On a problem of G. T. Whyburn.

By

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In his paper *Concerning irreducible cuttings of continua* (this volume) G. T. Whyburn gives an example of a continuum M which contains no indecomposable continuum, but which contains two points A and B, such that no cutting of M between A and B is irreducible. He proves that if M is an indecomposable continuum, then *every* two points A and B of M are such that no irreducible cutting of M between A and B exists. He raises the following questions:

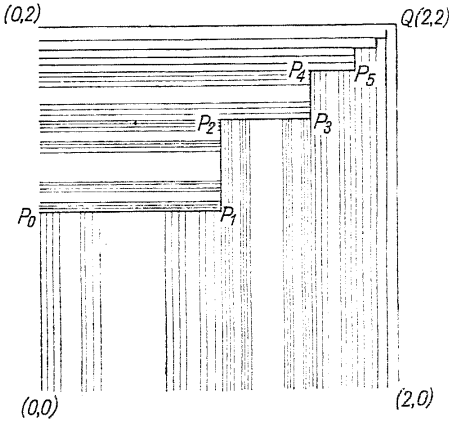
(1). *If a continuum M has the property that for every two points A and B of M it is true that no irreducible cutting of M between A and B exists, then is it necessarily true that M is indecomposable or that M contains an indecomposable continuum.* (2) *If every cutting of a continuum M is reducible, is M necessarily indecomposable?*

In the present paper it is shown by an example that the answer to both of these questions is in the *negative.* A bounded continuum every subcontinuum of which is indecomposable is characterized by irreducible cuttings of subcontinua.

The definitions used are those given by Dr. Whyburn (loc. cit.).

Example. Let $C, D, Q, P_0, P_1, P_2, P_3, P_4, \ldots$, denote the points which in a rectangular coordinate system have coordinates $(0, 2)$, $(2, 0)$, $(2, 2)$, $(0, 1)$, $(1, 1)$, $(1, 1\frac{1}{2})$, $(1\frac{1}{2}, 1\frac{1}{2})$, $(1\frac{1}{2}, 1\frac{3}{4}), \ldots$, respectively. Let P_0Q denote the arcs consisting of the point Q and the straight line intervals joining in succession the points $P_0, P_1, P_2, \ldots$ Let CQ and DQ denote the straight line intervals from C to Q and from D to Q, respectively. Let G_i denote a non-dense perfect set on the

interval $P_i P_{i+1}$ containing the points P_i and P_{i+1} $(i = 0, 1, 2, \ldots)$. Obviously $P_{2k} P_{2k+1}$ is a horizontal interval, and $P_{2k+1} P_{2k+2}$ is a vertical interval $(k = 0, 1, 2, \ldots)$. Let M_{2k} denote the point set consisting of the arc $P_{2k} P_{2k+1}$, together with all vertical intervals with one end point in G_{2k}, and the other end point on the x-axis. Let M_{2k+1} denote the point set consisting of the arc $P_{2k+1} P_{2k+2}$, together with all horizontal intervals with one end point in G_{2k+1} and the other on the y-axis. Clearly M_i is a continuum $(i = 0, 1, 2, \ldots)$. Let M be the point set $(CQ + DQ + M_0 + M_1 + M_2 + \ldots)$.



Evidently the bounded continuum M is arcwise connected. Also every subcontinuum of M is arcwise connected. For suppose A and B are two points of M, and K is a subcontinuum of M containing A and B, but not containing the arc AB of M which has A and B as end points. Then the set of all points common to the arc AB and the continuum K is not connected. Hence [1]) $AB + K$ separates the plane. But M does not separate the plane, nor does M contain any domain. Hence no subcontinuum of M separates the plane. Therefore K is arcwise connected, and every subcontinuum of M is decomposable.

Now let A and B denote any two points of M. Suppose K is an irreducible cutting of M between A and B. Then $M - K$ is the

[1]) Janiszewski: *Sur les coupures du plan faites par des continus,* Prace Matem.-fizyczne, 1913.

sum of two mutually separated sets S_A and S_B, containing A and B respectively. Then [1]) S_A+K and S_B+K are continua, and $(S_A+K)+(S_B+K)$ does not separate the plane. Hence K, the common part of these two continua, is a continuum. From the fact that K is an irreducible cutting of M between A and B it follows that every point of K is a limit point of S_A and of S_B.

Since it separates M, K contains a subinterval of the arc P_0Q. There exists no arc lying in M, containing both A and B and containing no subinterval of P_0Q. For if there exists such an arc t, then K must contain a point between A and B on t, and also points on the arc P_0Q; hence K must contain either A or B, contrary to the supposition that K is a cutting of M between A and B.

Let X be the first point of K on the arc P_0Q, and let P_m be the last point of the set $P_0, P_1, P_2, \ldots$ which preceeds X on the arc P_0Q. The segment P_0X belongs either to S_A or S_B. Suppose it belongs to S_A.

Suppose i is a positive integer such that K contains an arc on the interval P_iP_{i+1}. The continuum K contains an arc t lying on P_iP_{i+1} and containing no point of the set G_i. Hence if w_{i+1} denotes the maximal straight line interval that lies in M_{i+1} and contains P_i and P_{i+1}, then $M_{i+1}-w_{i+1}$ contains points of S_A and of S_B, Moreover $M_{i-1}-w_{i-1}$ is connected. Hence it contains a point of K. But K contains a point of P_iP_{i+1}. Therefore K contains a subinterval of P_iP_{i+1}.

Obviously the point B is not on the arc P_0Q. Let Y denote the first point that the arc BA of M has in common with P_0Q. The point Y must be identical with X. For suppose that it is not, and let $\overline{X}$ denote a point between X and Y on the arc XQ. Then the subarc $\overline{X}Q$ of XQ cuts M between A and B, contrary to the supposition that K is an irreducible cutting of M between A and B.

The point Y is not a limit point of points of G_m on the segment $P_0Y \equiv P_0X$; for if it were, the point B would be a limit point of S_A. Hence any arc of YQ with Y as one end point will cut M between A and B. Hence our supposition that there exists an irreducible cutting of M between some two of its points has led to a contradiction.

[1]) G. T. Whyburn, loc. cit., Theorem 2.

Therefore M is a bounded continuum every subcontinuum of which is decomposable, but such that there exists no irreducible cutting of M between any two of its points. Obviously, therefore, there is no irreducible cutting of M, for such a cutting would be an irreducible cutting between some two points.

Theorem 1. *If M is a bounded continuum and is expressible as the sum of two subcontinua H and K such that $H \neq M$ and $K \neq M$, then there exists a subcontinuum N of M and an irreducible cutting L of N such that L is a subset of $H.K$ and $N-L$ is the sum of two connected sets belonging to H and K respectively.*

Proof. Let C be a component [1]) of $H-H.K$. Then since H is a bounded continuum it follows by a well known theorem that $\overline{C}$ contains at least one point of $H.K$. Also $\overline{C}-C \subset H.K$. Hence $C+K$ is a continuum. Clearly $\overline{C}-C=\overline{C}.K \subset H.K$.

Likewise, if Q is a component of $K-\overline{C}.K$, $\overline{Q}$ contains at least one point of $\overline{C}.K$, and $\overline{Q}.\overline{C}=\overline{Q}-Q \subset \overline{C}.K \subset H.K$. Let N denote the continuum $Q+\overline{C}$ and let L denote the point set $\overline{Q}.\overline{C}=\overline{Q}-Q$. Then $N-L \equiv Q+(\overline{C}-\overline{Q}.\overline{C})$. The set Q is a connected subset of K and contains no limit point of $\overline{C}-\overline{C}.\overline{Q}$, for Q contains no point of $\overline{C}$. Clearly $\overline{C}-\overline{C}.\overline{Q}=C+(\overline{C}.K-\overline{Q}.\overline{C})$ is a connected subset of H and contains no limit point of Q. That L is an *irreducible* cutting of N follows from the fact that every point of L is a limit point of Q and also of $\overline{C}-\overline{C}.\overline{Q}$.

Corollary. *If M is a continuum and no bounded subcontinuum of M contains an irreducible cutting of itself, then every bounded subcontinuum of M is indecomposable.*

Theorem 2. *A necessary and sufficient condition that every subcontinuum of a bounded continuum M be indecomposable is that no subcontinuum of M contain an irreducible cutting of itself.*

That the condition of theorem 2 is necessary is proved by G. T. Whyburn (loc. cit., Corollary 3a). That it is also sufficient follows from the above Corollary.

[1]) By a *component* of a point set G is meant a maximal connected subset of G

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