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Homogeneous operators

by

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Abstract. A self-adjoint operator A defined on a separable Hilbert space is called homogeneous if, for every real λ, the operator $A-\lambda I$ is unitarily equivalent to A. Similarly a unitary operator U on a separable Hilbert space is called homogeneous if for each $\alpha \in R$ the operator $e^{i\alpha} U$ is unitarily equivalent to U. It is proved that every homogeneous self-adjoint operator is equivalent to the operator of multiplication by the independent variable in the space of square integrable vector-valued functions of a real variable. A similar characterization is given for homogeneous unitary operators.

The purpose of this paper is to present the concept of homogeneity of operators on a Hilbert space and to give a characterization of such operators. We begin with self-adjoint operators.

A self-adjoint operator A defined on a separable Hilbert space will be called *homogeneous* if, for every real $\lambda \in R$, the operator $A-\lambda I$ is unitarily equivalent to A, i.e. there exists a unitary operator U_λ such that

$$A-\lambda I = U_\lambda A U_\lambda^*.$$

From this definition it follows that a self-adjoint homogeneous operator A is unbounded and that the real line $\boldsymbol{R}$ is in the continuous spectrum of A.

A simple example of a self-adjoint homogeneous operator is the operator of multiplication by the independent variable t in $L_2(\boldsymbol{R})$,

$$(Ax)(t) = tx(t) \tag{1}$$

$x \in L_2(\boldsymbol{R})$. Another example, which is a slight generalization of the first one, is the operator A which acts on a direct sum of copies of $L_2(\boldsymbol{R})$, i.e. on the Hilbert space of vector-valued functions of a real variable and is defined by the same formula (1). We will prove that these are essentialy the only homogeneous self-adjoint operators.

THEOREM 1. *Let A be a self-adjoint homogeneous operator on a separable Hilbert space H and let A have spectral multiplicity k, $k = 1, 2, \ldots, n$, $n \leqslant \infty$. Then A is equivalent to the operator of multiplication by an independent variable in the direct sum of k copies of $L_2(\boldsymbol{R})$.*

We can prove similar results for homogeneous unitary operators. A unitary operator U in a separable Hilbert space will be called homo-

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geneous if, for each $\alpha \in \boldsymbol{R}$, $\alpha \in [0, 2\pi]$, the operator $e^{i\alpha}U$ is unitarily equivalent to U.

Let K denote the unit circle in the complex plane. The preceding definition implies that for the homogeneous unitary operator U the spectrum $\sigma(U)$ is continuous and equal to K.

Consider the unitary operator U in the Hilbert space $L_2(K)$ (with a Lebesgue measure on K) defined by

$$(Ux)(z) = zx(z) \tag{2}$$

$x \in L_2(K)$, $|z| = 1$. The operator V_a on $L_2(K)$ given by $(V_a x)(z) = e^{-ia/2}x(e^{ia}z)$ is unitary and $V_a U V_a^* = e^{ia}U$; hence U is a homogeneous unitary operator on $L_2(K)$. Again we can generalize this example by replacing the scalar function x in (2) by a vector-valued function. In analogy to Theorem 1 we have

THEOREM 2. *Let U be a unitary homogeneous operator in a separable Hilbert space H and let U have spectral multiplicity k, $k = 1, 2, \ldots, n$, $n \leqslant \infty$. Then U is equivalent to the operator of multiplication by the independent variable z ($|z| = 1$) in the direct sum of k copies of $L_2(K)$, where K is the unit circle in the complex plane and the measure in K is a Lebesgue measure.*

The tools we need to prove above theorems are: the theory of spectral and ordered representation of unbounded self-adjoint and bounded normal operators ([1], Ch. XII and Ch. X) and the theory of quasi-invariant measures ([2], [4]). We use the same notation as is used in [1]. We frequently refer to Definition XII.3.4 in [1].

We also introduce the notion of quasi-invariant measures [2].

DEFINITION. *Let G be a locally compact (compact) commutative topological group. A regular Borel measure μ defined on G is called quasi-invariant, if for every measurable set $D \subset G$ for which $\mu(D) = 0$, $\mu(-x+D) = 0$ for each $x \in G$.*

LEMMA 1. *If the finite positive non-zero regular Borel measures μ and ϱ on R are quasi-invariant, then $\mu \cong \varrho$.*

Proof. We will prove that μ and ϱ are equivalent to their convolution $\mu * \varrho$. Let μ and ϱ be finite Borel measures on a locally compact group; then their convolution $\mu * \varrho$ is defined by

$$(\mu * \varrho)(D) = \int_G \mu(-t+D)\,d\varrho(t) = \int_G \varrho(-t+D)\,d\mu(t),$$

where D is a Borel subset of G and $t \in G$. The convolution $\mu * \varrho$ is a finite regular Borel measure on G ([3], [5]). For our purpose G is the real line and the group operation is addition.

Now (cf. [4]) we take a Borel set $D \subset \boldsymbol{R}$ such that $\mu(D) = 0$, and form the convolution

$$(\mu * \varrho)(D) = \int_{-\infty}^{\infty} \mu(-t+D)\,d\varrho(t).$$

Since μ is quasi-invariant, $\mu(-t+D) = 0$ for every $t \in \boldsymbol{R}$. Then $(\mu * \varrho)(D) = 0$ and it follows that the measure $\mu * \varrho$ is absolutely continuous with respect to μ.

Let us choose a set $D \subset \boldsymbol{R}$ such that $(\mu * \varrho)(D) = 0$, i.e.

$$\int_{-\infty}^{\infty} \mu(-t+D)\,d\varrho(t) = 0.$$

Then $\mu(-t+D) \equiv 0$ a.e. $[\varrho]$ and hence there exists a t_0 such that $\mu(-t_0+D) = 0$. By quasi-invariance we have $\mu(D) = 0$. We have thus proved $\mu * \varrho \cong \varrho$. In a similar way we see that $\varrho \cong \mu * \varrho$, hence $\mu \cong \varrho$.

LEMMA 2. *The measures μ_k of the ordered representation corresponding to the homogeneous operator A are quasi-invariant.*

Proof. For $e \subset \boldsymbol{R}$ and $\lambda \in \boldsymbol{R}$ denote by $e+\lambda$ the set e translated by λ. First we prove that μ is quasi-invariant. To begin with, notice that the spectral measure E of operator A is quasi-invariant, i.e. $E(e) = 0$ implies $E(e+\lambda) = 0$ for every $\lambda \in \boldsymbol{R}$. Indeed, since A and $A - \lambda I$ are unitarily equivalent, E is equivalent to the spectral measure E_λ of $A - \lambda I$. Hence, if $E(e) = 0$, then $E_\lambda(e) = 0$. But $E_\lambda(e) = E(e+\lambda)$, which proves our claim.

Notice now that $\mu = \mu_1$ (see Lemma X.5.8 of [1]). In other words, the element x_1 of H corresponding to μ_1 is maximal. Now, if $\mu_1(e) = 0$, then $\mu_k(e) = 0$ for $k = 2, 3, \ldots$, which implies that $E(e) = 0$. It follows that $E(e+\lambda) = 0$ and consequently $\mu(e+\lambda) = \mu_1(e+\lambda) = \big(E(e+\lambda)x_1, x_1\big) = 0$.

Let e_k, $k = 1, 2, \ldots$, be the multiplicity sets of the ordered representation relative to A. Fix a $\lambda \in \boldsymbol{R}$. Define Borel measures ν_k by

$$\nu_k(e) = \mu\big(e \cap (\lambda + e_k)\big). \tag{3}$$

Denote

$$H_1 = \sum_{k=1}^{\infty} \oplus L_2(\mu_k) \quad \text{and} \quad H_2 = \sum_{k=1}^{\infty} \oplus L_2(\mu_k).$$

By Theorem XII. 3.16, [1], to the operator A corresponds in the space H_1 the operator of multiplication by t, and to the operator $A - \lambda I$ the operator of multiplication by $t - \lambda$. Define a map $V\colon H_1 \to H_2$ as

$$V\big(x_1(t), x_2(t), \ldots\big) = \big(x_1(t+\lambda), x_2(t+\lambda), \ldots\big)$$

where $(x_1(t), x_2(t), \ldots) \in H_1$. V is an isometric isomorphism:

$$\|V\{x_k(t)\}\|^2 = \|\{x_k(t+\lambda)\}\|^2 = \sum_{k=1}^{\infty} \int_{R} |x_k(t+\lambda)|^2 d\nu_k(t)$$
$$= \sum_{k=1}^{\infty} \int_{e_k+\lambda} |x_k(t+\lambda)|^2 d\mu(t) = \sum_{k=1}^{\infty} \int_{e_k} |x_k(\tau)|^2 d\mu(\tau)$$
$$= \sum_{k=1}^{\infty} \int_{R} |x_k(\tau)|^2 d\mu_k(\tau) = \|\{x_k(t)\}\|^2.$$

To the operator $A-\lambda I$ corresponds in the space H_2 the operator of multiplication by t:

$$V(t-\lambda)\big[V^{-1}(y_1(t), y_2(t), \ldots)\big] = V(t-\lambda)[y_1(t-\lambda), y_2(t-\lambda), \ldots]$$
$$= V[(t-\lambda)y_1(t-\lambda), (t-\lambda)y_2(t-\lambda), \ldots]$$
$$= (ty_1(t), ty_2(t), \ldots).$$

From the above fact and definition (3) it follows that H_2 is an ordered representation of H relative to the $A-\lambda I$.

The operator A is homogeneous, and hence unitarily equivalent to $A-\lambda I$. To unitary equivalent operators correspond equivalent ordered representations (cf. [1], Def. XII. 3.15). Hence we have $\mu_k \cong \nu_k$.

Let e be a Borel set with $\mu_k(e) = 0$, i.e. $\mu(e \cap e_k) = 0$. Since $\mu_k \cong \nu_k$, $\nu_k(e) = \mu\big(e \cap (\lambda+e_k)\big) = 0$. Since μ is quasi-invariant, $\mu\Big(\big(e\cap(\lambda+e_k)\big)-\lambda\Big) = 0$. But $\big(e\cap(\lambda+e_k)\big)-\lambda = (e-\lambda)\cap e_k$. Hence $\mu\big((e-\lambda)\cap e_k\big) = \mu_k(e-\lambda) = 0$. Thus from $\mu_k(e) = 0$ it follows that $\mu_k(e-\lambda) = 0$ for every k, i.e. the measure μ_k is quasi-invariant, as was to be proved.

Let the measure ϱ be defined by $d\varrho = dx/(1+x^2)$; ϱ is a positive finite regular quasi-invariant measure. It is easy to see that the measure ϱ and the Lebesgue measure m are equivalent, $\varrho \cong m$. From Lemmas 1 and 2 follows that $\mu_k \cong \varrho$; hence $\mu_k \cong m$ for all k.

Proof of Theorem 1. Since $\mu_k \cong m$, by the Radon–Nikodym theorem there exists a positive function $h(t)$ integrable on every finite interval and such that

$$d\mu_k = h(t)\,dt.$$

Also,

$$dt = 1/h(t)\,d\mu_k, \quad h(t) \in L_1^{\mathrm{loc}}.$$

Define V: $L_2(\mu_k) \to L_2(m)$ by

$$Vx(t) = x(t)\sqrt{h(t)},$$

$x(t) \in L_2(\mu_k)$. This map is obviously an isometric isomorphism between $L_2(\mu_k)$ and $L_2(m)$. Evidently $\tilde{A} = UAU^{-1}$ is equivalent to the operator of multiplication by t in $L_2(m)$:

$$(V\tilde{A}V^{-1}y)(t) = ty(t),$$

$y(t) \in L_2(m)$, which completes the proof.

The proof of Theorem 2 is similar to the proof of Theorem 1. Unitary operators are bounded and normal, and hence the theory of spectral and ordered representation ([1], Ch. X.5) applies. There exists an isometric isomorphism of the Hilbert space H onto $\sum \oplus L_2(\mu_k)$, where μ_k are positive regular measures on the circle K. If we replace the locally compact group R by the compact group K, Lemma 1 is also valid. Every measure corresponding to a homogeneous unitary operator is quasi-invariant, which can be proved in the same way as in Lemma 2. From these results it follows that every quasi-invariant measure defined on K is equivalent to the Lebesgue measure on K. Now we can complete the proof of Theorem 2 in a similar way to that followed in the case of homogeneous self-adjoint operators.

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