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Weak type estimates for the Hardy–Littlewood maximal functions

by

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Abstract. In this paper we give sharp estimates for the weak type constants for the Maximal Operator of Differentiation. This is done in the case of one parameter m-dimensional parallelopipeds as a differentiation basis. The dependence on the parameter is asked to be more general than the usual monotonic one.

Introduction. The purpose of this paper is to improve and to extend results which have been obtained by Cotlar in [3] and [4]. These results are going to be used in [1].

1. Statement of results.

1.1. $R(x,t)$ will denote an m-dimensional rectangle having edges parallel to the coordinate axes, centered at the point x and edges given by $h_j(t)$, $j = 1, 2, \ldots, m$. Here $h_j(t)$ will denote the edge length corresponding to the x_j axis. The functions $h_j(t)$ are assumed to be continuous and non-negative and sytisfying the following conditions:

(1.1.1) $$t_1 \geqslant t_2 \Rightarrow k_j \cdot h_j(t_1) \geqslant h_j(t_2), \quad j = 1, 2, \ldots, m,$$

here k_j depends only on j and $k_j > 0$, $j = 1, 2, \ldots, m$,

(1.1.2) $$h_j(t) > 0, \quad t > 0; \quad h_j(0) = 0, \quad j = 1, 2, \ldots, m,$$

(1.1.3) $$h_j(t) \to \infty, \text{ as } t \to \infty \quad \text{for } j = 1, 2, \ldots, m.$$

1.2. By $f^*(x)$ we denote the maximal function

(1.2.1) $$\sup_{t>0} \left| \frac{1}{\mu(R(x,t))} \int_{R(x,t)} f\,d\mu \right|$$

where $R(x,t)$ are rectangles under the conditions of $(1,1)$, μ is a non-negative σ-additive measure defined on the Borel subsets of $\boldsymbol{R}^m$ and f is any μ-measurable and μ-locally integrable function. In the same way we define $\nu^*(x)$ for any σ-additive measure defined on the Borel subsets of $\boldsymbol{R}^m$, having bounded variation there

$$\sup_{t>0}\left|\frac{1}{\mu[R(x,t)]}\int_{R(x,t)}\nu[R(x,t)]\right|. \tag{1.2.2}$$

THEOREM 1. *Under the conditions of* (1.2), *the maximal function satisfies the following inequalities:*

i) *if* $f\in L^1_\mu(\boldsymbol{R}^m)$ *then*

$$\mu\{E(f^*>\lambda)\}<\frac{2^m m!\prod\limits_{j=1}[2+\log_2(k_j+1)]}{\lambda}\int_{\boldsymbol{R}^m}|f|\,d\mu,$$

ii) *if the total variation* $V(\nu)(\boldsymbol{R}^m)$ *is bounded then*

$$\mu\{E(\nu^*>\lambda)\}<\frac{2^m m!\prod\limits_{j=1}^{m}[2+\log_2(k_j+1)]}{\lambda}V(\nu)(\boldsymbol{R}^m),$$

where the k_j *are the constants defined in* (1.1.1) *and* $\log_2 S$ *denotes the logarithm on basis* 2. *Here the symbol* $E(f>\lambda)$ *denotes the set of points* x *where* $f(x)>\lambda$.

2. The basic lemmas.

2.1. LEMMA 1. *Let* Q_j; $j=1,2,\ldots,k$ *be a family of cubes in* $\boldsymbol{R}^m$ *having edges parallel to the coordinate axes. Suppose that for* $i>j$ *the center of* Q_i *does not belong to* Q_j *and* $k\cdot l_j>l_i$ $(k\geqslant 2)$, *where* l_i *and* l_j *denote respectively the lengths of the edges of* Q_i *and* Q_j.

Then every $x\in\boldsymbol{R}^m$ *belongs to at most*

$$2^m\big[(2^m-1)\big(1+\log_2(k)\big)\big]+2^m$$

different cubes.

2.2. LEMMA 2. *Let* $\{R_j\}$; $j=1,2,\ldots,k,\ldots$ *be a family of* m*-dimensional rectangles having edges parallel to the coordinate axes. Suppose that for* $i>j$ *the center of* $\boldsymbol{R}_i$ *does not belong to* $\boldsymbol{R}_j$ *and there exist constants* $k_1,k_2,\ldots,k_m$ *such that:*

$$k_s\cdot l_j^s>l_i^s,\quad s=1,2,\ldots,m,$$

where l_i^s *denotes the length of the edge corresponding to the* x_s *axis of the rectangle* R_i.

Then, every x *in* $\boldsymbol{R}^m$ *belongs to at most*

$$2^m m!\prod_{j=1}^{m}[1+\log_2(k_j+1)]$$

different rectangles.

2.3. LEMMA 3. *Let* S *be a bounded set in* $\boldsymbol{R}^m$. *Suppose that for each* $x\in S$ *there exists a non-degenerate rectangle* $\boldsymbol{R}(x)$ *centered at* x *under the conditions of* (1.1).

Then, there exists a denumerable family subfamily $\{\boldsymbol{R}(x_n)\}$ *such that:*

i) $\bigcup\limits_1^\infty \boldsymbol{R}(x_n)\supset S$.

ii) *Each* $x\in\boldsymbol{R}^m$ *belongs to at most*

$$2^m m!\prod_{j=1}^{m}[2+\log_2(1+k_j)]$$

different rectangles.

3. Proof of Lemma 1. Let x_0 be a point belonging to the cubes $Q_{d_1},\ldots,Q_{d_n}$. Without loss of generality we may assume that x_0 is the origin, and in order to simplify notation let us write Q_j instead of Q_{d_j}. Among the Q_j consider the subset of cubes $Q_{j_1},Q_{j_2},\ldots$ such that their centers are located in $\boldsymbol{R}^m_+$. Consider the length l_{j_1} of the first cube whose center lies on $\boldsymbol{R}^m_+$ and let us introduce an auxiliary cube Q_0 defined by:

$$0\leqslant x_s\leqslant \tfrac{1}{2}l_{j_1},\quad s=1,2,\ldots,m. \tag{3.1.1}$$

Then, clearly $Q_0\subset Q_{j_1}$ and since no center of Q_{j_s}, $s>1$ is contained in Q_{j_1} there is at most one center in Q_0. Consider now the set of 2^m-1 cubes adjacent to Q_0 with edges of length $\frac{1}{2}l_{j_1}$ and lying in $\boldsymbol{R}^m_+$.

In each of the adjacent cubes there is at most one center. In fact, suppose that in one of the cubes there are two centers C_{j_s} and C_{j_t}. Then, for some i, $1\leqslant i\leqslant m$

$$x_i[C_{j_s}]\geqslant \tfrac{1}{2}l_{j_1},$$
$$x_i[C_{j_t}]\geqslant \tfrac{1}{2}l_{j_1}.$$

Where $x_i[C_{j_s}]$ designates the ith coordinate of C_{j_s}. So, l_{j_s} and l_{j_t} are bigger than l_{j_1} and since

$$|x_r[C_{j_s}]-x_r[C_{j_t}]|\leqslant \tfrac{1}{2}l_{j_1}$$

for $r=1,2,\ldots,m$; it means that C_{j_s} lies in Q_{j_t} and conversely, which is a contradiction. Consider now the auxiliary cube Q_1 defined by the inequalities:

$$0\leqslant x_s\leqslant l_{j_1},\ 1\leqslant s\leqslant m.$$

According to the preceding reasoning at most $(2^m-1)+1$ centers are in Q_1.

Now we shall be concerned with the family of cubes with centers not on Q_1. Consider now the 2^m-1 cubes adjacent to Q_1 such that their edges have length l_{j_1}. As in the preceding reasoning there is at most one center in each adjacent cube. Notice that here the role of Q_0 is played by Q_1.

Call Q_2 the cube defined by

$$0 \leqslant x_i \leqslant 2 l_{j_1}, \quad i = 1, 2, \ldots, m. \tag{3.1.2}$$

According to the preceding reasoning there are at most $1+(2^m-1)+ +(2^m-1)$ centers in Q_2; that is $1+(2^m-1)$ in Q_1 and (2^m-1) in Q_2-Q_1. As before, we consider the 2^m-1 cubes adjacent to Q_2 having edge length $2l_{j_1}$ and contained in $\boldsymbol{R}_+^m$. Again, each one of the adjacent cubes contains at most one center.

The induction is now clear; we have constructed $Q_0, Q_1, \ldots, Q_r$, here Q_s is defined by

$$\{x/0 \leqslant x_i \leqslant 2^{s-1} l_{j_1};\ 1 \leqslant i \leqslant m\}.$$

Q_s contains Q_{s-1} and its 2^m-1 adjacent cubes having the same size. So, there are at most $1+s(2^m-1)$ centers in Q_s. Recalling the fact that $l_{j_r} < k \cdot l_{j_1}$, every center lies in the cube $\bar{Q}$ defined by:

$$0 < x_s < k \cdot \tfrac{1}{2} l_{j_1}.$$

Then, if $2^{s-1} l_{j_1} > k \cdot \frac{1}{2} l_{j_1}$ (that is, $s > \log_2 k$), Q_s contains $\bar{Q}$. It can be readily seen that in $\boldsymbol{R}_+^m$ there are at most $1+(1+\log_2 k)(2^m-1)$ cubes containing $0 = x_0$. By a symmetry argument in $\boldsymbol{R}^m$ there are at most

$$2^m[1+(1+\log_2 k)(2^m-1)].$$

This finishes the proof.

4. Proof of Lemma 2. For $m = 1$ is the preceding lemma. We shall prove it for $m = 2$ a typical case.

As in the preceding lemma, let x_0 be a point belonging to $R_{d_1}, R_{d_2}, R_{d_j}, \ldots$

We may assume that x_0 is the origin and that the centers of $R_{d_1}, R_{d_2}, \ldots, R_{d_j}, \ldots$ are in $\boldsymbol{R}_+^m$. Consider now the rectangle R_0 defined by

$$0 \leqslant x \leqslant \tfrac{1}{2} l_x(R_{d_1}), \quad 0 \leqslant y \leqslant \tfrac{1}{2} l_y(R_{d_1}). \tag{4.1.1}$$

Clearly $R_0 \subset R_{d_1}$ and there is at most one center in it (that of R_{d_1}). We are going to define the following two families of strips. For the y-axis we have the following strips:

$$S_r = \{(x, y);\ 2^{r-2} l_y[R_{d_1}] < y \leqslant 2^{r-1} l_y[R_{d_1}],\ x \geqslant 0\}.$$

For the x-axis

$$T_t = \{(x, y);\ 2^{t-2} l_x[R_{d_1}] < x \leqslant 2^{t-1} l_x[R_{d_1}],\ y \geqslant 0\}.$$

Every center belongs to $\bar{R}$ which is defined in the following way:

$$\bar{R} = \{(x, y);\ 0 \leqslant x \leqslant k_1 \cdot \tfrac{1}{2} l_x[R_{d_1}],\ 0 \leqslant y \leqslant k_2 \cdot \tfrac{1}{2} l_y[R_{d_1}]\}. \tag{4.1.2}$$

On the other hand:

$$\bar{R} \subset \Big(\bigcup_1^{r_0} S_r\Big) \cup \Big(\bigcup_1^{t_0} T_t\Big) \cup R_0 \quad \text{if } r_0 \geqslant \log_2 k_2,\ t_0 \geqslant \log_2 k_1. \tag{4.1.3}$$

In particular, we may choose r_0 and t_0 to be the first integers satisfying the inequalities (4.1.3); therefore, it will be valid also that:

$$r_0 \leqslant 1+\log_2 k_2, \quad t_0 \leqslant 1+\log_2 k_1. \tag{4.1.4}$$

We are going to evaluate how many centers there are in each strip. Consider the strip S_r and the rectangles R'_{d_i} with centers in S_r; then

$$\tfrac{1}{2} l_y[R'_{d_i}] \geqslant 2^{r-2} l_y[R_{d_1}]. \tag{4.1.5}$$

Now, if we cut the rectangles with the line $y = 2^{r-2} l_y[R_{s_1}]$, we have a family of segments I'_{d_i} satisfying the following conditions:

1) $(0, 2^{r-2} l_y[R_{d_1}])$ belongs to I'_{d_i} for all d_i,
2) if $d_i < d_j$, then $k_1 \cdot \text{length}\ \{I'_{d_i}\} \geqslant \text{length}\ \{I'_{d_j}\}$,
3) if $d_i < d_j$, then the center of I'_{d_j} does not belong to I'_{d_i}.

Conditions 1) and 2) can be readily checked. Suppose 3) fails for I'_{d_i}, I'_{d_j}, $d_i < d_j$; then it means

$$|x[C_{d_i}] - x[C_{d_j}]| \leqslant \tfrac{1}{2} l_x(R'_{d_i}); \tag{4.1.6}$$

on the other hand,

$$|y[C_{d_j}] - y[C_{d_i}]| \leqslant 2^{r-2} l_y[R_{d_1}] < \tfrac{1}{2} l_y[R'_{d_i}]. \tag{4.1.7}$$

But (4.1.6) and (4.1.7) say that R'_{d_i} contains C_{d_j}, contrary to the hypothesis.

For $m = 1$ the lemma is valid, so in the strip S_r there are at most $(1+\log_2 k_1)$ centers.

In the union $\bigcup_1^{r_0} S_r$ there are at most $(1+\log_2 k_2)(1+\log_2 k_1)$ centers. By using the same argument, we see that in the union $\bigcup_1^{t_0} T_t$ there are at most $(1+\log_2 k_1)(1+\log_2 k_2)$ centers. In $\boldsymbol{R}_+^2$ there are at most

$$2[\log_2 k_2+1][\log_2 k_1+1] \quad \text{centers}.$$

Therefore in $\boldsymbol{R}^2$ there are at most

$$2^2\Big[2!\prod_{j=1}^{2}[1+\log_2 k_j]+1\Big] \quad \text{centers}.$$

Hence it has been shown that the case $m = 1$ implies the case $m = 2$. In general the case $m-1$ implies the case m. This is done by defining the following "strips":

$$(4.1.8)\qquad S_r^{x_j} = \{(x_1, x_2, \dots, x_m);\ 2^{r-2} l_j(R_{d_1}) < x_j \leqslant 2^{r-1} l_j(R_{d_1}),\ x_k \geqslant 0,\ k = 1, 2, \dots, m\}.$$

Cutting the rectangles centered in the strip by the hyperplane $x_j = 2^{r-2} l_j(R_{d_1})$. We reduce the m-dimensional case to the $(m-1)$-dimensional one and everything follows as in the 2-dimensional case.

5. Proof of Lemma 3.

5.1. Consider $t_1^* = \sup\{t_x\}$ where $R(t_x)$ is the rectangle associated with $x \in S$. If $t_1^* = \infty$, we can choose a rectangle so big that S is contained in it and we stop the process. If $t_1^* < \infty$, then it is possible to find a point x_1 such that the rectangle associated with it, $R(t_{x_1})$, satisfies:

$$(5.1.1)\qquad 2l_j(t_{x_1}) > l_j(t_1^*), \quad j = 1, 2, \dots, m.$$

If $R(t_{x_1})$ covers S we stop the process. Suppose that $R(t_{x_1})$ does not cover S. We shall consider then the sub-family of rectangles with centers outside of $R(t_{x_1})$. t_2^* will be the $\sup\{t_x\}$ such that $x \notin R(t_{x_1})$. Clearly $t_2^* \leqslant t_1^*$; t_{x_2} will be chosen so that

$$(5.1.2)\qquad 2l_j(t_{x_2}) > l_j(t_2^*),\ j = 1, 2, \dots m.$$

If $R(t_{x_1}) \cup R(t_{x_2}) \supset S$ we stop the process. If it is not so, we continue.

If after a finite number of steps $\bigcup_1^n R(t_{x_s}) \supset S$ then

1) if $d > b$ the center of $R(t_{x_d})$ is not contained in $R(t_{x_b})$,

2) $t_{x_d} \leqslant t_b^*$; thus $l_j(t_{x_d}) \leqslant k_j l_j(t_b^*) \leqslant 2k_j l_j(t_{x_b})$ for $j = 1, 2, \dots, m$,

then we are under the conditions of Lemma 2, except for the fact that the values of the constants are $2k_j$ instead of k_j. In this case Lemma 3 is already proved.

If the process has a denumerable set of steps, that is, we have $\bigcup_1^\infty R(t_{x_s})$, then in this case, we have already the properties 1) and 2) of the preceding discussion and in order to prove the lemma, it remains to be shown that:

$$(5.1.3)\qquad S \subset \bigcup_1^\infty R(t_{x_s}).$$

Consider the family $\{\bar{R}(t_{x_s})\}$ obtained from the family $\{R(t_{x_s})\}$ by contracting each $R(t_{x_s})$ about its center in such a way that the length $l_j(t_{x_s})$ is transformed into $\frac{1}{4k_j} l_j(t_{x_s})$, $j = 1, 2, \dots m$ from properties 1) and 2) we have that the $\bar{R}(t_{x_s})$ are pairwise disjoint and since they are bounded uniformly and S is bounded, it follows that:

$$(5.1.4)\qquad \sum_1^\infty |\bar{R}(t_{x_s})| < \infty.$$

This shows that $t_{x_s} \to 0$, since $h_j(t) > 0$ for $t > 0$ and they are continuous. Therefore, $t_s^* \to 0$ also. (Recall that $2h_j(t_{x_s}) \geqslant h_j(t_s^*)$.) Suppose that there exists a point x such that:

$$x \notin \bigcup_1^\infty R(t_{x_s}).$$

Then $t_x < t_n^*$ for all n, therefore $t_x = 0$ which is a contradiction since we have assumed that the rectangle associated with each point is not degenerate. This finishes the proof of the lemma.

6. Lemma 3 is the key for the proof of Theorem 1, which follows by using a standard type of argument (see for example [2], Part I, p. 126, Lemmas 1.9 and 1.10).

Remark. Lemma 1 is a sharpened version of the corresponding lemma proved by Cotlar (see [3], p. 59 and 60). Lemma 2 is a sharpened generalization of the same lemma to the case of rectangles.

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