On Continuous Curves which are Homogeneous except for a Finite Number of Points.

(Second Part)

By

T. C. Benton (Philadelphia).

In the first part of this paper the discussion of the general case was not completed. This part will finish the discussion. As it is a direct continuation the paragraph numbers will continue from the point where those of the first paper stopped and reference to previous paragraphs will be made simply by number. (Any number less than 10 being in the first paper in vol. 13 of this journal).

IV.

10.0 The second case of 5.9 will now be considered. Let M be a continuous curve homogeneous except for m points $c_{11}, c_{12}, \ldots, c_{1m}$, each of which have k_1 arcs meeting at it, and n points $c_{21}, c_{22}, \ldots, c_{2n}$, each of which have k_2 arcs meeting at it. The points c_{1i}, c_{1j} can be made to correspond under some π_M for all values of i and j; also the points c_{2i}, c_{2j}: but not c_{1i} and c_{2j}.

10.1 *Each arc of* $\overline{M}_i$ *(see 5.8.1) has a* c_{1j} *at one end and a* c_{2k} *at the other.*

Proof. Since M is connected there must be some arc $\overline{M}_i$ connecting some c_{1j} to some c_{2k} and having no other c-point on it because if every arc joined only points of $\sum_i c_{1i}$ or only points of $\sum_j c_{2j}$ there would be at least two mutually exclusive parts of M, for the two types of arc cannot meet anywhere but at c-points since all other points are ordinary points (5.8). If p_1 and p_2 are

homogeneous points such that p_1 is on this arc from c_{1i} to c_{2j} and p_2 is on an arc from c_{1k} to c_{2l} and if $\pi_M(p_1) = p_2$; then the correspondent of c_{2j} must be either c_{1k} or c_{1l} because π_M is continuous. This however is a contradiction to the assumption that points of $\sum_i c_{1i}$ could not be made to correspond to points of $\sum_j c_{2j}$, and therefore no arcs joining the two c-points of the same class can exist.

10.2 *Let* M_{ij} *be the set of all arcs of* M *which join* c_{1i} *to* c_{2j} *such that these arcs contain no other c-points of* M. *Then* M_{ij} *is a homogeneous continuous curve except for* c_{1i} *and* c_{2j}.

Proof. Let p and q be any two points of M_{ij} and let π_M give $\pi_M(p) = q$. Now the arc of M_{ij} which contains p must correspond to that which contains q and therefore $\pi_M(c_{1i}) = c_{1i}$ and $\pi_M(c_{2j}) = c_{2j}$. But then every arc of M_{ij} must correspond to an arc of M_{ij} since π_M is a continuous correspondence.

10.3 Let a new set be constructed which has exactly the same c-points as M but which has only two arcs joining the c-points which in M were joined by a set M_{ij}. Call this set $\mathfrak{M}'$. In $\mathfrak{M}'$ each complementary domain is bounded by an even number of arcs, for the arc must end in a c-point of the class of which its beginning point was not a member. Let a second new set $\mathfrak{M}$ be constructed by taking the points of $\sum_i c_{1i}$ and joining them by certain arcs constructed in the following way. In $\mathfrak{M}'$ take a complementary domain $\mathfrak{D}$ which has more than two c-points on its boundary. The c-points are alternately a point of $\sum_i c_{1i}$ and a point of $\sum_j c_{2j}$. If the points of $\sum_j c_{2j}$ be disregarded $F(\mathfrak{D})$ will be the sum of certain arcs joining points of $\sum_i c_{1i}$. If this be done for all domains of $\mathfrak{M}'$ which have more than two c-points on their boundaries the result is the set $\mathfrak{M}$ which was to be constructed.

10.4 $\mathfrak{M}$ *is a set with* $\sum_i c_{1i}$ *as its only non-homogeneous points and is therefore a set described in part III.*

Proof. By making each arc of $\mathfrak{M}$ correspond to the arcs of which were used in constructing it there will result a 1—1, continuous correspondence except at the points of $\sum_j c_{2j}$ and their correspondents. Since $\mathfrak{M}'$ was homogeneous except for $\sum_i c_{1i} + \sum_j c_{2j}$ it is obvious that a correspondence of $\mathfrak{M}$ into itself can be found such that any two particular points, neither of which are the correspondents in $\mathfrak{M}$ of c-points in $\mathfrak{M}'$, will correspond.

Suppose Φ is the continuous correspondence which gives $\Phi(\mathfrak{M}') = \mathfrak{M}$, then Φ is 1—1 and continuous except for the points $\sum_j c_{2j}$ of $\mathfrak{M}'$ and their correspondents in $\mathfrak{M}$. Also $\Phi\{\sum_i c_{1i}\} = \sum_i c_{1i}$ from the method in which Φ was constructed. Suppose that $\Phi(\sum_j c_{2j}) = \sum_{j,k} p_{jk}$.

Consider now the case where p_{ik} and p_{jl} are the two homogeneous points of $\mathfrak{M}$ which are to correspond under a correspondence of $\mathfrak{M}$ to itself. Then $\Phi^{-1}(p_{ik}) = c_{2i}$ and $\Phi^{-1}(p_{jl}) = c_{2j}$. Let p_{ik} be on an arc from $c_{1\alpha}$ to $c_{2\beta}$ in $\mathfrak{M}$ which was constructed from the arcs $c_{1\alpha} c_{2i}$ and $c_{2i} c_{1\beta}$ in $\mathfrak{M}'$. Also let p_{jl} be on $c_{1\gamma} c_{1\delta}$ of $\mathfrak{M}$ which was constructed from $c_{1\gamma} c_{2j}$, $c_{2j} c_{1\delta}$ of $\mathfrak{M}'$. Let x be a homogeneous point of $\mathfrak{M}$ on the arc $c_{1\alpha} p_{ik}$ and x' the point of $\mathfrak{M}'$ such that $\Phi^{-1}(x) = x'$. Then $\Phi^{-1}(c_{1\alpha} p_{ik}) = c_{1\alpha} c_{2j}$. Also let y be a homogeneous point of $\mathfrak{M}$ on the arc $c_{1\alpha} p_{ik}$ and y' that point of $\mathfrak{M}'$ such that $\Phi^{-1}(y) = y'$; then $\Phi^{-1}(c_{1\gamma} p_{jl}) = c_{1\gamma} c_{2j}$. Now let $\Psi_{\mathfrak{M}'}$ throw x' into y'. Then $\Psi_{\mathfrak{M}'}(c_{1\alpha}) = c_{1\gamma}$ and $\Psi_{\mathfrak{M}'}(c_{2i}) = c_{2j}$ since points of $\sum_i c_{1i}$ cannot correspond to points of $\sum_j c_{2j}$. Then $\Phi(y') = z$, a point on $c_{1\gamma} c_{1\delta}$ in $\mathfrak{M}$. But $z \equiv y$ since Φ is 1—1 for homogeneous points, but then for continuity $\Phi(c_{2j}) = p_{jl}$.

Hence by choosing the correspondence of points where Φ fails to be 1—1 in this fashion on the other arcs we can get the proper correspondents for all the $\Phi(c_{2i})$ and still have the result of $(\Phi^{-1}\Psi\Phi)$ a 1—1 correspondence of $\mathfrak{M}$ into itself which throws p_{ik} into p_{jl}.

Lastly consider the case where p_{ik} and q are the two points of $\mathfrak{M}$ which are to correspond, where q corresponds to a homogeneous point of $\mathfrak{M}'$. Let $\Phi^{-1}(q) = s$. Let s be on an arc $c_{2i} c_{1j}$ of $\mathfrak{M}'$ and q on an arc $c_{1l} c_{1j}$ of $\mathfrak{M}$. Let p_{ns} be the point of $c_{1l} c_{1j}$ for which $\Phi^{-1}(w_{ns}) = c_{2i}$. By the preceding case it is possible to construct a correspondence which will make p_{ik} correspond to p_{ns}. But then there exists a correspondence which leaves all of $\mathfrak{M}$ invariant except the arc $c_{1l} c_{1j}$ and on that arc puts q in correspondence with p_{ns}. So p_{ik} can be made to correspond to q.

Therefore any pair of points of $\mathfrak{M}$ except pairs one or both of which belong to $\sum_l c_{1l}$ can be made to correspond. Hence $\mathfrak{M}$ is homogeneous except for the points $\sum c_{1l}$.

10.5 Having shown that it is possible to construct a skeleton set with all of its c-points of the same type, a method is needed to reconstruct the sets of the new type from those of part III. Suppose a skeleton set $\mathfrak{N}$ of the type in part III be taken. A set $\mathfrak{M}$ (in the sense of 10.3) can be constructed by replacing each arc of $\mathfrak{N}$ by two arcs. Then in any domain not bounded by just two arcs let the arcs of the boundary be replaced by a set of pairs of arcs which have only one point c_{2j} inside the domain in common and each pair of which ends in one of the $\sum_i c_{1i}$ points on the boundary of the domain. The new set so constructed will be a set $\mathfrak{M}'$ of 10.3. Obviously the correspondences of 10.3—10.5 can be reversed except for the points c_{2j} at which the correspondences fail to be 1—1. Hence the set $\mathfrak{M}'$ is a set with the properties postulated in 10.0 and the extension to more complicated sets of the same type is easily made.

10.6 On performing the construction of 10.5 only two new skeleton sets are found.