, L. Vietoris, Math. Ann., 97 (1927), pp. 454—472, and P. Alexandroff, loc. cit.

king of continua by $(n-2)$-cycles. As an example of the sort of surface to which the theorem applies, the reader is referred to the third example indicated above.

Theorem 4. *In E_n, let a compact continuum C be a common boundary of (at least) two uniformly locally connected domains D_1 and D_2, and suppose that for every $\epsilon > 0$ there exists in D_1 an $(n-2)$-cycle of diameter $< \epsilon$ that links C. Then there exists a point A of C such that for every $\epsilon > 0$ there are in $S(A, \epsilon)$ a pair of linked cycles Γ^{n-2} and γ^1, where $\Gamma^{n-2} \subset D_1$ and $\gamma^1 \subset D_2$, as well as a simple closed curve of C that is linked by Γ^{n-2}. In addition, γ^1 links C.*

Proof. Under the conditions of the hypothesis there exists a point A of C such that in every $S(A, \epsilon)$ there is an $(n-2)$-cycle of D_1 that links C. Consider now an arbitrary fixed ϵ. We note that C is a Jordan continuum [15]). Let M denote the component of $C \cdot S(A, \epsilon)$ determined by A. There exists in M a Jordan continuum K which contains A as well as every point of M (and of C) which lies in some $S(A, \delta)$, $\delta > 0$ [16]).

The set $K - A$ is connected [17]) and consequently there is in $K \cdot S(A, \delta)$ a Jordan continuum L which contains A as well as every point of K (and of M and C) which lies in $S(A, \eta)$, $\delta > \eta > 0$, but which contains no point of $K \cdot \overline{C - K}$, and such that $K - L$ is connected [18]).

In $D_1 \cdot S(A, \eta)$ there is an irreducible cycle Γ^{n-2} that links C. Then Γ^{n-2} links K. For suppose not. Then there exists a complex K_1^{n-1} such that

$$K_1^{n-1} \to \Gamma^{n-2}, \qquad [E_n - K].$$

[15]) See R. L. Moore, *On the relation of a continuous curve to its complementary domains in space of three dimensions*, Proc. Nat. Acad. Sci., 8 (1922), 33-38, Th. 1, and R. L. Wilder, *A converse of the Jordan-Brouwer separation theorem in three dimensions*, Trans. Amer. Math. Soc., 32 (1930), pp. 632—657, part II of proof of Th. 3.

[16]) H. Hahn, *Mengentheoretische Charakterisierung der stetigen Kurve*, Wien. Akad. Sitz., 123, Abt. IIa, pp. 2433—2489; see Th. XXI, p. 2475.

[17]) A consequence of Theorem 10 of my paper referred to in [13]).

[18]) See W. L. Ayres, *On continua that are disconnected by the omission of any point and some related problems*, Mon. f. Math. u. Phys., 36, 135—148, Th. 2. Also R. L. Wilder, *On the imbedding of subsets of a metric space in Jordan continua*, Fund. Math. 19 (1932), pp. 45—64, Th. 10.

As Γ^{n-2} lies wholly in $S(A, \eta)$, there is a complex K_2^{n-1} such that

$$K_2^{n-1} \to \Gamma^{n-2} \qquad [S(A, \eta);\ E_n - \overline{C - K}].$$

As Γ^{n-2} links C, it follows from the Alexander Addition Theorem [19]) that the cycle $K_1^{n-1} + K_2^{n-1}$ links $K \cdot \overline{C - K}$. Thus, there exist points P and Q of $K \cdot \overline{C - K}$ that are separated in E_n by $K_1^{n-1} + K_2^{n-1}$.

However, consider the set $F = K \cdot (K_1^{n-1} + K_2^{n-1})$. Clearly $F \subset L$. Now $K \cdot \overline{C - K} \subset K - L$, and therefore P and Q are points of $K - L$. But then $K - L$ is a connected set containing P and Q and not meeting $K_1^{n-1} + K_2^{n-1}$. Thus we conclude that Γ^{n-2} must link K.

It follows [20]), then, that Γ^{n-2} links a simple closed curve J of K. Furthermore, since D_2 is uniformly locally connected, there exists in $D_2 \cdot S(A, \epsilon)$ an irreducible cycle γ^1 which is linked with Γ^{n-2} [21]). That γ^1 links C follows immediately, since if it bounded in $E_n - C$ it would bound in D_2 and hence in $E_n - \Gamma^{n-2}$.

Definition. Let C denote a subset of any topological space S. We say that there exists a *deformation of C during the course of which C does not meet itself*, provided there exists a continuous function $f(P, t)$, $P \subset C$, $0 \leqq t \leqq 1$, such that $f(P, t) \subset S$, $f(P, 0) = P \subset C$, and such that $C \cdot f(P, t) = 0$ if $t > 0$.

For the sake of brevity, we shall call a deformation of C during the course of which C does not meet itself, a Δ-deformation [22]). In case C is a set for which there exists a Δ-deformation, we shall call C Δ-deformable.

[19]) J. W. Alexander, *A proof and extension of the Jordan-Brouwer separation theorem*, Trans. Amer. Math. Soc., 23 (1922), pp. 333—349, Corollary W^i.

[20]) By Theorem 3 of my paper *On the linking of Jordan continua in E_n by $(n-2)$-cycles*, to appear soon in Annals of Math.

[21]) The cycle γ^1 is obtained by approximation to J, using the uniform local connectedness of D_2. See my paper referred to in [15]). Also see, in regard to the mutual linking of Γ^{n-2} and γ^1, L. Pontrjagin, loc. cit., Satz III.

[22]) Instead of requiring, in the definition of Δ-deformation, that $C \cdot f(P, t) = 0$ if $t > 0$, we might stipulate instead that for given P, $f(P, t_1) \subset C$ implies that $f(P, t) = P$ for $0 \leqq t \leqq t_1$, so that it is not necessary that all points of C leave their initial position simultaneously. However, the existence of such a deformation implies the existence of a Δ-deformation $\varphi(C, t)$: For given P, let t_1 be the greatest value of t such that $f(P, t) = P$. Then, let $\varphi(P, t') = f(P, t_1 + t'(1 - t_1))$ for $0 \leqq t' \leqq 1$. Then $\varphi(C, t')$ is a Δ-deformation of C.

Theorem 5. *In E_n, let C be a compact continuum which cuts E_n and is Δ-deformable in E_n. Then C is a Jordan continuum whose complement is the sum of two and only two uniformly locally connected domains of which C is the common boundary.*

Proof. That $E_n - C$ is the sum of just two domains D_1 and D_2 of which C is the common boundary follows from Theorem 1. We shall show that these domains are uniformly locally connected, from which the conclusion that C is a Jordan continuum follows[15]).

By hypothesis C is deformable into a set C' by means of a Δ-deformation, which we denote briefly by Δ. where C', being a connected point set, must lie in D_2, say. We note then that $f(P, t)$, where f is the function defining Δ, lies in D_2 for all t such that $0 < t$.

Suppose D_1 is not uniformly locally connected. Then there exist an $\epsilon > 0$, a point A of C, and a sequence of pairs of points $(P_i Q_i)$, $i = 1, 2, 3, \ldots$, belonging to $D_1 \cdot S(A, \epsilon)$, such that $\lim_{n\to\infty} P_i = A$, $\lim_{n\to\infty} Q_i = A$, and each pair of points $(P_i Q_i)$ is separated in E_n by $C + F(A, \epsilon)$.

It is well-known[23]) that Δ may be extended into a deformation Δ' of the set $H = C + F(A, \epsilon)$. Although Δ' is not a Δ-deformation of H, it agrees, on C, with Δ. There exists a value of $t = t_1 > 0$ such that during that part of the deformation Δ' which takes place over the interval $0 \leqq t \leqq t_1$, $F(A, \epsilon)$ does not meet A, and, indeed, does not enter a certain neighborhood $S(A, \eta)$. Let us denote the part of the deformation Δ' just referred to by Δ''. Then Δ'' carries C into a set C'' of D_2, and H into a set H'.

Let δ be a positive number such that $\delta < \eta$ and there are no points of C'' in $S(A, \delta)$. Consider a fixed pair of points (P_k, Q_k) in $S(A, \delta)$. As stated above, H separates (P_k, Q_k) in E_n; but H' has no points in $S(A, \delta)$, so that H' does not separate (P_k, Q_k). Then by the theorem of Pontrjagin referred to above, if this is the case, H must meet either P_k or Q_k during the deformation Δ''. However, C deforms entirely in D_2 (whereas P_k and Q_k lie in D_1), and $F(A. \epsilon)$ does not enter $S(A, \delta)$ at all. Consequently the supposition that D_1 is not uniformly locally connected leads to a contradiction.

Let us now consider D_2. Let A denote any point of C. We again extend Δ into a deformation Δ' as defined above, and we now select $t = t_1 > 0$ so that during the portion Δ'' of Δ' over the interval $0 \leqq t \leqq t_1$, C deforms into a set C'', A goes into a point A'' in $S(A, \epsilon)$ in such a way that its path of deformation, $\overrightarrow{AA''}$, does not leave $S(A, \epsilon)$, and $F(A, \epsilon)$ does not meet A. As the path $\overrightarrow{AA''}$ is connected, it lies, except for A, wholly in one component, D, of $D_2 \cdot S(A, \epsilon)$. Also, there exists an $\eta > 0$ such that 1) $F(A, \epsilon)$ does not enter $S(A. \eta)$ during the deformation Δ'' and 2) every point in $C \cdot S(A, \eta)$ is deformed into a point in D without leaving $S(A, \epsilon)$.

Since no point of C meets C again during the deformation Δ'' every point of $C \cdot S(A, \eta)$ must lie on the boundary of D. But this clearly implies that either a) D is the only component of $D_2 \cdot S(A, \epsilon)$ that has points in $S(A, \eta)$, or b) if another such component exists, such points of its boundary as lie in $S(A, \eta)$ are also on the boundary of D. If case a) holds, D_2 is clearly uniformly locally connected.

Consider case b). There exists $\eta' < \eta$ such that no point of C exterior to $S(A, \eta)$ enters $S(A, \eta')$ during the deformation Δ'' and $H' \cdot S(A, \eta') = 0$. Suppose D' another component of $D_2 \cdot S(P, \epsilon)$ having points in $S(A, \eta')$. Let P and P' denote points of D_1 and D', respectively, in $S(A, \eta')$. Since, after the deformation Δ'', the set H' does not separate P and P', and $F(A, \epsilon)$ does not enter $S(A, \eta')$, C must meet either P or P' during the deformation. Suppose x a point of C that meets P' during the deformation (we recall that no point of C can meet P). Then x lies in $S(A, \eta)$, and we have its path of deformation $\overrightarrow{xP'x''}$ passing through P', with x'' in D, yet not leaving $S(A, \epsilon)$ nor meeting C. This is impossible, and consequently there are no components of $D_2 \cdot S(P, \epsilon)$ with points in $S(A, \eta')$ other than D, and accordingly, since A is any point of C, D_2 is uniformly locally connected.

Corollary. *In the plane, let C be a compact continuum which cuts E_2 and is Δ-deformable. Then C is a simple closed curve.*

As another result of Theorem 5, it is easily shown by a stan-

[23]) By virtue of extension theorems for continuous functions. See Hahn, *Reele Funktionen*, 1, (Berlin, 1920), pp. 137 and 140. Also see the proof-method used by O. Haupt, *Über die Erweiterung stetiger Abbildungen*, J. für d. reine u. angew, Math., 168 (1932), pp. 129—130. Also see L. E. J. Brouwer, Math. Ann., 71, p. 309; ibid., 79, p. 209, and P. Urysohn, Math. Ann., 94, p. 293.

dard procedure that every point of C is regularly accessible from its complement. Is the same true in case C does not cut E_n? By Theorem 1, every point of C is a boundary point of the one domain constituting the complement of C in this case. By following through the proof for the uniform local connectedness of D_2 in Theorem 5 above, we see that in this case $(E_n - C) \cdot S(A, \epsilon)$ has at most two components with points in $S(A, \eta')$, and from this that C is regularly accessible. Thus we have the theorem

Theorem 6. *In E_n let C be a Δ-deformable continuum. Then every point of C is regularly accessible from its complement.*

For the special case of $n=3$ we now prove the following theorem:

Theorem 7. *In E_3 let C be a compact and Δ-deformable continuum which cuts E_3. Then C is a closed 2-dimensional manifold.*

Proof. Since, by Theorem 5, C is the common boundary of two uniformly locally connected domains D_1 and D_2, it is necessary only to prove that for some $\epsilon > 0$ there is no 1-cycle of diameter $< \epsilon$ that links C[24]). Suppose this not to be the case. Since, by Theorem 1, $E_3 - C$ consists of just the two domains D_1, D_2, the hypothesis of Theorem 4 is satisfied for at least one of these domains, say D_1, and we accordingly obtain the point A and the properties stated in the conclusion of Theorem 4.

By the hypothesis there exists a Δ-deformation $f(C, t)$, $0 \leqq t \leqq 1$, and we denote $f(C, 1)$ by C'. The set C' lies wholly in one of the domains complementary to C, say in D_1. Because of the continuity of f, we note that all of the sets $f(C, t)$ lie in D_1 in this case.

By Theorem 4, there exists in $D_2 \cdot S(A, \eta)$, where η is such that $C' \cdot S(A, \eta) = 0$, a 1-cycle γ^1 which links C. Obviously, however, γ^1 does not link C' since it bounds in $S(A, \eta)$. But by the deformation theorem of Pontrjagin[8]), γ^1 must link C', since during the deformation C does not meet γ^1.

If C' lies in D_2, a contradiction is established in the same manner. We must conclude, then, that there exists an $\epsilon > 0$ such that no 1-cycle of $E_3 - C$ links C, and accordingly C is a closed two-dimensional manifold.

Problem: In E_n, $n > 3$, is a compact and Δ-deformable continuum an $(n-1)$-dimensional closed manifold?

[24]) By Theorem 21 of my paper *On the properties*... referred to in footnote [13]).

In conclusion we might note the following result, which follows from Theorem 4 and Theorem 21 of my paper referred to in footnote [13]):

Theorem 8. *In order that a compact continuum C in E_3 should be a closed two-dimensional manifold it is necessary and sufficient that C be a common boundary of (at least) two uniformly locally connected domains D_1 and D_2 and that there exist an $\epsilon > 0$ such that no 1-cycle of D_1 of diameter less than ϵ links C.*