A theorem on continua.

By

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In his paper *A characterization of continuous curves*[1]) R. L. Moore establishes the following property of continuous curves:

Theorem A. *Of two concentric circles C_1 and C_2, let C_1 be the smaller. Denote by H the point-set which is the sum of C_1, C_2, and the annular domain bounded by C_1 and C_2. Let M be a continuous curve which contains a point A interior to C_1 and a point B exterior to C_2. If N is any connected subset of M containing A and B, N will contain at least one point of some continuum which is a subset of M and H, and which has at least one point in common with each of the circles C_1 and C_2.*

The question has been raised by Mr. Moore as to whether Theorem A will hold true if M is any continuum. It is the purpose of this note to show that this question may be answered in the affirmative.

Theorem B. *Theorem A holds true, if M, instead of being a continuous curve, is any continuum.*

Proof. Denote by NH the set of points common to N and H, and by MH the set of points common to M and H. Since the set of points common to two closed sets is closed, it is clear that MH is a closed set.

For any point x of NH, let $N(x)$ be the set of all points which lie with x in a connected subset of MH. Since MH is closed, $N(x)$ is a continuum.

For every point x. $N(x)$ contains at least one point of the set $C_1 + C_2$. If x is a point of the set $C_1 + C_2$, this is self-evident.

1) This volume, pp. 302—306.

Suppose that x is a point of NH which is not a point of either C_1 or C_2. Then $C_1 + C_2$ and x are two closed mutually exclusive sets. By a theorem due to Miss Mullikin [1]), there exists a point set L which is a subset of M, is connected, contains no point of $C_1 + C_2$, and does not contain x, but such that x is a limit point of L and $C_1 + C_2$ contains at least one limit point, P, of L. It follows that the set $x + L + P$ is a subset of $N(x)$, and that $N(x)$ contains a point of at least one of the circles C_1, C_2.

If there exists any point x of NH such that $N(x)$ contains a point of C_1 and a point of C_2, the theorem is proved. Suppose, however, that no such point exists. Then NH can be divided into two mutually exclusive point sets, N_1 and N_2 as follows: Every point x of NH such that $N(x)$ contains a point of C_1 assign to the set N_1; every point x of NH such that $N(x)$ contains a point of C_2 assign to the set N_2. Neither N_1 nor N_2 can be vacuous, since NH contains at least one point of each of the circles C_1 and C_2 Furthermore, every point x of NH belongs to either N_1 or to N_2, since, as shown above, $N(x)$ contains at least one point of $C_1 + C_2$.

The sets N_1 and N_2 are mutually separated [2]). For suppose N_2 contains a limit point, t, of N_1. Then from the point set N_1 can be selected a countable sequence of points, $x_1, x_2, x_3, \ldots$, such that every circle that encloses t also encloses all except a finite number of these points. Consequently every circle that encloses t encloses points from all except a finite number of the sets $N(x_1)$, $N(x_2)$, $N(x_3), \ldots$ Hence by a well known theorem of Janiszewski [3]), the limit set, N, of these sets is connected. But every set of the sequence $N(x_1)$, $N(x_2)$, $N(x_3), \ldots$ contains at least one point of C_1, and therefore N contains a point of C_1. As MH is closed, it follows that N is a subcontinuum of MH which contains x and a point on C_1 Then N is a subset of $N(t)$. This is clearly a contradiction of the fact that t is a point of N_2. Therefore the set N_2 can contain no limit point of the set N_1.

[1]) Anna M. Mullikin, *Certain theorems relating to plane connected point sets*. Trans. Amer. Math. Soc., XXIV (1922), pp. 144—162, Th. 1.

[2]) That is, they are not only mutually exclusive, but neither contains a limit point of the other.

[3]) See S. Janiszewski, *Sur les continus irréductibles entre deux points*, Journal de l'Ecole Polytechnique (2), XVI (1912), p. 109, Th. 1

In a similar manner it can be shown that N_1 contains no limit point of N_2.

Denote by N_I the set of all points of N that lie interior to C_1 and by N_E the set of all points of N that lie exterior to C_2.

Then

$$N = (N_I + N_1) + (N_E + N_2).$$

But $N_I + N_1$ and $N_E + N_2$ are clearly mutually separatad. But N is by hypothesis a connected set. Hence the supposition that there does not exist any point x of NH such that $N(x)$ contains a point of C_1 and a point of C_2 leads to a contradiction, and the theorem is proved.