Potentially regular point sets.

By

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PART I.

1. A separable metric space S will be called *potentially regular* if for each point p of S there exists a monotone decreasing sequence of neighborhoods $[U_i]$, $(i = 1, 2, 3, \ldots)$ of p whose boundaries $[B_i]$ are finite point sets and such that

$$p = \prod_1^\infty (U_i + B_i),$$

that is, p is the only point common to the closed neighborhoods $[\overline{U}_i]$. The least integer $O_s(p)$ for which such a sequence $[U_i]$ exists such that $O_s(p) = \operatorname{Lim} \overline{\overline{B}}_i$, where $\overline{\overline{B}}_i$ is the power of B_i, will be called the *order* of p in S.

For compact or locally compact spaces S, obviously the notion of potential regularity and the corresponding order of points is equivalent to the notion of a regular space and the corresponding order or index of points in the sense of Menger and Urysohn[1]. Indeed any space which is regular in the Menger-Urysohn sense is also *potentially regular*. In general, however, the converse of this statement is not true. For example, the set S of all points (x, y) in the plane such that $y = 0$ when $x = 0$ and $y = \sin 1/x$ when $x \neq 0$ and $-1 \leqslant x \leqslant 1$ is potentially regular, as is indeed every set which is connected and irreducibly connected between two

[1]) See K. Menger, *Grundzüge einer Theorie der Kurven*, Math. Ann., Bd. 95, p. 277; P. Urysohn, Comptes Rendus, vol. 175, p. 440

points, but not regular in the Menger-Urysohn sense. However, part I of this paper is devoted to the proof of the following

Theorem. *Let R be any separable metric space which is connected and potentially regular. Then there exists a biunivalued and continuous transformation of R into a separable metric space R^* which is connected and regular in the Menger-Urysohn sense. Furthermore, under this transformation, the property of a finite point set to separate [2]) two given points in R is invariant.*

It follows from the second part of this theorem that the order $O_r(p)$ of any given point of R is equal to the Menger-Urysohn order of the image point p^* of p in R^*.

The proof for the theorem will be based on the sequence of lemmas which follows below (§§ 2—5) which are valid in the space R.

2. *A separable metric space S is potentially regular if and only if every two points of S may be separated in S by a finite set of points.*

That every potentially regular space S has this property is an immediate consequence of the definition of potential regularity. For if p and q are points of such a space, we take the sequence $[U_i]$ of neighborhoods for p and choose i so large that $q \cdot [U_i + B_i] = 0$. Then the finite point set B_i separates p and q in S, for $S - B_i = U_i + [S - \overline{U}_i]$. Let us suppose, then, that the space S has the property of this lemma, and prove that S is potentially regular.

Since S is separable, it therefore contains a countable sequence of points $p_1, p_2, p_3, \ldots$ such that every point of S either belongs to this sequence or is a limit point of it. Now let p be any point whatever of S. Since S obviously is potentially regular at each of its isolated points, we may assume that $p_i \neq p$, for each i. For each i there exists by hypothesis a finite subset X of S which separates p_i and p in S. Clearly there exists a number $r > 0$ such that X also separates p and the set $V_r(p_i)$ of all points whose distance

[2]) A subset X of R is said to separate two points or point sets A and B in R provided that $R - X = R_a + R_b$, where the sets R_a and R_b are mutually separated (i. e., mutually exclusive and neither contains a limit point of the other) and contain A and B respectively.

from p_i is $<r$ in S. For each i there exists a number s_i, $0<s_i<\varrho(p_i, p)$, such that for every number r, $0<r<s_i$, there exists a finite subset of S which separates p and the set $V_r(p_i)$ in S but for no $r>s_i$ does such a set exist. For each i, let r_i be a number so that

2.1) $$s_i > r_i > s_i - 1/i$$

and let X_i be a finite set separating p_i and $V_{r_i}(p_i)$ in S. Then $S-X_i = S_i^p + S_i^v$, where the sets S_i^p and S_i^v, are mutually separated and contain the sets p_i and $V_{r_i}(p_i)$ respectively. For each i, let

$$U_i = S_1^p \cdot S_2^p \cdot S_3^p \ldots S_i^p.$$

Clearly U_i is an open neighborhood of p and the sequence $[U_i]$ is monotone, (i. e., $j>i$ implies $U_j \subset U_i$). And since the boundary B_i of U_i is a subset of $\sum\limits_1^i X_n$, obviously B_i is finite. It remains, then, to show that p is the only point common to the closed neighborhoods $[\overline{U}_i]$. Suppose, on the contrary, that some other point q is common to all the neighborhoods $[\overline{U}_i]$. There exists, by hypothesis, a finite subset X of S which separates p and q in S Hence $S-X = S_p + S_q$, where S_p and S_q are mutually separated and contain p and q respectively. There exists a positive number r such that

2.2) $$V_{4r}(q) \subset S_q.$$

Since q is a limit point of the sequence $p_1, p_2, \ldots$, there exists an integer i such that

2.3) $$1/i < r \quad \text{and} \quad V_r(p_i) \supset q.$$

Now 2.2) and 2.3) together give that $V_{2r}(p_i) \subset S_q$, and therefore

2.4) $$s_i \geqslant 2r.$$

Then 2.1) and 2.4) give that $r_i \geqslant r$. Hence $V_{r_i}(p_i) \supset V_r(p_i) \supset q$ [by 2.3)]. Thus $q \subset S_i^v$, which contradicts the supposition that q belongs to $\overline{U}_i$, because $\overline{U}_i \cdot S_i^v = 0$. Therefore $p = \prod\limits_1^\infty (U_i + B_i)$, and our lemma 2) is proved.

3. *In the space R, every component of the complement of a finite set is open.*

Let $B = p_1 + p_2 + p_3 + \ldots p_n$ be any finite subset of R, let C be any component of $R-B$, and let x be any point of C. For each i, $1 \leqslant i \leqslant n$, by virtue of § 2, there exists a finite subset K_i of R which separates p_i and x in R. Therefore, $M - K_i = M_i(x) + M_i(p_i)$, where the sets $M_i(x)$ and $M_i(p_i)$ are mutually separated and contain the points x and p_i respectively. Now let

$$M(x) = \prod_1^n M_i(x) \quad \text{and} \quad N(x) = \sum_{i=1}^n M_i(p_i).$$

Then clearly

$$M = M(x) + N(x) + \sum_1^n K_i$$

and

$$\overline{M(x)} \cdot \overline{N(x)} = K \subset \sum_1^n K_i.$$

Therefore K is finite and contains, say k, points. But by a theorem proved by Knaster and Kuratowski[3]) and independently by the author, the set $M(x)+K$ is the sum of a finite number $m \leqslant n$ of mutually separated connected sets $C_1, C_2, \ldots, C_m$. One of these sets, say C_j, contains the point x. Since $B \subset N(x)$, therefore $C_j \cdot B = 0$ and hence $C_j \subset C$. Clearly x is not a limit point of $M - C_j$, and thus it follows that a neighborhood U of x exists such that $U \subset C_j \subset C$. Therefore C is open in R; and our lemma is established.

4. *The set Q of all local separating points*[4]) *of R of order >2 is countable.*

Suppose, on the contrary, that Q is uncountable. Now for each point q of Q, a connected neighborhood R_q of q exists whose boundary B_q is finite and such that q is a cut point of R_q. For each such point q of Q, there exists a positive number r such that the

[3]) See *A remark on a theorem of R. L. Moore*, Proc. Ntl. Acad. of Sciences, vol. 13 (1927); also see an abstract of a paper by the author in the Bull. Amer. Math. Soc., vol. 33 (1927), p. 388.

[4]) A point p of a connected potentially regular set R will be called a local separating point of R provided that p is a cut point of some connected open subset of R whose boundary is finite.

set $V_r(q)$ of all points of R whose distance from q is less than r is a subset of R_q. Then since, by supposition, Q is uncountable, it follows that there exists a point x of Q and an uncountable subset K of Q such that for each point y of K, $R_y \supset x$. Now there exists a monotone decreasing family of neighborhoods $[U_i(x)]$ $(i = 1, 2, 3, \ldots)$ of x with finite boundaries B_i such that

$$x = \prod_1^\infty [U_i(x) + B_i].$$

By virtue of § 3 we may assume that each of the neighborhoods $[U_i(x)]$ is connected; for we may replace $U_i(x)$ by the component of $U_i(x)$ which contains x. Now for each positive integer i, let K_i denote the set of all points q of K such that $B_q \cdot U_i(x) = 0$. Then since $K = \sum_1^\infty K_i$, and since K is uncountable it follows that for some i, say $i = m$, K_m is uncountable.

Now for each point p of K_m, $B_p \cdot U_m(x) = 0$. And since $R_p \cdot U_m(x) \supset x$ and $U_m(x)$ is connected, it follows that for each point p of K, $R_p \supset U_m(x)$. Since p is a cut point of R_p, therefore[5]) p is also a cut point of $U_m(x)$. But since K_m is uncountable, it follows[6]) that at least one point h of K_m is a point of order two in the sense defined in this paper of $U_m(x)$ i. e., $O_m(x) = 2$. But since $U_m(x)$ is an open subset of R and the boundary of $U_m(x)$ is finite, it is readily seen that h is a point of order x of R, i. e., $O_r(h) = 2$ This contradicts our supposition, and therefore Q is countable.

5. *Let K be any definite countable subset of R which is dense in R* (i. e., $\overline{K} = R$) *and which contains the set Q of all local separating points of R of order > 2. Let p be any point of R and let R_e be any neighborhood of p whose boundary B_e is a finite point set every point of which is a limit point of $R - R_e$. Then there exists a connected neighborhood R_g of p with boundary B_g belonging to K such that $R_g \subset R_e$ and $\overline{\overline{B}}_g < \overline{\overline{B}}_e$.*

[5]) See R. L. Moore, Proceedings Ntl. Acad of Sci., vol. 14 (1928) pp. 85—88.

[6]) See my paper *On Non-separated Cuttings of Connected Point Sets* (to appear), where it is shown that all save a countable number of the cut points of any connected separable metric space S are points of order 2 in S in the sense of this paper.

Proof. By virtue of § 3, we may suppose, without loss of generality, that R_e is connected. Let x be any point of B_e. It readily follows from § 3 that x is a limit point of some single component C of $R - \overline{R}_e$; and therefore it follows immediately that x is a local separating point of R — for there exists a connected neighborhood U of x with a finite boundary and such that $U \cdot (B_e - x) = 0$, and therefore $U - x = R_e \cdot U + (R - \overline{R}_e) \cdot U$. Thus since $Q \subset K$, either x belongs to K or else x is a point of order 2 of R.

Let $x_1, x_2, \ldots, x_m$ $(m \leqslant \overline{\overline{B}}_e)$ be the points, if any such exist, of B_e which do not belong to K. For each $n \leqslant m$ there exists a monotone decreasing sequence of connected neighborhoods $[U_i^n]$ $(i = 1, 2, 3, \ldots)$ of x_n whose boundaries contain exactly two points A_i^n and B_i^n and such that

$$x_n = \prod_{i=1}^\infty \overline{U}_i^n, \quad \text{and} \quad p \cdot \overline{U}_i^n = 0, \qquad (1 \leqslant n \leqslant m; \quad i = 1, 2, 3, \ldots).$$

It is readily shown that there exists an integer j such that for each pair of integers r and $s \leqslant m$, $\overline{U}_j^r \cdot \overline{U}_j^s = 0$, and such that for each $n \leqslant m$, one of the points A_i^n and B_i^n $(i \geqslant j)$, say A_i^n, belongs to R_e and the other to $R - \overline{R}_e$. For each $n \leqslant m$, the point x_n separates the points A_j^n and B_j^n in the connected set $\overline{U}_j^n$. Thus $\overline{U}_j^n - x_n = U^n(a) + U^n(b)$, where the sets $U^n(a)$ and $U^n(b)$ are mutually separated and contain A_j^n and B_j^n respectively. Let V_n denote the connected[7]) point set $U^n(a) + x_n$. Now let $h_n = h$ denote the set of all points of $V_n = V$ which separate $A_j^n = a$ and $x_n = b$ in V_n. Clearly h_n exists and indeed contains $\sum_{i>j} A_i^n$ I shall now show that it contains at least one point which belongs to K.

If on the contrary, no point of h belongs to K, then since clearly every point of h is a local separating point of R and $Q \subset K$, it follows that every point of h is a point of order two of R. I shall show that under these conditions $h = V - (a + b)$. Suppose, on the contrary that there exists a point q of $V - (a + b)$ which does not belong to h. Now for each point y of h, $V - y = V_a(y) + V_b(y)$, where the sets $V_a(y)$ and $V_b(y)$ are mutually separated

[7]) See Knaster and Kuratowski, Fund. Math., vol. 2., pp. 206—253, see Theorem 6.

and contain a and b respectively. Now let h_1 be the set of all points y of h such that q belongs to the set $V_b(y)$, and let $h_2 = h - h_1$. Then in the order from a to b there must exist a last point [8]) of h_1, i. e., a point f of h_1 such that $h - f \subset V_a(f)$. For if not, then using the fact that h_1 is separable, it is easily shown that a sequence $f_1, f_2, f_3, \ldots$ of points of h_1 exists such that $h_1 \subset A = \sum_{i=1}^{\infty} V_a(f_i)$ and such that, for each i, $f_{i+1} \subset V_b(f_i)$. Since A is an open subset of V, there exists at least one point p_1 of $V - A$ which is a limit point of A. No other point p_2 of $V - A$ is a limit point of A; for if so, there exist, by § 3, mutually exclusive connected neighborhoods U_1 and U_2 of p_1 and p_2 respectively; and then there exists an i such that $V_a(f_i) \cdot U_1 \neq 0 \neq V_a(f_i) \cdot U_2$; and since $p_1 + p_2 \subset V_b(f_i)$, it follows that $f_i \subset U_1 \cdot U_2$, contrary to the fact that $U_1 \cdot U_2 = 0$. Thus p_1 is the only limit point of A in $V - A$; and since, as is easily seen, b belongs to $V - A - p_1$, it follows that p_1 separates a and b in V and therefore belongs to h. Furthermore, since q belongs to $V - A - p_1$, which is identically the set $V_b(p_1)$, then p_1 belongs to h_1, contrary to the fact that $h_1 \subset A$. Thus the supposition that h_1 has no last point is false, and accordingly there exists a last point f of h_1.

Since f is point of order 2 of V, there exists a sequence of neighborhoods $[U_i]$ $(i = 1, 2, \ldots)$ of f with boundaries B_i containing just two points, say r_i and s_i, and such that $f = \prod_{1}^{\infty} (U_i + B_i)$. Now for each i sufficiently large clearly r_i and s_i belong to h; and if we agree that r_i always precedes s_i in the order a, b in V, then r_i belongs to h_1 and s_i to h_2. For each i,

$$V = V_a(r_i) + V_b(r_i) \cdot V_a(s_i) + V_b(s_i) + r_i + s_i.$$

Now by the definition of the sets h_1 and h_2, it follows that q belongs to neither $V_a(r_i)$ nor $V_b(s_i)$; and since $r_i + s_i \subset h$, therefore $q \subset V_b(r_i) \cdot V_a(s_i)$. But the set $V_b(r_i) \cdot V_a(s_i)$ is connected [9]) and contains the point f of U_i but no boundary point of U_i, and hence is a subset of U_i. Thus q belongs to U_i, for every i, contrary to the

[8]) The points of h are linearly ordered in V from a to b. The point y of h precedes or follows the point x of h according as y belongs to $V_a(x)$ or to $V_b(x)$. See my paper in the Bull. Amer. Math. Soc., vol. 35 (1925), pp. 87—104.

[9]) See my paper in the Bull. Amer. Math. Soc., loc. cit.

fact that f is the only point common to $\overline{U}_i$. Thus the supposition that h is not identical with the set $V - (a + b)$ leads to a contradiction. Now since by hypothesis K is dense in R and $h = V - (a + b)$ is an open subset of R, therefore h contains at least one point y which belongs to K.

Returning to the points $x_1, x_2, \ldots, x_m$, let us select, for each integer $n \leqslant m$, a point y_n from the set $h_n \cdot K$. For each n, let us replace the point x_n in B_e by the point y_n, and call the set thus obtained B_g; i. e., $B_g = B_e - \sum_1^m x_n + \sum_1^m y_n$. Then clearly $B_g \subset K$ and $\overline{\overline{B}}_g = \overline{\overline{B}}_e$. If R_g is the component of $R - B_g$ containing the point p, then since, for each n, y_n separates A_f^n and x_n in V_n it follows that $R_g \subset R_e$ and that B_g is the boundary of R_g. This completes the proof of the lemma in this section.

6. **Proof of the Theorem.** Let us select some definite subset K of R satisfying the conditions on the set K in § 5. Let G be the system of all connected open subsets U of M such that the boundary of U consists of a finite number of points of K. Now since the set of all finite subsets of K is countable, since no two connected open sets having the same boundary can have a point in common without being identical, and since R is separable, it follows that the system G is countable. Now let E be the system of all open sets in R whose boundaries are finite point sets. Let p be any point of R and R_e' any set of the system E which contains p. Now R_e' contains a neighborhood R_e of p which also belongs to E and every point of the boundary of which is a limit point of $R - \overline{R}_e$. For by § 1 it follows that there exists a neighborhood U of p whose boundary B is finite and such that $U + B \subset R_e'$. Now add to U all the points of B which are not limit points of $R - \overline{U}$, and the set R_e thus obtained has the desired properties. Then by § 5, there exists a set R_g of the system G which contains p and is a subset of R_e and hence also of R_e'. Clearly every set of the system G is itself an element of the system E. Therefore the neighborhood systems G and E are equivalent.

Now let R^* be the space whose points are identically the same as the points of R but in which „limit point“ is defined by means of the system of neighborhoods E. That is, in the space R^* any set R_e of the system E is a neighborhood of the point p^* if and only if R_ε contains p^*; and a point p^* of R^* is a limit point of a set of points M^* if and only if every neighborhood of p^* belonging to E contains at least one point of M^* distinct from p^*. The space R^* as thus defined obviously satisfies the four axioms of Hausdorff and therefore it is a topological space in the sense of Hausdorff[10]) Since, as shown above, the countable subsystem G of E is equivalent to E, therefore R^* is perfectly separable. From § 1 it follows that within every neighborhood U belonging to E of a point p^* of R^* there exists a neighborhood V of p^* also belonging to E (since every neighborhood in R with a finite boundary belongs to E) which lies together with its boundary in U. Thus the space R^* is „regular" in the sense of Alexandroff-Urysohn. Therefore R^* is metric[11]). When we suppose that a distance function is defined in R^*, then it follows immediately from the definition of the system E that R^* is regular in the sense of Menger. And since by definition every set which is open in R^* is also open in R, it follows that R^* is a biunivalued and continuous image of R. This completes the proof.

7. **Remarks. A general problem.** If K is any subset of R of the type considered in § 5, then it follows by § 5 that the order $O_r(p)$ of every point p of R is unchanged if we restrict ourselves in defining order of points in § 1 only to neighborhoods of the system G of all connected open subsets of R whose boundaries are finite subsets of K. Hence the order of every point p of R *relative to* K is the same as the order of p relative to R. Thus the set K is a very simple and well defined countable subset of R (or G is an equally well defined countable family of neighborhoods in R) relative to which the true order of every point of R is determined.

In the special case[12]) in which R is a regular curve in the sense of Menger, such a set K may be defined for R in the following manner. Let H be the set of all ramification points of R, i. e., all points of order >2 of R, and let $h_1, h_2, h_3, \ldots$ be the components of $R-\overline{H}$. Then for each i, h_i is either an open or

[10]) See Hausdorff, *Grundzüge der Mengenlehre*, 1914.

[11]) See Alexandroff-Urysohn, Math. Ann., vol. 93 (1924), p. 263; and Tichonoff, Math. Ann., vol. 98 (1925), p. 301.

[12]) For a discussion of this special case, see a note by the author, „*Über die Struktur regulärer Kurven*", Wiener Akademie Anzeiger, 1930, N° 6.

semi-open arc. For each i, let k_i be a countable subset of h_i which is dense on h_i, such as, for example, the image set of the set of all rational points on the open or semi-open unit interval when this interval is put into a topological correspondance with h_i; and let $K=Q+\sum_{i=1,2,\ldots} k_i$. Then K satisfies all the conditions on the set K in § 5, and accordingly *the order of every point of* R *relative to* K *is the same as its order relative to the entire curve* R.

Let Z be any system of closed sets in a separable metric space S which is monotone and additive, i. e., every closed subset of an element of Z is itself an element of Z and the sum of every two elements of Z is also an element of Z. Suppose we say that a space S is uniordered[13]) relative to such a system Z provided that for each point p of S there exists a monotone decreasing sequence of neighborhoods $[U_i]$ $(i=1,2,3,\ldots)$ of p with boundaries $[U_i]$ which belong to the system Z such that $p=\prod_1^\infty(U_i+B_i)$. Then the method of proof given in § 2 may be employed to prove the following more general theorem.

In order that a space S *should be uniordered relative to the system* Z *it is necessary and sufficient that every two points of* S *may be separated by a set of the system* Z.

The lemma in § 2 gives only the special case of this general theorem in which Z is the system of all finite subsets of S. Another interesting special case of this theorem is that in which the system Z consists of exactly the null set. In this case we have the conclusion that *every set* T *which is separated between each pair of its points is „zero-dimensional" in the sense that there exists a sequence of neighborhoods* $[U_i]$ *of* p *with vacuous boundaries such that* $p=\prod_1^\infty U_i$, although there exist[14]) such sets T which are one dimensional in the Menger-Urysohn sense. It would be interesting to determine if every such set T may be transformed by a biunivalued and con-

[13]) This notion resembles the notion of „übergeordnet" due to Menger and Hurewicz (See Menger, *Dimensiontheorie*, pp. 123—125) in the same way as the notion of potential regularity resembles that of regularity in the Menger-Urysohn sense. Monotone and additive set-systems have also been studied by Menger, cf., for example, *Grundzüge einer Theorie der Kurven*, loc. cit.

[14]) See Mazurkiewicz, Fund. Math., vol. 2, p. 201; for a quite simple example, see Kuratowski, Ann. Soc. Math. Pol., vol. 5 (1926), p. 109.

tinuous transformation into a set which is zero-dimensional in the Menger-Urysohn sense or, in general, to solve the following general

Problem. *Can every space S which is uniordered relative to a system Z be transformed by a biunivalued and continuous transformation into a separable metric space S^* in which every point p^* is contained in arbitrarily small neighborhoods with boundaries belonging to Z?*

PART II.

Completely Partitionable Connected Sets.

We here consider a special type of potentially regular connected point sets. A connected set M, lying in a separable metric space, will be called *completely partitionable* provided that every two points of M may be separated in M by some third point of M. We give below a characterization of these sets and also characterize the connected sets which are homeomorphic with a subset of an acyclic continuous curve as sets which are completely partitionable and at the same time regular in the Menger sense. I note here the fact that, contrary to a statement made in the abstracts of this paper [15]), it is not necessarily true that in a completely partitionable connected set M, every two points may be joined by an irreducibly connected set. For let I_1 be the set of all points (x, y) in the plane such that $y=0$, $0 \leqslant x < 1$; let I_2 be the set such that $y=1$, $1 \leqslant x \leqslant 2$; for each positive integer n, let L_n be the set such that $x=(n-1)/n$, $0 \leqslant y \leqslant 1$. Finally, let $M = I_1 + I_2 + \sum_1^\infty L_n$, and let A and B denote the points $(0,0)$ and $(2,1)$ respectively. Then M is connected and completely partitionable, but there exists in M no irreducible connected set between A and B.

A_1. *In order that a connected set M (in a separable metric space) should be completely partitionable, it is necessary and sufficient that there exist a biunivalued and continuous transformation of M into a subset of an acyclic continuous curve.*

[15]) Wiener Akademie Anzeiger, 1930, N° 2; also Bull. Amer. Math. Soc., March., 1930. The second characterization of completely partitionable connected sets announced in these abstracts is therefore not valid.

The condition is necessary. For let M be any completely partitionable connected set. By § 2, M is potentially regular. Accordingly, by the theorem stated in § 1, there exists a biunivalued and continuous transformation T_1 of M such that $T_1(M)$ is connected, regular, and completely partitionable [16]). But then $T_1(M)$ is connected im kleinen, and by a theorem of the author's [17]) there exists a biunivalued and continuous transformation T_2 of $T_1(M)$ into a subset H of an acyclic continuous curve, i. e., $T_2[T_1(M)] = H$. Now for each point p of M, let $T(p) = T_2[T_1(p)]$. Then it is easily seen that T is a biunivalued and continuous transformation of M into H.

The condition is also sufficient. For obviously every connected subset of an acyclic continuous curve is completely partitionable, and indeed, *if for the connected set M, there exists a biunivalued and continuous transformation T of M into a completely partitionable set $T(M)$, then M itself must be completely partitionable.* For let A and B be any two points of M. There exists a point X in $T(M)$ which separates $T(A)$ and $T(B)$ in $T(M)$. Thus $T(M) - X = N_1 + N_2$, where N_1 and N_2 are mutually separated and contain $T(A)$ and $T(B)$ respectively. But since T is biunivalued and continuous, it follows that $T^{-1}(N_1)$ and $T^{-1}(N_2)$ are mutually separated and contain A and B respectively and that $T^{-1}(N_1) + T^{-1}(N_2) = M - T^{-1}(X)$. Therefore, the point $T^{-1}(X)$ separates the points A and B in M, and thus M is completely partitionable.

A_2. *In order that a connected set M should be topologically contained in an acyclic continuous curve it is necessary and sufficient that M be completely partitionable and regular.*

That the conditions are necessary follows from the well known facts that every subset of an acyclic continuous curve is regular and that the properties *regularity* and *complete partitionability* are topological invariants. That the conditions are sufficient follows from result A_1 and the following lemma.

[16]) That $T_1(M)$ is completely partitionable follows from the last part of the theorem in § 1; because for each pair of points A and B of $T_1(M)$ there exists a point x of M which separates $T_1^{-1}(A)$ and $T_1^{-1}(B)$ in M, and by the last part of the theorem in § 1, it follows that the point $T_1(x)$ separates A and B in $T_1(M)$.

[17]) See my paper *„On the structure of connected and connected im kleinen point sets"* (to appear in Trans. Amer. Math. Soc.), result 8.2 a).

Lemma. *If for the completely partitionable, regular and connected sets H and N there exists a biunivalued and continuous transformation T of H into N, then the inverse T^{-1} of T is also continuous, and therefore H and N are homeomorphic.*

Proof. Let t be any simple continuous arc in N joining two points A and B of N. Now there exists [18]) in H one and only one arc t^* between the points $T^{-1}(A)$ and $T^{-1}(B)$. Since T is biunivalued and continuous, the image $T(t^*)$ of t^* under T is an arc in N from A to B. But [18]) there is only one arc in N from A to B. Therefore $T(t^*) = t$, (or $T^{-1}(t) = t^*$), and hence it follows that the inverse T^{-1} of T is continuous on every simple continuous arc t in N.

Now let p be any point of N and ε any positive number. Since H is regular, there exists a connected ε neighborhood R of the point $P = T^{-1}(p)$ in H whose boundary B in H is finite. Since [18]) N is arcwise connected im kleinen, there exists a neighborhood V of p in N such that every point x of V can be joined to p by an arc t in N which contains no point whatever of $T(B)$. But by what we have just proved, $T^{-1}(t) = t^*$ is an arc in H from $T^{-1}(x)$ to P; and since t^* contains the point P of R but contains no point whatever of the boundary B of R, it follows that t^* is a subset of R. Thus for each point x of V, $T^{-1}(x) \subset R$, and therefore T^{-1} is continuous.

[18]) See § 7 of my paper just cited above, ref. [17]).

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Sur une propriété des ensembles G_δ.

Par

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M. Aronszajn a étudié une classe particulière des transformations continues des ensembles de points, notamment des fonctions $f(x)$, définies et continues sur un ensemble donné E qui trasforment tout ensemble ouvert relativement à E en un ensemble ouvert relativement à l'image $f(E)$ de E [1]) [2]). Il m'a posé récemment le problème si toute transformation de cette sorte d'un ensemble G_δ particulier donne un ensemble G_δ. Je donnerai ici une solution positive de ce problème, en démontrant le théorème général que voici:

Théorème: *Si E est un ensemble G_δ* (d'un espace à m dimensions) *et si $f(x)$ est une fonction définie et continue sur E qui transforme tout ensemble ouvert relativement à E en un ensemble ouvert relativement à $f(E)$, $f(E)$ est aussi un ensemble G_δ* [3]).

Démonstration. Il suffira évidemment de démontrer notre théorème pour les ensembles bornés. Nous supposerons, pour fixer les idées, que E et $f(E)$ sont des ensembles plans.

Soit donc E un ensemble G_δ plan borné, $f(x)$ — une fonction définie et continue sur E qui transforme tout ensemble ouvert dans E en un ensemble ouvert dans $f(E)$.

E étant un ensemble G_δ plan borné, nous pouvons poser

$$E = G_1 G_2 G_3 \dots, \tag{1}$$

[1]) Un sous-ensemble H de E est dit *ouvert relativement à E*, ou *ouvert dans E*, s'il est un produit de E par un ensemble ouvert.

[2]) Cf. S. Stoïlow: *Fund. Math.* t. XIII, p. 186.

[3]) Quant à une application de ce théorème, voir la Note de M. Aronszajn qui paraîtra dans ce volume.