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Integration of evolution equations in a locally convex space

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Abstract. Let $H = H(\boldsymbol{R}^m)$ be the space of all real-valued functions in $C^\infty(\boldsymbol{R}^m)$ having every partial derivative in $L_2(\boldsymbol{R}^m)$ and topologized by the seminorms defined as follows:

$$p_i(\varphi) = \Big(\sum_{|\nu|=0}^{i} \int_{\boldsymbol{R}^m} |D^{(\nu)}\varphi(\underset{\sim}{t})|^2 d\underset{\sim}{t}\Big)^{1/2}, \quad \varphi \in H,\ i = 0, 1, 2, \ldots$$

Let A be an elliptic differential operator with coefficients possessing bounded derivatives of all orders. This paper solves the Cauchy problem for the system:

$$\frac{\partial u(\xi, \underset{\sim}{t})}{\partial \xi} = (Au)(\xi, t), \quad \xi > 0,\ \underset{\sim}{t} \in \boldsymbol{R}^m,$$

$$u(0, \underset{\sim}{t}) = f(\underset{\sim}{t}), \quad f \in H,\ \underset{\sim}{t} \in \boldsymbol{R}^m.$$

1. Introduction. The present paper is a follow-up to [2], and its knowledge is assumed here. Let Ω be an open subset of a Euclidean space. For convenience we shall denote by $C^\infty = C^\infty(\Omega)$ the space of all infinite times continuously differentiable real-valued functions on Ω and by $C_0^\infty(\Omega)$ the space of functions in $C^\infty(\Omega)$ having compact support in Ω.

Now let A be the partial differential operator of $2n$th order in m-dimensional Euclidean space $\boldsymbol{R}^m$ given by

$$A = -(-1)^n \sum_{|\varrho|,|\nu|=0}^{n} D^{(\varrho)} a_{\varrho;\nu}(\underset{\sim}{t}) D^{(\nu)}, \tag{1.1}$$

where the coefficients $a_{\varrho;\nu}$ belong to $C^\infty(\boldsymbol{R}^m)$ with bounded partial derivatives of all orders. We assume further that $a_{\varrho;\nu}(\underset{\sim}{t}) = a_{\nu;\varrho}(\underset{\sim}{t})$ for $|\varrho| = |\nu| = n$ and there is a constant $\varepsilon_0 > 0$ such that

$$\sum_{|\varrho|=|\nu|=n} a_{\varrho;\nu}(\underset{\sim}{t}) t_1^{\varrho_1} \ldots t_m^{\varrho_m} \cdot t_1^{\nu_1} \ldots t_m^{\nu_m} \geqslant \varepsilon_0 \Big(\sum_{j=1}^{m} t_i^2\Big)^n \tag{1.2}$$

for each $(t_1, \ldots, t_m) \in \boldsymbol{R}^m$; so that A is an elliptic differential operator.

Let $H = H(\boldsymbol{R}^m)$ be the space of all functions in $C^\infty(\boldsymbol{R}^m)$ with every partial derivative in $L_2(\boldsymbol{R}^m)$ and denote by H also the topological vector space obtained by imposing the topology determined by the set $\{p_i\colon i \geqslant N\} = \{p_a\colon a \in \mathscr{A}\}$ of semi-norms on this family of functions, where

$$(1.3)\qquad p_i(\varphi) = \Big(\sum_{|\nu|=0}^{i} \int_{\boldsymbol{R}^m} |D^{(\nu)}\varphi(\underset{\sim}{t})|^2\, d\underset{\sim}{t}\Big)^{1/2}, \qquad \varphi \in H.$$

Note that H is a Fréchet Space.

In this paper we prove the following theorem:

1.4 THEOREM. *The Cauchy problem for the equation*

$$(1.5)\qquad \begin{aligned} \frac{\partial u(\xi, \underset{\sim}{t})}{\partial \xi} &= (Au)(\xi, \underset{\sim}{t}), \qquad \xi > 0,\ \underset{\sim}{t} \in \boldsymbol{R}^m,\\ u(0, \underset{\sim}{t}) &= f(\underset{\sim}{t}), \qquad \underset{\sim}{t} \in \boldsymbol{R}^m, \end{aligned}$$

is solvable in the following sense: For any given $f \in H$ the equation (1.5) *admits a solution $u = u(\xi, \underset{\sim}{t}) = u(\xi, \underset{\sim}{t}; f) \in C^\infty((0, \infty) \times \boldsymbol{R}^m)$ satisfying the following:*

(i) $u(\xi+\eta, \underset{\sim}{t}; f) = u(\xi, \underset{\sim}{t}; u(\eta, \cdot; f))$ $\quad(\xi, \eta > 0$ *and* $\underset{\sim}{t} \in \boldsymbol{R}^m)$,

(ii) $u(\xi, \cdot; f) \in H$ *for each* $\xi > 0$,

(iii) $\lim\limits_{\xi \to 0+} u(\xi, \cdot; f) = f(\cdot)$ *in* H.

Moreover, the solution $u(\xi, \underset{\sim}{t}; f)$ satisfying (ii) *and* (iii) *is uniquely determined for $f \in H$.*

2. Preliminaries.

2.1. Notations. For each non-negative integer i,

(1) $(\varphi, \psi)_i = \sum\limits_{|\nu|=0}^{i} \int\limits_{\boldsymbol{R}^m} D^{(\nu)}\varphi(\underset{\sim}{t}) D^{(\nu)}\psi(\underset{\sim}{t})\, d\underset{\sim}{t}$ for all $\varphi, \psi \in H = H(\boldsymbol{R}^m)$

and $i = 0, 1, 2, \ldots$

(2) H_i is the pre-Hilbert space formed by H under the inner product (1).

(3) $\bar{H}_i$ is the completion of H_i with respect to the norm $\|\cdot\|_i = p_i(\cdot)$.

Indeed the p'_is are norms on H. Hence, under the topology induced by $p_i(\cdot)$, the normed linear space formed by the elements of $H/p_i^{-1}(0)$ can and will be replaced by H_i in the sequel. Whenever necessary $f_i(=f)$ will denote an element of H_i seen as a coset. Note that the operator A defined by (1.1) with domain H and range in H has the property that the linear operator $A_i\colon H_i \to H_i$ defined by

$$A_i f_i = (Af)_i \qquad f_i \in H_i$$

is well-defined. For, clearly, $f_i = g_i$ in $H_i \Rightarrow f(\underset{\sim}{t}) \equiv g(\underset{\sim}{t}) \Rightarrow A(f-g)(\underset{\sim}{t})$

$\equiv 0 \Rightarrow \sum\limits_{|\nu|=0}^{i} \int\limits_{\boldsymbol{R}^m} |D^{(\nu)} A(f-g)(\underset{\sim}{t})|^2 d\underset{\sim}{t} = 0 \Rightarrow (Af - Ag)_i = 0 \Rightarrow (Af)_i = (Ag)_i$ $\Rightarrow A_i f_i = A_i g_i$. Hence A_i is well-defined on H_i. Observe that A_i is effectively the operator A taken as acting on the normed linear space H_i into itself.

We define the adjoint A^* of A by

$$(2.2)\qquad A^* = -(-1)^n \sum_{|\varrho|, |\nu|=0}^{n} (-1)^{|\varrho|+|\nu|} D^{(\nu)} a_{\varrho;\nu}(\underset{\sim}{t}) D^{(\varrho)}.$$

Note that $f \in H$ can be approximated by a sequence of functions in $C_0^\infty(R^m)$ (cf. [4] page, 58). Thus, since (a) the inner product $(\cdot,\cdot)_0$ defined by 2.1(1) on $H \times H$ is continuous, (b) A^* is continuous on H and (c) by partia integration $(Af, \varphi)_0 = (f, A^*\varphi)_0$ for all $\varphi \in C_0^\infty(\boldsymbol{R}^m)$, it follows, in the limit that

$$(2.3)\qquad (Af, g)_0 = (f, A^* g)_0 \qquad \text{for all } f, g \in H.$$

Similarly

$$(2.4)\qquad \Big(\sum_{|\gamma|=0}^{i} (-1)^{|\gamma|} D^{(\gamma)} D^{(\gamma)} Af, g\Big)_0 = \Big(f, \sum_{|\gamma|=0}^{i} (-1)^{|\gamma|} A^* D^{(\gamma)} D^{(\gamma)} g\Big)_0$$

for all $f, g \in H$.

Extend the inner product on H_i by continuity to $\bar{H}_i$. Still denote the extension by $(v, w)_i$ for all $v, w \in \bar{H}_i$ and the corresponding norm by $\|\cdot\|_i$. Since $C_0^\infty(\boldsymbol{R}^m)$ is $\|.\|_i$-dense in $\bar{H}_i$, the following is immediate.

2.5. LEMMA. *Let $f, g \in \bar{H}_i$. Then $(f-g, \varphi)_i = 0$ for all $\varphi \in C_0^\infty(\boldsymbol{R}^m)$ implies $f = g$ in $\bar{H}_i$.*

We now state a suitable form of Gårding's inequality.

2.6. LEMMA. *Suppose the differential operator*

$$(2.7)\qquad \check{A} = (-1)^j \sum_{|\beta|, |\gamma|=0}^{j} D^{(\beta)} a_{\beta,\gamma}(\underset{\sim}{t}) D^{(\gamma)}$$

defined on H into H satisfies the following:

(1) *The $a_{\beta;\gamma}$'s are functions in $C^\infty(\boldsymbol{R}^m)$ with bounded partial derivatives of all orders.*

(2) $a_{\beta;\gamma}(\underset{\sim}{t}) = a_{\gamma;\beta}(\underset{\sim}{t})$ *for* $|\gamma| = |\beta| = j$.

(3) *There exists a positive constant ε_0 such that*

$$\sum_{|\beta|=|\gamma|=j} a_{\beta;\gamma}(\underset{\sim}{t}) t_1^{\beta_1} \ldots t_1^{\beta_m} \cdot t_1^{\gamma_1} \cdots t_m^{\gamma_m} \geqslant \varepsilon_0 \Big(\sum_{i=1}^{m} t_i^2\Big)^j$$

for all $(t_1, \ldots, t_m) \in \boldsymbol{R}^m$ (that is, $\check{A}$ is strongly elliptic).

Then there exist positive constants c and C such that

(2.8) $$(\check{A}f, f)_0 \geqslant c\|f\|_j^2 - C\|f\|_0^2 \quad \textit{for all } f\in H.$$

Proof. By [1], Theorem 7.6,

(2.9) $$(\check{A}\varphi, \varphi)_0 \geqslant c\|\varphi\|_j^2 - C\|\varphi\|_0^2$$

for all $\varphi\in C_0^\infty(\boldsymbol{R}^m)$. This class of functions is dense in H. Hence, for any $f\in H$, there exists a sequence $\{\varphi_k\}\subset C_0^\infty(\boldsymbol{R}^m)$ such that $\lim_{k\to\infty}\varphi_k = f$ in H. Now observe that the operator $\check{A}$, the norms $\|\cdot\|_j$, $\|\cdot\|_0$ are continous on H and the inner product $(\cdot,\cdot)_0$ is continuous on $H\times H$. Hence, by taking $\varphi = \varphi_k$ in (2.9) and letting $k\to\infty$, we obtain the lemma.

2.10. Corollary. *Let A be the differential operator defined by* (1.1). *There exist positive constants c_i and C_i such that if $\lambda > C_i$ then*

(2.11) $$\Big(\sum_{|\gamma|=0}^{i}(-1)^{|\gamma|}D^{(\gamma)}D^{(\gamma)}(\lambda I - A)f, f\Big)_0 = \Big(f, \sum_{|\gamma|=0}^{i}(-1)^{|\gamma|}(\lambda I - A^*)D^{(\gamma)}D^{(\gamma)}f\Big)_0 \geqslant c_i\|f\|_{n+i}^2$$

for all $f\in H$. Further, for each positive λ, there exists a positive constant $K_{\lambda,i}$ such that

(2.12) $$\Big|\Big(\sum_{|\gamma|=0}^{i}(-1)^{|\gamma|}D^{(\gamma)}D^{(\gamma)}(\lambda I - A)f, g\Big)_0\Big| = \Big|\Big(f, \sum_{|\gamma|=0}^{i}(-1)^{|\gamma|}(\lambda I - A^*)D^{(\gamma)}D^{(\gamma)}g\Big)_0\Big| \leqslant K_{\lambda,i}\|f\|_{n+1}\|g\|_{n+1}$$

for all $f, g\in H$.

Proof. The equality in (2.11) is true by (2.4). It is easy to show that the differential operator $(-1)\sum_{|\gamma|=0}^{i}(-1)^{|\gamma|}D^{(\gamma)}D^{(\gamma)}A$ satisfies the hypotheses of Lemma 2.6. Thus there exist positive constants c_i, C_i such that

$$\Big((-1)\sum_{|\gamma|=0}^{i}(-1)^{|\gamma|}D^{(\gamma)}D^{(\gamma)}Af, f\Big)_0 \geqslant c_i\|f\|_{n+i}^2 - C_i\|f\|_0^2 \quad \text{for all } f\in H.$$

Now

$$\Big(\sum_{|\gamma|=0}^{i}(-1)^{|\gamma|}D^{(\gamma)}D^{(\gamma)}(\lambda I - A)f, f\Big)_0 = \lambda\Big(\sum_{|\gamma|=0}^{i}(-1)^{|\gamma|}D^{(\gamma)}D^{(\gamma)}f, f\Big)_0 - \Big(\sum_{|\gamma|=0}^{i}(-1)^{|\lambda|}D^{(\lambda)}D^{(\gamma)}Af, f\Big)_0$$

for all $f\in H$. Thus

(2.13) $$\Big(\sum_{|\gamma|=0}^{i}(-1)^{|\gamma|}D^{(\gamma)}D^{(\gamma)}(\lambda I - A)f, f\Big)_0 \geqslant c_i\|f\|_{n+i}^2 + (\lambda - C_i)\|f\|_i^2$$

for all $f\in H$. Therefore, provided $\lambda > C_i$, we have

(2.14) $$\Big(\sum_{|\gamma|=0}^{i}(-1)^{|\gamma|}D^{(\gamma)}D^{(\gamma)}(\lambda I - A)f, f\Big)_0 \geqslant c_i\|f\|_{n+i}^2$$

for all $f\in H$, and (2.11) is true.

Each $\binom{\gamma}{\beta}D^{(\beta)}a_{\beta;\nu}$ is bounded. Hence (2.12) follows as a consequence of the Schwartz inequality and the corollary is proved.

3. Proof of Theorem 1.4. We would have proved Theorem 1.4 if we had shown that the differential operator A with domain $D(A) = H$ generates an $L_{\mathscr{A}}(H)$-operator semi-group of class $(C_0, 1)$. For this purpose we shall employ a variant of a technique of Yosida ([4], pp. 413–416).

It is clear that the linear operator A with $D(A) = H$ is continuous and therefore closed in H. It is also clear that $D(A) = H$ is dense in H. We have already noted that $A_i = A\colon H_i\to H_i$ is well-defined. Now to show that A with $D(A) = H$ generates an $L_{\mathscr{A}}(H)$-operator semi-group of class $(C_0, 1)$ we still need, according to [2], establish the following:

(3.1)
(i) For each i, A_i is closable in $\bar{H}_i$.
(ii) For each i, there exist positive numbers σ_i, M_i such that the rosolvent $R(\lambda; \bar{A}_i)$ of the closure $\bar{A}_i$ of A_i in $\bar{H}_i$ exists for all $\lambda > \sigma_i$ and
$$\|[R(\lambda; \bar{A}_i)]^k\|_i \leqslant M_i(\lambda - \sigma_i)^{-k} \quad \text{for all } \lambda > \sigma_i \text{ and } k = 1, 2, \ldots$$

We first take up (3.1)(i).

3.2. Lemma. *The linear operator $A_i = A\colon H_i\to H_i$ is closable in $\bar{H}_i$.*

Proof. Let $\{f_k\}\subset D(A_i) = H_i\subset \bar{H}_i$ be such that $\lim_{k\to\infty}\|f_k\|_i = 0$ and $\lim_{k\to\infty}\|A_if_k - g\|_i = 0$, $g\in\bar{H}_i$. It remains to show that $g = 0$ in $\bar{H}_i$ to prove the lemma. Now for any $\varphi\in C_0^\infty(\boldsymbol{R}^m)$,

(3.3) $$(A_if_k, \varphi)_i = \Big(\sum_{|\gamma|=0}^{i}(-1)^{|\gamma|}D^{(\gamma)}D^{(\gamma)}A_if_k, \varphi\Big)_0 = \Big(f_k, \sum_{|\gamma|=0}^{i}(-1)^{|\gamma|}A_i^*D^{(\gamma)}D^{(\gamma)}\varphi\Big)_0$$

for all $k = 1, 2, \ldots$ and $\varphi\in C_0^\infty(\boldsymbol{R}^m)$. Note that the inner products $(\cdot,\cdot)_i$

and $(\cdot,\cdot)_0$ are continuous on $\bar{H}_i\times\bar{H}_i$. Hence, passing to the limit in (3.3), we have

$$(g,\varphi)_i = \Big(0,\ \sum_{|\gamma|=0}^{i}(-1)^{|\gamma|}A_i^* D^{(\gamma)}D^{(\gamma)}\varphi\Big)_0 = 0.$$

Hence, by Lemma 2.5, $g=0$ in $\bar{H}_i$. This proves the lemma.

To establish (3.1)(ii) we need a few preparatory results.

3.4 Lemma. *Let a positive number λ_i be so chosen that Corollary 2.10 is valid for $\lambda\geqslant\lambda_i$. Then, for any $f\in H$, the equation*

(3.5) $$\lambda u - Au = f, \qquad (\lambda\geqslant\lambda_i),$$

has a solution $u_{f,i}\in\bar{H}_{n+i}\cap C^\infty$ in the sense that

(3.6) $$\big((\lambda I - A)u_{f,i},\varphi\big)_i = (f,\varphi)_i \quad \textit{for all } \varphi\in C_0^\infty(\boldsymbol{R}^m).$$

Moreover, $u_{f,i}$ is unique in $\bar{H}_{n+i}\cap C^\infty$.

Proof. Observe that $\lambda I - A$ is strongly elliptic. Define a bilinear functional

$$B_{\lambda,i}(u,v) = \Big(\sum_{|\gamma|=0}^{i}(-1)^{|\gamma|}(\lambda I - A^*)D^{(\gamma)}D^{(\gamma)}u,\ v\Big)_0$$

for all $u,v\in H$. From Corollary 2.10,

$$|B_{\lambda,i}(u,v)|\leqslant K_{\lambda,i}\|u\|_{n+i}\|v\|_{n+1} \quad\text{and}\quad B_{\lambda,i}(u,u)\geqslant c_i\|u\|_{n+i}^2.$$

Hence we may extend $B_{\lambda,i}(u,v)$, by continuity, to a bilinear functional $\bar{B}_{\lambda,i}(u,v)$ defined for $u,v\in\bar{H}_{n+i}$ and such that

(3.7) $$|\bar{B}_{\lambda,i}(u,v)|\leqslant K_{\lambda,i}\|u\|_{n+i}\|v\|_{n+i} \quad\text{and}\quad \bar{B}_{\lambda,i}(u,u)\geqslant c_i\|u\|_{n+i}^2.$$

The linear functional $F_i(u)=(u,f)_i$ defined on $\bar{H}_{n+i}$, is bounded since $|(u,f)_i|\leqslant\|u\|_i\|f\|_i\leqslant\|u\|_{n+i}\|f\|_i$. Hence, by the Riesz representation theorem, in the Hilbert space $\bar{H}_{n+i}$, (see [4], page 90), there exists a uniquely determined $v=v(f)\in\bar{H}_{n+i}$ such that $(u,f)_i=\big(u,v(f)\big)_{n+i}$ for all $u\in\bar{H}_{n+i}$. Thus, by the Lax–Milgram theorem ([4], page 92),

$$(u,f)_i = \big(u,v(f)\big)_{n+i} = B_{\lambda,i}\big(u,S_{\lambda,i}v(f)\big) \quad\text{for all } u\in\bar{H}_{n+i},$$

where $S_{\lambda,i}$ is a bounded linear operator from $\bar{H}_{n+i}$ onto $\bar{H}_{n+i}$. Let $\{v_k\}\subset H$ be a sequence such that $\lim\limits_{k\to\infty}\|v_k - S_{\lambda,i}v(f)\|_{n+i}=0$. Then for $u\in C_0^\infty(\boldsymbol{R}^m)\subset H$,

$$\begin{aligned}\bar{B}_{\lambda,i}\big(u,S_{\lambda,i}v(f)\big) &= \lim_{k\to\infty}\bar{B}_{\lambda,i}(u,v_k) = \lim_{k\to\infty}B_{\lambda,i}(u,v_k)\\ &= \lim_{k\to\infty}\Big(\sum_{|\gamma|=0}^{i}(-1)^{|\gamma|}(\lambda I - A^*)D^{(\gamma)}D^{(\gamma)}u,\ v_k\Big)_0\\ &= \Big(\sum_{|\gamma|=0}^{i}(-1)^{|\gamma|}(\lambda I - A^*)D^{(\gamma)}D^{(\gamma)}u,\ S_{\lambda,i}v(f)\Big)_0,\end{aligned}$$

so that

$$\begin{aligned}(u,f)_i &= \Big(u,\ \sum_{|\gamma|=0}^{i}(-1)^{|\gamma|}D^{(\gamma)}D^{(\gamma)}f\Big)_0\\ &= \Big(\sum_{|\gamma|=0}^{i}(-1)^{|\gamma|}(\lambda I - A^*)D^{(\gamma)}D^{(\gamma)}u,\ S_{\lambda;i}v(f)\Big)_0\end{aligned}$$

for all $u\in C_0^\infty(\boldsymbol{R}^m)$. Hence by the strong ellipticity of $(\lambda I - A)$ and by the fact that $f\in C^\infty(\boldsymbol{R}^m)$, we see, from the Weyl–Schwartz theorem ([3], page 136), that $u_{f,i}=S_{\lambda,i}v(f)\in\bar{H}_{n+i}$ is a solution in $C^\infty(\boldsymbol{R}^m)$.

The uniqueness of such a solution $u=u_{f,i}$ of (3.5) is proved as follows. Let a function $u\in\bar{H}_{n+i}\cap C^\infty$ satisfy $\lambda u - Au = 0$ in the sense of (3.6). Thus $Au=\lambda u\in\bar{H}_{n+i}\cap C^\infty$ and so the expression $(\lambda u - Au, u)_i$ is defined and has value 0. As a consequence of Corollary 2.10, there exist C_i, c_i and $K_{\lambda,i}$ all positive such that $\lambda > C_i$ implies

$$K_{\lambda,i}\|u\|_{n+i}^2\geqslant(\lambda u - Au,\ u)_i = \Big(\sum_{|\gamma|=0}^{i}(-1)^{|\gamma|}D^{(\gamma)}D^{(\gamma)}(\lambda u - Au),\ u\Big)_0\geqslant c_i\|u\|_{n+i}^2$$

for all $u\in H$. This means that $\big((\lambda I - A)\cdot,\cdot\big)_i$ is an equivalent norm in H_{n+i}. Thus taking $\{u_k\}\subset H$ such that $\lim\limits_{k\to\infty}\|u-u_k\|_{n+i}=0$ gives

$$0 = (\lambda u - Au,\ u)_i = \lim_{k\to\infty}(\lambda u_k - Au_k,\ u_k)_i\geqslant c_i\|u\|_{n+i}^2.$$

Hence $\|u\|_{n+i}=0$, that is, $u=0$ in $\bar{H}_{n+i}\cap C^\infty$. We have proved the lemma.

3.8. Corollary. *For each i, a positive constant σ_i exists such that for any $f\in H$, the equation*

$$\lambda u - Au = f, \qquad (\lambda\geqslant\sigma_i),$$

admits a uniquely determined solution $u=u_f\in\bar{H}_{n+i}\cap C^\infty$ in the sense of Lemma 3.4. Moreover,

(3.9) $$\|u_f\|_i\leqslant(\lambda-\sigma_i)^{-i}\|f\|_i.$$

Proof. It is clear from Lemma 3.4 that if $\lambda\geqslant\sigma_i > C_i$ then the solution $u=u_f$ exists in the sense of (3.6) and is unique. We now obtain the estimate (3.9). The inequality (2.13) implies

(3.10) $$\Big(\sum_{|\gamma|=0}^{i}(-1)^{|\gamma|}D^{(\gamma)}D^{(\gamma)}(\lambda I - A)u,\ u\Big)_0\geqslant(\lambda - C_i)\|u\|_i^2$$

for all $u \in H$ and $\lambda > C_i$. Now, for each $u \in H$, we have, by the Schwartz inequality,

(3.11) $$\Big|\Big(\sum_{|\gamma|=0}^{i}(-1)^{|\gamma|}D^{(\gamma)}D^{(\gamma)}(\lambda I-A)u,\,u\Big)_0\Big|^2$$
$$\leqslant\Big(\sum_{|\gamma|=0}^{i}\int_{R^m}|D^{(\gamma)}(\lambda I-A)u(t)|^2dt\Big)\Big(\int_{R^m}|D^{(\gamma)}u(t)|^2dt\Big)$$
$$\leqslant\Big(\sum_{|\gamma|=0}^{i}\int_{R^m}|D^{(\gamma)}(\lambda I-A)u(t)|^2dt\Big)\Big(\sum_{|\gamma|=0}^{i}\int_{R^m}|D^{(\gamma)}u(t)|^2dt\Big)$$
$$=\|(\lambda I-A)u\|_i^2\|u\|_i^2.$$

Combining (3.10) and (3.11) gives

(3.12) $$\|(\lambda I-A)u\|_i\geqslant(\lambda-C_i)\|u\|_i\quad\text{whenever } u\in H.$$

Since the solution $u = u_f = u_{f,i} \in \bar{H}_{n+i} \cap C^\infty$ of (3.5) is approximated in $\|\cdot\|_{n+i}$-norm by a sequence of functions in H and since the norm $\|\cdot\|_{n+i}$ is larger than the norm $\|\cdot\|_i$, we obtain, passing to the limit,

$$\|u_f\|_i\leqslant(\lambda-\sigma_i)^{-1}\|f\|_i,\qquad \lambda\geqslant\sigma_i,$$

which concludes the proof.

3.13. Corollary. *The closure $\bar{A}_i$ of A_i in $\bar{H}_i$ possesses, for $\lambda>\sigma_i$, the resolvent $R(\lambda;\,\bar{A}_i)$ defined on $\bar{H}_i$ into $\bar{H}_i$ such that*

(3.14) $$\|[R(\lambda;\,\bar{A}_i)]^k\|_i\leqslant(\lambda-\sigma_i)^{-k},\qquad k=1,2,\ldots$$

Proof. Let $f\in H$. Now $(\lambda I-A)f\in H$. Set $(\lambda I-A)f=g$. Note that $g\in H$ is unique. If $\lambda>\sigma_i$, then as a consequence of this uniqueness and Lemma 3.4, the map $f\to g$ is one-one from H onto H. Thus if we set $f=R(\lambda)g$, then $R(\lambda)$ is a linear operator from H onto H. Now

(3.15) $$(\lambda I-A)R(\lambda)g=g\quad\text{for all } g\in H.$$

Furthermore, consider $R(\lambda)(\lambda I-A)g$, $g\in H$. We have just seen that $g\in H$ implies that there exists $h\in H$ such that $R(\lambda)h=g$. Hence we have $R(\lambda)(\lambda I-A)R(\lambda)h=R(\lambda)h=g$; that is,

(3.16) $$R(\lambda)(\lambda I-A)g=g\quad\text{for all } g\in H.$$

Now by Corollary 3.8

$$\|R(\lambda)g\|_i\leqslant(\lambda-\sigma_i)^{-1}\|g\|_i\quad\text{for all } g\in H.$$

Thus $R(\lambda)$ is a continuous linear operator on H_i onto H_i. Hence it is uniquely extensible to a continuous linear operator $\bar{R}_i(\lambda)$ on $\bar{H}_i$ into

$\bar{H}_i$ such that

(3.17) $$\|\bar{R}_i(\lambda)g\|_i\leqslant(\lambda-\sigma_i)^{-1}\|g\|_i\quad\text{for all } g\in\bar{H}_i.$$

By Lemma 3.2, $A_i=A$ is closable in $\bar{H}_i$. Its closure is denoted by $\bar{A}_i$. Since H_i is $\|\cdot\|_i$-dense in $\bar{H}_i$ and $\bar{R}_i(\lambda)$ is continuous on $\bar{H}_i$ we see that, for any $g\in\bar{H}_i$, there exists a sequence $\{g_k\}\subset H_i$ such that $\|g_k-g\|_i\to 0$ and $\|\bar{R}_i(\lambda)g_k-\bar{R}_i(\lambda)g\|_i\to 0$ as $k\to\infty$. Clearly $\{\bar{R}_i(\lambda)g_k\}\subset D(\bar{A}_i)$ and $\bar{A}_i$ being closed in $\bar{H}_i$, we have, for any $g\in\bar{H}_i$,

$$(\lambda I-\bar{A}_i)\bar{R}(\lambda)g=\lim_{k\to\infty}(\lambda I-\bar{A}_i)\bar{R}_i(\lambda)g_k=\lim_{k\to\infty}(\lambda I-A)R(\lambda)g_k=\lim_{k\to\infty}g_k=g$$

in the topology of $\bar{H}_i$ (consequence of (3.15)). Similarly,

$$\bar{R}_i(\lambda)(\lambda I-\bar{A}_i)g=g\quad\text{for all } g\in D(\bar{A}_i).$$

Thus $\bar{R}_i(\lambda)$ is the resolvent $R(\lambda;\,\bar{A}_i)$ of $\bar{A}_i$ over the space $\bar{H}_i$. Moreover, (3.17) gives

$$\|R(\lambda;\,\bar{A}_i)g\|_i\leqslant(\lambda-\sigma_i)^{-1}\|g\|_i\quad\text{for all } g\in\bar{H}_i \text{ and } \lambda>\sigma_i,$$

from where it follows that

$$\|[R(\lambda;\,\bar{A}_i)]^k\|_i\leqslant(\lambda-\sigma_i)^{-k}\quad\text{for all } \lambda>\sigma_i \text{ and } k=1,2,\ldots$$

This proves the corollary.

We have thus established (3.1)(ii). It follows that the differential operator $A\colon H\to H$, defined by (1.1), is the infinitesimal generator of an $L_{\mathscr{A}}(H)$-operator semi-group of class $(C_0, 1)$ and thus Theorem 1.4 is established.

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