Concerning continuous curves of certain types[1]).

By

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A number of authors[2]) have discussed continuous curves[3]) which contain no simple closed curve and have shown that they possess a number of interesting properties. It is the purpose of this paper to show that many of these properties remain true in more general types of continuous curves.

We shall consider the following five types of continuous curves M: (1) M contains at most a finite number of simple closed curves, (2) if ε is any positive number, M contains at most a finite number of simple closed curves of diameter greater than ε, (3) M contains only a finite number of arcs between any two points of M, (4) every connected subset of M is arc-wise connected, (5) every closed and connected subset of M is a continuous curve. A continuous curve which satisfies the first condition will be said to be *simply cyclic*, one which satisfies the second condition will be said to be *almost simply cyclic* and one which satisfies the third condition will be said to be *simply joined*.

Of the twenty possible relations between the five types I have been able to settle all except two. The questions as to whether, or not, (2) implies (4) and that (5) implies (4) are not settled in

[1]) Presented to the American Mathematical Society May 1, 1926.

[2]) S. Mazurkiewicz, *Un théorème sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 2 (1921), pp. 119—130, R. L. Wilder, *Concerning continuous curves*, Fund. Math., vol. 7 (1925), pp. 340—377, and others.

[3]) For definitions and theorems concerning continuous curves, see R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, Bulletin of the American Mathematical Society, vol. 27 (1923), pp. 289—302.

this paper[1]) Excepting these two cases every implication between the five types is given in this paper. In every other case an example may be found to show that there is no implication.

Theorem 1. *If every connected subset of a continuous curve is arc-wise connected, then every closed and connected subset is a continuous curve*[2]).

Proof. Let K denote any closed and connected subset of the continuous curve M and let T denote any open subset of K. If x and y are any two points which lie in a connected subset of T, by hypothesis there exists an arc xy which lies wholly in T. Therefore K is a continuous curve by a theorem due to R. L. Wilder[3]).

Theorem 2. *Every closed and connected subset of an almost simply cyclic continuous curve is a continuous curve*

Proof. Let K be any closed and connected subset of an almost simply cyclic continuous curve M Let us suppose that K is not a continuous curve. Then by the Moore-Wilder Lemma[4]) there exist two concentric circles C_1 and C_2 and an infinite sequence of continua, K_∞ $K_1, K_2, K_3, \ldots$, all belonging to K such that (1) each of these continua contains a point on C_1 and a point on C_2 and lies entirely in $L = C_1 + C_2 + I$, where I denotes the annular domain bounded by C_1 and C_2, (2) no two of the continua have a point in common and no one of them except possibly K_∞ is a proper subset of any connected point set which is common to K and L, (3) the set K_∞ is the sequential limiting set of the sequence of sets $K_1, K_2, K_3, \ldots$[5]).

[1]) B. Knaster and C. Kuratowski have given an example which satisfies condition (5) but not condition (4) and thus show that (5) does *not* imply (4). See *A connected and connected im kleinen point set which contains no perfect subset*, Bull Amer. Math. Soc., vol 33 (1927), pp. 106—9. The question as to whether (2) implies (4) remains as an unsettled question.

[2]) To make the statements of the theorems simpler we will consider a single point as a special case of an arc or of a continuous curve.

[3]) Loc. cit., Theorem 18, p. 373.

[4]) R. L. Moore, loc. cit., p. 296, and R. L. Wilder, loc. cit., p. 371.

[5]) The point set K_∞ is said to be the *limiting set* of the sequence of point sets $K_1, K_2, K_3, \ldots$ provided that (a) each point of K_∞ is the sequential limit point of an infinite subsequence of some sequence of points $p_1, p_2, p_3, \ldots$ such that, for every n, p_n belongs to K_n, (b) if $p_1, p_2, p_3, \ldots$ is a sequence of points such that, for every n, p_n belongs to K_n, then K_∞ contains the sequential limit point of every subsequence of $p_1, p_2, p_3, \ldots$ that has a sequential limit point. If

Let $\overline{K} = K_\infty + K_1 + K_2 + \ldots$ The sets K_1 and $\overline{K} - K_1$ are both closed and have no common points. Then for each point of K_1 there exists a circle whose interior contains this point but no point of $\overline{K} - K_1$. The set of all such interiors of circles for all points of K_1 forms a connected domain containing K_1 but no points of $\overline{K} - K_1$. The set of points of M lying in this domain is an open subset of M. The continuum K_1 contains a point A_1 on C_1 and a point B_1 on C_2. Then M contains an arc $A_1 B_1$ which lies wholly in the open subset of M[1]) and thus contains no point of $\overline{K} - K_1$. The arc $A_1 B_1$ has a last point x_1 on C_1 and the subarc $x_1 B_1$ of $A_1 B_1$ has a first point y_1 on C_2[2]).

Let us continue this process. In general, the sets K_n and $\overline{K}_n = \overline{K} - (K_1 + K_2 + \ldots + K_n) + x_1 y_1 + x_2 y_2 + \ldots + x_{n-1} y_{n-1}$ are closed and have no common points. Then for each point P of K_n there exists a circle whose interior contains P but no point of $\overline{K}_n$. The set of all such interiors for all points P of K_n forms a connected domain D_n containing K_n but no point of $\overline{K}_n$. The set $M.D_n$, the intersection of M and D_n, is an open subset of M. The continuum K_n contains a point A_n on C_1 and a point B_n on C_1 and thus M contains an arc $A_n B_n$ which lies wholly in the open subset $M.D_n$ The arc $A_n B_n$ contains a last point x_n on C_1 and the subarc $x_n B_n$ of $A_n B_n$ contains a first point y_n on C_2. The arc $x_n y_n$ lies, except for its end-points, wholly in the set I.

Let $N_i = x_i y_i$. Then M contains an infinite sequence of arcs $N_1, N_2, N_3, \ldots$ such that no two of the arcs have a point in common and, for every value of i, N_i contains a point x_i on C_1 and a point y_i on C_2 and, except for these two points, N_i lies wholly in I. There exists a sequence of positive integers $n_1, n_2, n_3, \ldots$ and two points X and Y such that (1) X lies on C_1 and is the sequential limit point of the sequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ and Y lies on C_2 and is the sequential limit point of the sequence $y_{n_1}, y_{n_2}, y_{n_3}, \ldots$ (2) all of the points x_{n_i} lie on one of the two arcs of C_1 from x_{n_1} to X and in the order $x_{n_1}, x_{n_2}, x_{n_3}, \ldots X$ and all of the points y_{n_i} lie on one of the two arcs of C_2 from y_{n_1} to Y and in the order $y_{n_1}, y_{n_2}, y_{n_3}, \ldots Y$.

The limiting set N_∞ of the sequence $N_{n_1}, N_{n_2}, N_{n_3}, \ldots$ is connected[1]) and contains X and Y. It thus contains a point on every circle concentric with and lying between C_1 and C_2. Let the radius of C_i be denoted by r_i and suppose η is a number such that

$$r_1 - r_2 \geqslant 10\eta > 0 \quad \text{and} \quad r_2 > \eta.$$

Let C_3 and C_4 be circles concentric with C_1 and with radii $r_2 + \eta$ and $r_1 - \eta$ respectively. The set N_∞ contains a point Z_1 on C_3 and a point Z_2 on C_4. Since M is connected im kleinen at Z_1 and Z_2. there exists a positive number δ such that every point of M within a distance δ of either Z_1 or Z_2 can be joined to that point by an arc of M every point of which is within a distance η of Z_1 or Z_2 as the case may be. There exists an integer n_k such that N_{n_k} contains two points p_1 and p_2 such that

$$d(p_1, Z_1) < \delta \quad \text{and} \quad d(p_2, Z_2) < \delta.$$

Then p_1 can be joined to Z_1 by an arc U_1 of M and p_2 can be joined to Z_2 by an arc U_2 of M such that every point of U_1 is within a distance η of Z_1 and every point of U_2 is within a distance η of Z_2.

The arc $U_i (i = 1, 2)$ has at least one point in common with N_{n_j} for ever $j \geqslant k$. The arc U_i has a last point L_{oi} in common with N_{n_k}. The subarc $L_{oi} Z_i$ of U_i has a first point F_{1i} in common with $N_{n_{k+1}}$ and the arc $F_{1i} Z_i$ of U_i has a last point L_{1i} in common with $N_{n_{k+1}}$. Continuing we hawe, for each i and j $(i = 1, 2;\ j = 1, 2, 3, \ldots)$, the subarc $L_{j-1,i} Z_i$ of U_i has a first point F_{ji} in commo with $N_{n_{k+j}}$ and the subarc $F_{ji} Z_i$ of U_i has a last point L_{ji} in common with $N_{n_{k+j}}$. For every j, the arc $F_{ji} L_{j-1,i}$ has one end-point on $N_{n_{k+j}}$ and the other on $N_{n_{k+j-1}}$ and no other point in common with either arc. Then the arcs $F_{j1} L_{j-1,1}$ of U_1, L_{j-1}, $L_{j-1,2}$ of $N_{n_{k+j-1}}$, $L_{j-1,2} F_{j2}$ of U_2 and $F_{j2} F_{j1}$ of $N_{n_{k+j}}$ form a simple closed curve J_j which is of diameter greater than 6η since U_1 lies in the interior of a circle of radius $r_2 + 2\eta$ and U_2 lies in the exterior of a concentric circle of radius $r_1 - 2\eta \geqslant r_2 + 8\eta$.

the further condition is satisfied that every infinite subsequence of the sequence $K_1, K_2, K_3, \ldots$ has the same limiting set K_∞, then K_∞ is said to be the *sequential limiting set* of the sequence $K_1, K_2, K_3, \ldots$

[1]) Cf. R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), pp. 254—260.

[2]) In referring to the first or last points of a set on an arc xy, the order from x to y is implied.

[1]) Z. Janiszewski, *Sur les continus irréductibles entre deux points*, Journal de l'Ecole Polytechnique, Ser. 2, vol. 16 (1912), p. 98, Theorem 1.

Since for $i \neq j$, J_i and J_j are different simple closed curves, M contains an infinite number of simple closed curves of diameter greater than 6η. But this contradicts the hypothesis that M is an almost simply cyclic continuous curve. Therefore every closed and connected subset K of M is a continuous curve.

Theorem 3. *Every connected subset of a simply joined continuous curve in arc-wise connected.*

Proof. Let K be any connected subset of a simply joined continuous curve M and let A and B be any two points of K. Since M is a continuous curve there exists at least one simple continuous arc of M from A to B and by hypothesis there are not more than a finite number of these arcs. Let $C_1, C_2, \dots C_n$ be the set of all arcs of M from A to B. If any arc of this set lies wholly in K our theorem is established. If not, then each arc C_i must contain a point P_i which does not belong to K. But $M - (P_1 + P_2 + \dots + P_n)$ is an open subset of M and A and B lie in a connected subset, namely K, of this open subset. Therefore $M - (P_1 + P_2 + \dots + P_n)$ contains an arc $\overline{C}$ from A to B[1]. But as $\overline{C}$ contains no one of the points $P_1, P_2, \dots P_n$ it is different from any of the arcs of the set $C_1, C_2, \dots C_n$. But this is contrary to the hypothesis that this set contains every arc of M from A to B.

Theorem 4. *Every closed and connected subset of a simply joined continuous curve is a continuous curve.*

This result is a consequence of Theorems 1 and 3.

Theorem 5. *A simply joined continuous curve is almost simply cyclic.*

Proof. Let M denote a simply joined continuous curve. Suppose there exists a positive number ε such that M contains an infinite set $J_1, J_2, J_3, \dots$ of simple closed curves each of which is of diameter greater than ε. For each i, J_i contains two points A_i and B_i such that

$$d(A_i B_i) > \varepsilon.$$

There exists two points A and B and an increasing sequence of integers $n_1, n_2, n_3, \dots$ such that A is the sequential limit point of the sequence $A_{n_1}, A_{n_2}, A_{n_3}, \dots$ and B is the sequential limit point of the sequence $B_{n_1}, B_{n_2}, B_{n_3}, \dots$

[1]) Cf. R. L. Moore, *Concerning continuous curves in the plane*, loc. cit.

Case I. Suppose that $[J_{n_i}]$ contains an infinite subsequence $J_{11}, J_{12}, J_{13}, \dots$ (if $i > j$, $J_{1i} = J_{n_k}$ and $J_{1j} = J_{n_m}$, then $k > m$) such that each curve J_{1i} contains both A and B. By hypothesis M contains only a finite number of arcs $C_1, C_2, C_3 \dots C_n$ from A to B such that no two have any points in common except A and B. Let L_j denote the set $\sum_{i=1}^{i=j} C_i$. The set L_n contains only a finite number of simple closed curves so there exists an integer r_1 such that for $i \geqslant r_1$, J_{1i} contains at least one point not in L_n. Then J_{1r_1} contains a point p_1 not in L_n. On the arc Ap_1B of J_{1r_1} let x_1 be the first point of L_n on p_1A and y_1 the first point of L_n on p_1B. Suppose x_1 belongs to C_{s_1} and y_1 to C_{t_1}, s_1 and t_1 being not necessarily different but if $s_1 = t_1$ then we will suppose the order Ax_1y_1B. Let C_{n+1} be the arc formed of Ax_1 of C_{s_1}, $x_1p_1y_1$ of J_{1r_1} and y_1B of C_{t_1}.

Continue this process with L_{n+1} in place of L_n. In general the set L_{n+i-1} contains only a finite number of simple closed curves. Thus there exists a number r_i such that if $j \geqslant r_i$ then J_{1j} contains at least one point not in L_{n+i-1}. Then J_{1r_i} contains a point p_i not in L_{n+i-1}. On the arc Ap_iB of J_{1r_i} let x_i be the first point of L_{n+i-1} on the arc p_iA and y_i be the first point of L_{n+i-1} on p_iB. Suppose x_i belongs to C_{s_i} and y_i to C_{t_i}. If $s_i = t_i$ then we will suppose the order Ax_iy_iB on C_{s_i}. Let C_{n+i} be the arc of M from A to B composed of Ax_i of C_{s_i}, $x_ip_iy_i$ of J_{1r_i} and y_iB of C_{t_i}.

The set L_{n+i} contains at least i arcs from A to B in M and, since we may continue the process indefinitely, the hypothesis that M is simply joined is contradicted.

Case II. Suppose that $[J_{n_i}]$ contains an infinite subsequence $J_{21}, J_{22}, J_{23}, \dots$ such that each curve J_{2i} contains A but does not contain B. Let K_1 be a circle with center at B and radius $\frac{1}{2}\varepsilon$. The exterior of K_1 contains A. Since M is connected im kleinen at B there exists a circle $\overline{K_1}$ with center at B such that every point of M in the interior of $\overline{K_1}$ can be joined to B by an arc of M every point of which lies in the interior of K_1. As B is the sequential limit point of the sequence $[B_{n_i}]$, there exists an integer r_1 such that J_{2r_1} contains a point q_1 in the interior of $\overline{K_1}$. Let C_1 be an arc of M from q_1 to B lying entirely in the interior of K_1. Let p_1 be the first point C_1 has in common with J_{2r_1} in the order from B to q_1. Let $\overline{C_1}$ be the arc from B to A which consists of Bp_1 of C_1 together with either of the arcs of J_{2r_1} from p_1 to A.

Let K_2 be a circle with center at B whose exterior contains J_{2r_1}. There exists a circle $\overline{K_2}$ with center at B such that every point of M in the interior of $\overline{K_2}$ can be joneid to B by an arc of M which lies wholly inside K_2. There exists a curve J_{2r_2} which contains a point q_2 inside $\overline{K_2}$. Let C_2 denote an arc of M from q_2 to B which lies wholly interior to K_2 and let p_2 be the first point in the order B to q_2 that the arc C_2 has in common with J_{2r_2}. At least one of the two arcs of J_{2r_2} from p_2 to A is not a subset of $\overline{C_1}$. Let $p_2 A$ be an arc of J_{2r_2} having this property. Let $\overline{C_2}$ be the arc of M from B to A composed of Bp_2 of C_2 and $p_2 A$ of J_{2r_2}. Let t_{12} be the first point the arc $p_2 B$ has in common with the arc $p_1 B$ in the order p_2 to B if it is distinct from B and let t_{21} be the first point the arc $p_1 B$ has in common with the arc $p_2 B$ in the order p_1 to B if it is distinct from B. If either of the points is not distinct from B then t_{12} or t_{21} as the case may be, denotes a vacuous set.

Let K_3 be a circle with center at B whose exterior contains $J_{2r_2} + t_{12} + t_{21}$. Repeat the above process with K_3 in place of K_2 In general let K_n be a circle with center at B whose exterior contains $J_{2r_{n-1}} + \sum_{i=1}^{n-2}(t_{n-1,i} + t_{i,n-1})$, where t_{ij} denotes the first point the arcs $p_j B$ and $p_i B$ have in common in the order p_j to B if it is distinct from B and, it not, t_{ij} is a vacuous set There exists a circle $\overline{K_n}$ with center at B such that every point of M in the interior of $\overline{K_n}$ can be joined to B by an arc of M which lies wholly interior to K_n. There exists a curve J_{2r_n} which contains a point q_n interior to $\overline{K_n}$. Let C_n denote an arc of M from q_n to B which lies interior to K_n and let p_n be the first point the arc C_n has in common with J_{2r_n} in the order B to q_n. At least one of the two arcs of J_{2r_n} from p_n to A is not a subset of any one of the arcs $\overline{C_1}, \overline{C_2}, .. \overline{C_{n-1}}$ and let $p_n A$ denote this arc. Let $\overline{C_n}$ be the arc of M from B to A consisting of the arc Bp_n of C_n and the arc $p_n A$ of J_{2r_n}.

As this process may be continued indefinitely, there exist an infinite number of arcs of M from A to B, $\overline{C_1}, \overline{C_2}, \overline{C_3}, \ldots$ But this contradicts the hypothesis that M is simply joined.

Case III. Suppose that $[J_{n_i}]$ contains an infinite subsequence $J_{31}, J_{32}, J_{33}, \ldots$ such that each curve J_{3i} contains B but does not contain A.

Case IV. Suppose that $[J_{n_i}]$ contains an infinite subsequence J_{41}, J_{42}, J_{43}. such that each curve J_{4i} contains neither A not B.

Cases III and IV may be proved impossible by methods similar to those of Case II. Thus all four cases are impossible. But if M is not almost simply cyclic we must have one of the four cases. Therefore the continuous curve M is almost simply cyclic.

Theorem 6. *A simply cyclic continuous curve is almost simply cyclic.*

This is an obvious consequence of the definitions.

Theorem 7. *A simply cycltc continuous curve is simply joined.*

This result may be proved by methods very similar to those used in Case I of the proof of Theorem 5.

Theorem 8. *Every connected subset of a simply cyclic continuous curve is arc-wise connected.*

This theorem is a consequence of Theorems 3 and 7.

Theorem 9. *Every closed and connected subset of a simply cyclic continuous curve is a continuous curve.*

Theorem 9 follows from Theorems 1 and 8.

Theorem 10. *Every boundary point of an S-domain* [1]) *of a simply joined continuous curve is accessible from the domain.*

Proof. Let P be a boundary point of an S-domain D of a continuous curve S which is simply joined. Then $D + P$ is connected and is therefore arc-wise connected by Theorem 3.

Theorem 11. *If ε is any positive number, then a simply joined continuous curve contains at most a finite number of mutually exclusive connected sets of diameter greater than ε.*

Suppose that a simply joined continuous curve M contains an infinite set, $K_1, K_2, K_3, \ldots$, of mutually exclusive connected subsets each of diameter greater than some positive number ε. Each set K_i contains two points x_i and y_i which are at a distance apart greater

[1]) A connected subset D of a continuous curve S is said to be a *S-domain* if for every point P of D there exists a circle K with center at P such that the set of all points of S which (1) lie interior to K, and (2) lie with P in a connected subset of S that lies wholly interior to K, is a subset of D. Cf. R. L. Wilder, loc. cit., p. 341. A point P is said to be a *boundary point* of a *S-domain* D if P is a limit point of D but does not belong to D. A boundary point P of a domain D is said to be *accessible from the domain* if for every point Q of the domain there exists an arc PQ which lies except for the point P entirely in the domain D.

than ε. By Theorem 3, eacht set K_i contains an arc C_i from x_i to y_i. But this is impossible by Theorem 4 and a theorem due to H. M. Gehman[1]).

Theorem 12. *If ε is a positive number, then an almost simply cyclic continuous curve contains at most a finite number of mutually exclusive closed and connected sets of diameter greater than ε.*

This theorem is a consequence of Theorem 2 and a theorem due to H. M. Gehman[2]).

[1]) *Concerning the subsets of a plane continuous curve*, Annals of Mathematics, vol. 27 (1925), p. 39, Theorem V.

[2]) Loc cit, Theorem V.

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Beweis des Satzes, dass jede abgeschlossene Menge positiver Dimension in einem lokal zusammenhängenden Kontinuum von derselben Dimension topologisch enthalten ist.

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1. Unter einer *abgeschlossenen Menge* wird im Folgenden ein beliebiger kompakter metrisierbarer topologischer Raum[1]) verstanden.

Eine zusammenhängende abgeschlossene Menge heisst ein *Kontinuum.*

Bekanntlich ist ein Kontinuum dann und nur dann stetiges Bild der Einheitsstrecke $0 \leqslant t \leqslant 1$, wenn es *lokal* (oder *im Kleinen*) *zusammenhängend* ist[2]); (im letzteren Satze ist auch die Bedeutung des Begriffes des lokalen Zusammenhanges enthalten).

Der *Dimensionsbegriff* wird im allgemein üblichen Urysohn-Menger'schen Sinne verstanden[3]).

Endlich heisst ein topologischer Raum R_0 in einem anderen

[1]) d. h. ein kompakter topologischer Raum in dem das zweite Abzählbarkeitsaxiom erfüllt ist. Vgl. hierzu Hausdorff, Grundzüge der Mengenlehre (Leipzig, 1914), Kap. VII, sowie P. Urysohn, Zum Metrisationsproblem (Math. Ann., 94, S. 309) und vor allem P. Urysohn, Mémoire sur les multiplicités cantoriennes (Fund. Math., VII, S. 30—137 und VIII, S. 225—359).

[2]) siehe Hahn, Wiener Berichte, 123, (1924), S. 2433, Mazurkiewicz, Fund. Math., I, (1920), S. 167, Sierpiński, Fund. Math. I, (1920), S. 44, wo sich auch verschiedene Fassungen des Begriffes des lokalen Zusammenhanges finden. Als zusammenfassende Darstellung der ganzen Theorie des lokalen Zusammenhanges sei insbesondere das Buch von Hausdorff, „Mengenlehre" (neue Auflage, Berlin, 1927) erwähnt.

[3]) siehe P. Urysohn's unter 1) zitiertes „Mémoire" sowie K. Menger, Monatshefte f. Math. u. Phys., 33, 34.