(14) *zwei Bögen* $(\bar{p}_\nu\ p_\nu^*)$ *und* $(\bar{p}_\mu^2\ p_\mu^*)$ *haben höchstens den Endpunkt* $p_\nu^* = p_\mu^*$ *gemein.*

Analog zeigt man die Existenz von n Bögen $(\bar{q}_\nu^2\ q_\nu^*)$ mit den Endpunkten $\bar{q}_\nu^2$ und q_ν^*, sodass gilt:

(12′) $$(\bar{q}_\nu^2\ q_\nu^*) \subset \{q_\nu^*\} + B_1 + B_2 + \dots;$$

(13′) $$(\bar{q}_\nu^2\ q_\nu^*) \subset V_1;$$

(14′) *je zwei Bögen* $(\bar{q}_\nu^2\ q_\nu^*)$ *und* $(\bar{q}_\mu^2\ q_\mu^*)$ *haben höchstens den Endpunkt* $q_\nu^* = q_\mu^*$ *gemein.*

Aus (4), (13) und (13′) folgt:

(15) *je zwei Bögen* $(\bar{p}_\nu^2\ p_\nu^*)$ *und* $(\bar{q}_\mu^2\ q_\mu^*)$ *sind zu einander fremd.*

Wir bezeichnen nun mit $(p_\nu^2\ q_\nu^2)$ den Teilbogen von C_ν mit den Endpunkten p_ν^2 und q_ν^2. Dann sind die Punkte $\bar{p}_\nu^2$ und $\bar{q}_\nu^2$ miteinander verbunden durch einen Teilbogen von $(\bar{p}_\nu^2\ p_\nu^2) + (p_\nu^2\ q_\nu^2) + (q_\nu^2\ \bar{q}_\nu^2)$, der zu allen Bögen $(\bar{p}_\mu^2\ p_\mu^*)$ und $(\bar{q}_\mu^2\ q_\mu^*)$ mit $\mu \neq \nu$ wegen (6) fremd ist; und je zwei dieser Teilbögen sind ebenfalls wegen (6) zueinander fremd. Wegen (1) kann man also für jedes ν einen Bogen $(\bar{p}_\nu^2\ \bar{q}_\nu^2)$ mit den Endpunkten $\bar{p}_\nu^2$ und $\bar{q}_\nu^2$ finden, sodass folgendes gilt:

(16) $$(\bar{p}_\nu^2\ \bar{q}_\nu^2) \subset B_1 + B_2 + B_3 \dots;$$

(17) $(\bar{p}_\nu^2\ \bar{q}_\nu^2)$ *ist zu* $(\bar{p}_\mu^2\ \bar{q}_\mu^2)$, $(\bar{p}_\mu^2\ p_\mu^*)$ *und* $(\bar{q}_\mu\ q_\mu^*)$ *fremd* $(\mu \neq \nu)$.

Die Bogensumme $(p_\nu^*\ \bar{p}_\nu^2) + (\bar{p}_\nu^2\ \bar{q}_\nu^2) + (\bar{q}_\nu^2\ q_\nu^*)$ enthält einen Bogen C_ν^* mit den Endpunkten p_ν^* und q_ν^*. Aus (14), (14′), (15) und (17) folgt erstens:

C_μ^* und C_ν^* haben höchstens Endpunkte gemein $(\mu \neq \nu)$.

Aus (12), (12′) und (16) ergibt sich zweitens:

$$C_\nu^* \subset P + Q + B_1 + B_2 + \dots.$$

Damit ist auch der Zusatz bewiesen.

Wien, 1930.

A point set characterization of closed 2-dimensional manifolds[1]).

By

J. H. Roberts[2]) (Philadelphia).

Much work has been done on the problem of characterizing various point sets by internal properties. For example R. L. Moore has given[3]) a set of axioms in terms of *point* and *region* which determine the euclidean plane. C. Kuratowski has characterized[4]) the topological sphere as a Peano space with no cut point and having the property of Janiszewski[5]). This result is also contained in work of Leo Zippin[6]).

Miss I. Gawehn has given[7]) a set of four conditions in terms of *point* and *neighborhood* which are necessary and sufficient that a point set be a *closed* 2-*dimensional manifold*[8]). It readily follows

[1]) Presented to the American Mathematical Society, Feb. 22, 1930.

[2]) National Research Fellow.

[3]) *On the foundations of plane analysis situs*, Transactions of the American Mathematical Society, vol. 17 (1916), pp. 131—164.

[4]) *Une caractérisation topologique de la surface de la sphère*, Fundamenta Mathematicae, vol. 13 (1929), pp. 307—318.

[5]) A space M has the property of Janiszewski if, given a continuum C in M which does not cut M, for every decomposition of C into two continua K and L the product $K \cdot L$ is a continuum.

[6]) *A study of continuous curves and their relation to the Janiszewski-Mullikin theorem*, Transactions of the American Math. Soc., vol. 31 (1929), pp. 744—770.

[7]) *Über unberandete 2-dimensionale Mannigfaltigkeiten*, Mathematische Annalen, vol. 98 (1927) pp. 321—354.

[8]) For a definition of this term see O. Veblen, *Analysis situs*, Cambridge Colloquium Lectures, vol. 5, part II, pp. 44—45; or Kerékjártó, *Vorlesungen über Topologie*, p. 132.

from her work that a closed 2-dimensional manifold is a compact metric continuum which is locally a plane [1]). In the present paper this result is strongly used.

I wish to thank Prof. J. R. Kline for his help and encouragement in the writing of this paper.

Definitions: Any closed point set which is homeomorphic with a subset of some arc is said to be an *arc set*. A point set M is said to have the *arc property* provided that every arc set which is a subset of M lies on an arc which is a subset of M.

Moore and Kline have proved [2]) that a number plane has the arc property, and their method can be used to show the same result for every closed, 2-dimensional manifold.

Theorem. *In order that a compact continuous curve* [3]) *M containing at least ONE simple closed curve be a closed 2-dimensional manifold it is necessary and sufficient that M have the arc property and, for some k ($k \geqq 0$) contain $2k$ simple closed curves $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_k, \beta_k$ such that* (a) *$\alpha_i \cdot \beta_i$ is at most one point,* (b) *$(\alpha_i + \beta_i) \cdot (\alpha_j + \beta_j) = 0$ $(i \neq j)$ and* (c) *if γ_i denotes α_i or β_i $(i = 1, 2, \ldots, k)$ and K denotes $M - \Sigma_1^k \gamma_i$, then K is connected but every simple closed curve in K separates K.*

I shall show that the conditions are sufficient by showing that M is locally a plane.

1. Let P denote any point of M which does not belong to $\alpha_i \cdot \beta_i$, and for each i choose γ_i to be α_i or β_i so that γ_i does not contain P. Let K denote $M - \Sigma_1^k \gamma_i$. Now Leo Zippin has shown [4]) that a continuous curve which satisfies the Jordan curve theorem non-vacuously is homeomorphic with the complement on a sphere of a closed and totally disconnected (or vacuous) point set. I shall show that K is a continuous curve which satisfies the Jordan curve theorem non-vacuously.

[1]) A point set M is said to be *locally a plane at the point P of M* if M contains a neighborhood of P which is homeomorphic with a plane. If M is locally a plane at each of its points it is said to be *locally a plane*.

[2]) Annals of Mathematics, vol. 20 (1918–19) pp. 218–223.

[3]) A *continuous curve* is here understood to mean a metric, locally compact, connected and connected im kleinen space.

[4]) *On continuous curves and the Jordan curve theorem*, American Journal of Math., vol. 52, (1930), pp. 331–350. A continuous curve S satisfies the Jordan curve theorem non-vacuously if it contains at least one simple closed curve, and every simple closed curve which it contains has exactly two complementary domains and is the boundary of each of these domains.

First, K is metric and locally compact, as it is a subset of such a set. By hypothesis K is connected. As K is the complement of a closed (or vacuous) set in a continuous curve it is connected im kleinen. Thus K is a continuous curve. If $k = 0$, so that $K = M$, then by hypothesis K contains at least one simple closed curve. Suppose $k \neq 0$ and γ_1 is β_1. Then if $\alpha_1 \cdot \beta_1 = 0$ the simple closed curve α_1 lies in K. If $\alpha_1 \cdot \beta_1$ is a point Q then the set $\alpha_1 - Q$ lies in K. As M has the arc property it contains an arc which contains a subarc QV belonging to β_1 and two infinite sequences of points converging on Q from both sides on α_1. This arc must contain a subarc which lies in K and has just one point (an end point) on α_1. I have thus shown that K contains a triod [1]). Again using the fact that M has the arc property it readily follows that K contains a simple closed curve.

It remains to show that every simple closed curve in K has exactly two complementary domains and is the complete boundary of each (in K). Let J be any simple closed curve in K. By hypothesis $K - J = S_1 + S_2$, where S_1 and S_2 are mutually separated sets. Let AB denote an arc which lies in S_1 except that B is on J. Let C be a point of $J - B$, and let CB denote either arc which is a subset of J. Consider an arc in M which contains CB and two infinite sequences of points converging on B on the arc AB and on the set $J - CB$, respectively. Such an arc contains infinitely many subarcs with one point on $J - CB$, and otherwise lying in S_1, such that B is a limit point of the sum of their end points. Thus every limit point of S_1 on J is a limit point from both sides on J of limit points of S_1. But the set $\overline{S}_1 \cdot J$ is closed. Hence it is identical with J. Suppose next that S_1 is not connected. Then $K - J$ has at least three distinct components s_1, s_2, and s_3, each of which is bounded (in K) by the simple closed curve J. As M

[1]) A *triod* is the sum of three arcs which have a common end point and the arcs of each pair are otherwise mutually exclusive. See Moore, *Concerning triods in the plane and the junction points of plane continua*, Proceedings of the National Academy of Sciences, vol. 14 (1928), pp. 85–88.

has the arc property it readily follows that K contains two simple continuous arcs $A_1X_1B_1$ and $A_2X_2B_2$ lying in s_1 and s_2, respectively, except for their end points, which lie on J in the order $A_1A_2B_2B_1$. By hypothesis the simple closed curve $A_1X_1B_1B_2X_2A_2A_1$ separates K into two sets, t_1 and t_2. Suppose t_1 contains the connected set s_3. Then t_1 contains every point of J which does not belong to $A_1X_1B_1B_2X_2A_2A_1$. But it is easily seen that t_2 must contain some point of J. Thus we have reached a contradiction, which shows that K is homeomorphic with the complement on a sphere of a closed and totally disconnected (or vacuous) set, and is therefore locally a plane. Then M is locally a plane at the point P[1].

2. Now let P denote any point $\alpha_i\cdot\beta_i$ $(i=1,2,\dots,k)$, and let K denote $M-\Sigma_1^k\alpha$. For each h $(h\leqq k)$ let X_h be a point of α_h not on β_h. Let ε be any positive number. As M is locally a plane at X_h, and K is connected and connected im kleinen, it readily follows that K contains a connected set T_ε such that (1) T_ε contains, for each h $(h\leqq k)$, a set F_h such that F_h+X_h is a simple closed curve on which X_h can be approached from both sides of α_h, (2) $\overline{T}_\varepsilon$ does not contain P, but does contain every point of M at a distance greater than ε from P, and (3) $\overline{T}_\varepsilon$ contains an open subset containing the curve α_j $(j\neq i)$. Let H denote the component of $M-\overline{T}_\varepsilon$ which contains P.

As H is locally a plane at every point except possibly P it follows that it contains points G and L on α_i on either side of P (in H) and that there exist in $K\cdot H$ sets S_1, S_2, S_3, and S_4, each homeomorphic with a plane, and such that (1) $\overline{S}_1\cdot\overline{S}_2=GP$ and $\overline{S}_3\cdot\overline{S}_4=LP$ (where GP and LP are arcs in α_i and in H), and (2) $(\overline{S}_1+\overline{S}_2)\cdot(\overline{S}_3+\overline{S}_4)=P$. As M has the arc property it readily follows that we can let a denote 1 or 2 and b denote 3 or 4 so that H contains infinitely many distinct arcs A_1B_1, A_2B_2, $A_3B_3,\dots$, which lie

[1] We now have the result that M is locally a plane at all but a finite number of its points. It does not follow that such a set is locally a plane at all of its points, even if we assume the arc property. A simple example showing the truth of this is obtained by building a *sphere with infinitely many handles*, taking care that the second handle is on the first, the third on the second, etc. A set is thus obtained which has the arc property and is locally a plane at every point except one, but is not locally a plane at that point.

in K except for their end points, which are on the arcs GP and LP respectively, and such that for each i the arc A_iB_i has two segments A_iC_i and B_iD_i lying in S_a and S_b, respectively. Let Q be a point on the segment A_1B_1.

Suppose that the segment A_1QB_1 does not separate K. Then there exists in K a simple closed curve W on which Q can be approached from both sides of A_1QB_1, and $W\cdot A_1QB_1=Q$. The curve W separates K, and thus separates the segments A_1Q and B_1Q in K. Now for some n the arc A_nB_n has no point on W. Moreover for every e the set of points of $S_a(S_b)$ at a lower distance less than e from α_i contains a connected subset containing arc segments which are subsets of $A_1Q(B_1Q)$ and A_nB_n, respectively. But for some e such a set contains no point of W. Thus $K-W$ contains a connected set containing the segments A_1Q and B_1Q. Thus we have reached a contradiction, which means that the segment A_1QB_1 separates K.

Let U_1 and U_2 denote mutually separated sets whose sum is K minus the segment A_1QB_1. Suppose U_1 contains the connected set T. It is readily seen that every point of the arc A_1QB_1 is a limit point both of U_1 and of U_2. Let V denote an arc in M containing the arc A_1QB_1 and infinite sequences converging on A_1 and lying in U_1 and U_2, respectively. The arc V contains infinitely many subarcs lying in $H-A_1QB_1$, such that the end points of each arc belong to U_1 and U_2, respectively. Each such arc contains a first point in the order from U_2 to U_1 which is not in U_2. This point is on α_i, as it is in H but not in K. It readily follows that one of the two arcs A_1PB_1 and $A_1X_iB_1$ (subsets of α_i) is on the boundary of U_2. Now X_i is not a limit point of U_2 since T_ε belongs to U_1 and $\overline{T}_\varepsilon$ contains a neighborhood of X_i. Thus we see that U_2 is bounded by the sum of the arcs A_1QB_1 and A_1PB_1. It is easily seen that U_2 is connected, and that if J is any simple closed curve in U_2 then $U_2-J=Y_1+Y_2$, where Y_1 and Y_2 are mutually separated connected sets, and Y_1 has no limit point on the simple closed curve $A_1PB_1QA_1$. Thus it follows that the axioms of R. L. Moore's set[1] Σ_1 are satisfied, and U_2 is homeomorphic with a plane.

Let EOF denote any arc which lies in H, and except for its

[1] Loc. cit.

end points lies in U_2, such that E and F lie on the arcs GP and LP, respectively. Then the arc EPF which is a subset of α_i lies in H. As proved above the segment EOF separates K into two sets Z_1 and Z_2. Suppose that Z_1 contains T_ε. Then it follows that Z_2 is bounded by the curve $EPFOE$. But as Z_1 also has limit points on the arc EPF it follows that Z_1 contains U_2, so that U_2 and Z_2 have no common point. As above it follows that Z_2 is homeomorphic with a plane. Now let J denote the curve $A_1QB_1+B_1F+FOE+EA_1$, where B_1F and EA_1 are arcs which are subsets of $\alpha_i \cdot H$. Let N_1 be U_2+Z_2 plus the segment EPF, and let N_2 denote the remainder of $M-J$. It is obvious that no point of N_2 is a limit point of N_1, as the sets $U_2+A_1QB_1+A_1PB_1$ and $Z_2+EOF+EPF$ are closed, their sum contains N_1, but contains no point of N_2. As M is locally a plane at every point of the segment EPF except possibly P, and each such point is a limit point of U_2 and of Z_2, it follows that no point of the segment EPF except P is a limit point of N_2. Now the sets U_2 and Z_2 are open with respect to M. Thus either P is a limit point of N_2 and is the only such point in N_1, or N_1 and N_2 are mutually separated. As the fact that M has the arc property makes the first situation impossible the second obtains. Thus we have established the following

Lemma: *If ε is any positive number then there exist points A_ε, E_ε, F_ε and B_ε on α_i in the order $A_\varepsilon E_\varepsilon PF_\varepsilon B_\varepsilon$. and arcs $A_\varepsilon Q_\varepsilon B_\varepsilon$ and $E_\varepsilon O_\varepsilon F_\varepsilon$ having only their end points not in K such that the simple closed curve J_ε ($J_\varepsilon = A_\varepsilon Q_\varepsilon B_\varepsilon F_\varepsilon O_\varepsilon E_\varepsilon A_\varepsilon$) divides M into two connected sets one of which is N_ε, where* (1) *N_ε contains P,* (2) *N_ε is of diameter less than ε, and* (3) *the segment $E_\varepsilon PF_\varepsilon$ divides N_ε into two mutually separated sets each homeomorphic with a plane.*

Now Schoenflies has proved [1]) the following theorem: „If J_1 and J_2 are simple closed curves in a plane and s_1 and s_2, respectively, denote their interiors, then any continuous $1-1$ correspondence between J_1 and J_2 can be extended to a continuous $1-1$ correspondence between s_1+J_1 and s_2+J_2". It is not difficult to show that the following analogous proposition is true: „If M is a continuous curve in which simple closed curves J_1 and J_2 bound sets s_1 and s_2 which are homeomorphic with a plane, M is locally a plane at every point of J_1 and of J_2, and J_1 and J_2 contain limit points of $M-s_1$ and $M-s_2$, respectively, then any continuous $1-1$ correspondence between J_1 and J_2 can be extended to a continuous $1-1$ correspondence between J_1+s_1 and J_2+s_2".

[1]) *Beiträge zur Theorie der Punktmengen*, Mathematische Annalen, vol. 62 (1906), pp. 286—328. See also J. R. Kline, *A new proof of a theorem due to Schoenflies*, Proceedings of the National Academy of Sciences, vol. 6 (1929), pp. 529—531.

Now let $J_1, J_{1/2}, J_{1/3}, \ldots$ denote an infinite sequence of curves each with the property of the curve J_ε of the lemma, and the additional property that $N_{1/k}$ contains $J_{1/(k+1)}$. The point set $(N_{1/k}+J_{1/k})-N_{1/(k+1)}$ is readily divisible into two sets, each of which, by theorem of the preceding paragraph, is homeomorphic with a circle plus its interior, and whose common part is the sum of the two arcs $E_{1/k}A_{1/(k+1)}$ and $F_{1/k}B_{1/(k+1)}$ on α_i. With the facts here presented one can set up a continuous $1-1$ correspondence between any $J_\varepsilon+N_\varepsilon$ and a circle plus its interior.

Thus M is locally a plane at every point and is therefore [1]) a closed, 2-dimensional manifold.

The necessity of the first condition has been considered in the introduction. From § 25, p. 48 of Veblen's *Analysis Situs* [2]) it is easily seen [2]) that M contains mutually exclusive simple closed curves $\alpha_1, \alpha_2, \ldots, \alpha_k$ whose sum does not separate M but such that $M-\Sigma_1^k \alpha_i$ is separated by every simple closed curve which it contains. The necessity of the second condition of the theorem now follows from the fact that M is locally a plane.

Example: Let J_1 be the circle in 4-dimensional space with equation $x^2+y^2=16$, $z=0$, $w=0$. Let $P(\theta)$ be the point of J_1 such that the angle between the positive part of the x-axis and the radius vector through $P(\theta)$ is θ $(0 \leqq \theta \leqq 2\pi)$. Let $A_i(\theta)$ $(i=1,2,3)$ denote a continuous point function of θ such that (1) for every θ $(0 \leqq \theta \leqq 2\pi)$ the interval $A_i(\theta)\ P(\theta)$ is of unit length, (2) the set $\Sigma_1^3 A_i(\theta)\, P(\theta)$ is a triod lying in the plane normal to J_1 at $P(\theta)$ and (3) $A_1(2\pi)=A_2(0)$, $A_2(2\pi)=A_3(0)$, and $A_3(2\pi)=A_1(0)$. Let H be the point set containing the arc $A_i(\theta)\ P(\theta)$ for every i $(i=1,2,3)$ and θ $(0 \leqq \theta \leqq 2\pi)$. Then H contains the circle J_1. Moreover the collection of points $A_i(\theta)$ $(i=1,2,3;\ 0 \leqq \theta \leqq 2\pi)$ is a simple

[1]) See the introduction.

[2]) Loc. cit. One sees that M is the sum of a subset of a sphere and k distinct sets, each being an anchor ring or a projective plane. From each such anchor ring and projective plane is obtained one curve of the set $\alpha_1, \alpha_2, \ldots, \alpha_k$.

closed curve which I shall call J_2. There exists (in 4 dimensions) a set K which is homeomorphic with a plane, contains no point of H, but is such that $K+J_2$ is homeomorphic with a circle plus its interior in a plane. Let M denote $H+K$. Then obviously M is not a manifold. But it has the arc property, and contains a simple closed curve J_1 such that $M-J_1$ is connected and every simple closed curve in $M-J_1$ separates $M-J_1$.

Consider the following condition: „The set M contains k mutually exclusive simple closed curves $\alpha_1, \alpha_2, \ldots, \alpha_k$ whose sum does not separate M, but such that $M-\Sigma_1^k \alpha_i$ is separated by every simple closed curve which it contains". The example given above shows that if this condition replaces the last condition of the theorem the conclusion no longer follows.

The University of Pennsylvania.

Concerning the proposition that every closed, compact, and totally disconnected set of points is a subset of an arc.

By

G. T. Whyburn (Baltimore).

1. The theorem that every closed, compact and totally disconnected set of points in a space Σ is a subset of a simple continuous arc in Σ was stated by Riesz[1]) in 1906, and by Denjoy[2]) in 1910 and was generalized and proved by Moore and Kline[3]) in 1919 for the case where Σ is the plane. It has been well recognized among topologists that this theorem holds true in case Σ is a euclidean space of any number of dimensions[4]). Evidently it is not valid in case Σ is the space composed of the points of a continuous curve M [= a connected, locally connected, locally compact, metric and separable space] unless some restriction be placed on the continuous curve. For if M is the sum of three arcs ax, bx and cx, where $ax \cdot bx = bx \cdot cx = ax \cdot cx = x$, then obviously no arc in M contains the set $a+b+c$.

The problem of finding a simple and not too restrictive condition on a continuous curve M in order that this proposition be valid in M has been the source of considerable discussion among topologists in recent years. In this article I shall give a solution to this problem embodied in the condition that the continuous curve M

[1]) Comptes Rendus, vol. 141, pp. 650—655.
[2]) Ibid, vol. 151, pp. 138—140.
[3]) Ann. of Math., vol. 20, pp. 218—223.
[4]) So far as the author knows, however, no proof has been given, up to the present time, even for this case of the theorem.