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Algebraic theory of right invertible operators

by

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Abstract. A notion of an initial operator for a right invertible operator acting in a linear space is introduced. Using this notion a Taylor Formula for right invertible operators is proved and the definition of definite integrals is given. Furthermore, initial value problems and mixed boundary value problems for equations with right invertible operators are solved. There are given applications to ordinary and partial differential equations (in particular to hyperbolic equations) and to difference equations, all of them with variable coefficients.

In a series of papers ([1], [3]–[6]) the author has given properties and applications of equations with an algebraic derivative defined in [1]. However, all the results obtained follow only from the fact, that each algebraic derivative (in sense of [1]) is a right invertible operator. We therefore give in the present paper an algebraic theory of right invertible operators acting in a linear space. This theory generalizes the results of the previous papers [2]–[6].

1. Notations and definitions. Polynomials in right invertible operators. Let X be a linear space (over an algebraically closed field $\mathscr{F}$ (1) of scalars). We consider a linear (i. e. additive and homogeneous) operator A defined in a linear subset $\mathscr{D}_A \subset X$, called the domain of A, and mapping $\mathscr{D}_A$ into X. We denote by $L(X)$ the collection of all such operators. Z_A will stand for the kernel of A, i. e. $Z_A = \{x \in \mathscr{D}_A:\ Ax = 0\}$.

DEFINITION 1. An operator $D \in L(X)$ is said to be *right invertible* if there exists an operator $R \in L(X)$ such that

(1) $RX \subset \mathscr{D}_D$ and $\mathscr{D}_R = X$,

(2) $DR = I$, where I denotes the identity operator.

The operator R is called a *right inverse* of D. The set of all right invertible operators belonging to $L(X)$ will be denoted by $\boldsymbol{R}(X)$.

DEFINITION 2. An operator $A \in L(X)$ is said to be a *Volterra operator*

(1) The assumption that F is algebraically closed is necessary only if we consider equations with scalar coefficients.

if the operator $I-\lambda A$ is invertible for every $\lambda\in\mathscr{F}$. The set of all Volterra operators belonging to $L(X)$ will be denoted by $\boldsymbol{V}(X)$.

DEFINITION 3. An operator $D\in\boldsymbol{R}(X)$ is said to be *V-right invertible* if its right inverse R is a Volterra operator. We then say that R is a V-right inverse of D. The set of all V-right invertible operators belonging to $L(X)$ will be denoted by $\boldsymbol{VR}(X)$.

Let $D\in\boldsymbol{R}(X)$. The kernel Z_D is called the *space of constants* for D and every element $z\in Z_D$ is called a *constant*. We also write $a_D=\dim Z_D$.

The proofs of the following properties can be found either in [2] or in [3], Chapter IV.

PROPERTY 1.1. *If R is a right inverse of $D\in\boldsymbol{R}(X)$, then $D^kR^k=I$ for $k=1,2,\ldots$*

PROPERTY 1.2. *If R is a right inverse of $D\in\boldsymbol{R}(X)$, m is an arbitrary positive integer and $z_k\in Z_D$, $z_k\neq 0$ $(k=0,1,\ldots,m)$, then the elements z_0, $Rz_1,\ldots,R^mz_m$ are linearly independent.*

PROPERTY 1.3. *If R is a right inverse of $D\in\boldsymbol{R}(X)$ and elements $z_1,\ldots,z_m\in Z_D$ are linearly independent, then the elements $R^kz_1,\ldots,R^kz_m$ are linearly independent for $k=1,2,\ldots$*

PROPERTY 1.4. *If R is a right inverse of $D\in\boldsymbol{R}(X)$ and N is arbitrary positive integer, then*

$$Z_{D^N}=\Big\{z\in X:\ z=\sum_{k=0}^{N-1}R^kz_k,\ z_k\in Z_D\Big\}.$$

In the sequel we shall use the following notations: Let R be a right inverse of an operator $D\in\boldsymbol{R}(X)$. Let

$$Q(D)=\sum_{k=0}^{N}Q_kD^k,\quad \text{where } Q_k\in L(X)\ (k=0,1,\ldots,N-1) \text{ and } Q_N=I. \tag{1.1}$$

We write

$$Q_\lambda^*(R)=(Q_0-\lambda I)R^N+\sum_{k=1}^{N}Q_kR^{N-k}\quad \text{for } \lambda\in\mathscr{F}. \tag{1.2}$$

THEOREM 1.1. *Suppose that R is a right inverse of an operator $D\in\boldsymbol{R}(X)$. Then*

(1) *If $Q_0^*(R)$ is invertible, then $Q(D)\in\boldsymbol{R}(X)$ and its right inverse is*

$$\hat{R}=R^N[Q_0^*(R)]^{-1}. \tag{1.3}$$

(2) *If $Q_\lambda^*(R)$ is invertible for every $\lambda\in\mathscr{F}$, then $\hat{R}\in\boldsymbol{V}(X)$, so that $Q(D)\in\boldsymbol{VR}(X)$.*

(3) *If $Q_0^*(R)$ is invertible, then all solutions of the equation*

$$Q(D)x=y,\quad y\in X \tag{1.4}$$

are of the form

$$x=\hat{R}\Big[y-\sum_{m=0}^{N-1}Q_m\sum_{k=m}^{N-1}R^{k-m}z_k\Big]+\sum_{k=0}^{N-1}R^kz_k,$$

where $z_k\in Z_D$ are arbitrary and $\hat{R}$ is defined by Formula (1.3).

Proof. (1). By assumption we have from Property 1.1.

$$Q(D)\hat{R}=Q(D)R^N[Q_0^*(R)]^{-1}=\Big(\sum_{k=0}^{N}Q_kD^kR^N\Big)[Q_0^*(R)]^{-1}$$
$$=\Big(\sum_{k=0}^{N}Q_kR^{N-k}\Big)Q_0^*(R)^{-1}=[Q_0^*(R)][Q_0^*(R)]^{-1}=I,$$

which means that $Q(D)\in\boldsymbol{R}(X)$ and its right inverse is $\hat{R}$.

(2). Since the operator $Q_\lambda^*(R)$ is invertible for $\lambda\in\mathscr{F}$, then

$$I-\lambda\hat{R}=I-\lambda R^N[Q_0^*(R)]^{-1}=[Q_0^*(R)-\lambda R^N][Q_0^*(R)]^{-1}$$
$$=\Big(\sum_{k=0}^{N}Q_kR^{N-k}-\lambda R^N\Big)[Q_0^*(R)]^{-1}=Q_\lambda^*(R)[Q_0^*(R)]^{-1}$$

is invertible for $\lambda\in\mathscr{F}$. This implies that $\hat{R}\in\boldsymbol{V}(X)$, and, by conclusion (1) of this theorem, $Q(D)\in\boldsymbol{VR}(X)$.

(3). Properties 1.1 and 1.4 together imply that all solutions of the equation $D^Nx=[Q_0^*(R)]^{-1}u$ are of the form

$$x=R^N[Q_0^*(R)]^{-1}u+\sum_{k=0}^{N-1}R^kz_k,\quad \text{where } z_k\in Z_D \text{ are arbitrary}.$$

Thus, by conclusion (1) of this Theorem

$$Q(D)x=$$
$$=Q(D)\Big\{R^N[Q_0^*(R)]^{-1}u+\sum_{k=0}^{N-1}R^kz_k\Big\}=u+Q(D)\sum_{k=0}^{N-1}R^kz_k$$
$$=u+\sum_{m=0}^{N}Q_mD^m\sum_{k=0}^{N-1}R^kz_k=u+\sum_{m=0}^{N-1}Q_m\Big(\sum_{k=0}^{m-1}D^{m-k}z_k+\sum_{k=m}^{N-1}R^{k-m}z_k\Big)+$$
$$+Q_ND^N\sum_{k=0}^{N-1}R^kz_k=u+\sum_{m=0}^{N-1}Q_m\sum_{k=m}^{N-1}R^{k-m}z_k+\sum_{k=0}^{N-1}D^{N-k}z_k$$
$$=u+\sum_{m=0}^{N-1}Q_m\sum_{k=m}^{N-1}R^{k-m}z_k.$$

Hence, from Equation (1.4), we conclude that

$$u = y - \sum_{m=0}^{N-1} Q_m \sum_{k=m}^{N-1} R^{k-m} z_k,$$

which gives the required form of x.

THEOREM 1.2. *Suppose that R is a V-right inverse of an operator $D \in \boldsymbol{VR}(X)$ and that $Q_k = q_k I$, where $q_k \in \mathscr{F}$ $(k = 0, 1, \ldots, N-1)$. Then*

(1) *$Q_0^*(R)$ is invertible.*

(2) *$Q(D)[Q_0^*(R)]^{-1} = D^N$.*

(3) *$Q(D) \in \boldsymbol{R}(X)$ and its right inverse is*

$$(1.5)\qquad [Q_0^*(R^N)]^{-1}R^N = R^N[Q_0^*(R)]^{-1} = \hat{R}.$$

(4) *$\hat{R} \in \boldsymbol{V}(X)$, hence $Q(D) \in \boldsymbol{VR}(X)$.*

(5) *$\alpha_{Q(D)} = N\alpha_D$ and*

$$Z_{Q(D)} = [Q_0^*(R)]^{-1} Z_{D^N} = \Big\{ z \in X \colon z = [Q_0^*(R)]^{-1} \sum_{k=0}^{N-1} R^k z_k,\ z_k \in Z_D \Big\}.$$

(6) *All solutions of Equation (1.5) are of the form*

$$x = [Q_0^*(R)]^{-1}\Big[R^N y_y + \sum_{k=0}^{N-1} R^k z_k\Big], \quad \text{where } z_k \in Z_D \text{ are arbitrary}.$$

Proof. The proofs of (1), (2), (5), (6) can be found either in [2] or in [3], Chapter IV. Having the identity (2) already proved, we act on both sides of this identity by the operator R^N and from Property 1.1 we obtain

$$Q(D)[Q_0^*(R)]^{-1}R^N = D^N R^N = I,$$

which implies that $Q(D) \in R(X)$. Since $Q_0^*(R)$ is a polynomial with scalar coefficients, we conlude that

$$[Q_0^*(R)]^{-1}R^N = R^N[Q^*(R)]^{-1} = \hat{R}$$

and that $\hat{R}$ is a right inverse of D. In [3], Chapter IV we have proved that the operator $I - \lambda\hat{R}$ is invertible for every $\lambda \in \mathscr{F}$, so that $\hat{R} \in \boldsymbol{V}(X)$ and $Q(D) \in \boldsymbol{VR}(X)$.

PROPOSITION 1.1 (cf. Mażbic-Kulma, [3], Chapter IV). *If $D_1, \ldots, D_m \in \boldsymbol{VR}(X)$ and the operator $R = R_m^{k_m} \ldots R_1^{k_1} \in \boldsymbol{V}(X)$ for some positive integers $k_1, \ldots, k_m$, where R_j denotes a V-right inverse of D_j $(j = 1, \ldots, m)$, then the operator $D = D_1^{k_1} \ldots D_m^{k_m}$ is V-right invertible having R as a V-right inverse.*

Indeed, $DR = D_1^{k_1} \ldots D_m^{k_m} R_m^{k_m} \ldots R_1^{k_1} = I$ and by assumption $R \in \boldsymbol{V}(X)$.

2. Calculus of right invertible operators. In the sequel the following notion will play a fundamental role.

DEFINITION 4. An operator $F \in L(X)$ is said to be an *initial operator* for an operator $D \in \boldsymbol{R}(X)$ corresponding to a right inverse R of D if

(i) $FX = Z_D$, $F^2 = F$,

(ii) $FR = 0$ on X.

This definition immediately implies that

$$(2.1)\qquad DF = 0 \quad \text{on } X,$$

$$(2.2)\qquad Z_D \cap Z_F = \{0\}.$$

THEOREM 2.1. *Let R be a right inverse of $D \in \boldsymbol{R}(X)$. Then $F \in L(X)$ is an initial operator for D (corresponding to R) if and only if the following identity*

$$(2.3)\qquad F = I - RD$$

holds on $\mathscr{D}_D$.

Proof. Necessity. Let F be an initial operator for $D \in R(X)$ corresponding to a right inverse R of D. Let $x \in \mathscr{D}_D$ and put $u = RDx$. Then by definition, $Du = D(RD)x = (DR)Dx = Dx$. Hence $D(u-x) = Du - Dx = 0$, which implies $u - x = z \in Z_D$. On the other hand, since $FR = 0$, we have $Fu = FRDx = 0$. Then, since $z = u - x \in Z_D$, we have $Fz = z$ and

$$(I-RD)x - RDx = x - u = -z = -Fz$$
$$= -F(u-x) = -Fu + Fx = Fx,$$

which implies Formula (2.3).

Sufficiency. Let $F = I - RD$ on $\mathscr{D}_D$. Then $F^2 = (I-RD)^2 = I - RD - RD + RDRD = I - 2RD + RD = I - RD = F$. We have to show that $FX = Z_D$. Indeed, $DF = D(I-RD) = D - DRD = D - D = 0$. Hence $FX \subset Z_D$. Moreover, if $z \in Z_D$, then $Dz = 0$ and $Fz = z - RDz = z$. Thus F maps X onto Z_D. Finally, we have $FR = (I-RD)R = R - RDR = R - R = 0$. This proves that F is an initial operator for D corresponding to R.

However, in applications, to any right invertible operator D there corresponds a family $\{R_\gamma\}_{\gamma\in\Gamma}$ of its right inverses. This fact and Theorem 2.1 together imply

THEOREM 2.2. *Let $\{R_\gamma\}_{\gamma\in\Gamma}$ be a family of right inverses of an operator $D \in \boldsymbol{R}(X)$. Then this family induces uniquely a family of initial operators $\{F_\gamma\}_{\gamma\in\Gamma}$ defined by*

$$(2.4)\qquad F_\gamma = I - R_\gamma D \quad \text{on } \mathscr{D}_D \quad (\gamma \in \Gamma).$$

THEOREM 2.3 (The Taylor Formula). *Let $\{R_\gamma\}_{\gamma\in\Gamma}$ be a family of right inverses of an operator $D\in\mathbf{R}(X)$. Let $\{F_\gamma\}_{\gamma\in\Gamma}$ be the family of initial operators induced by $\{R_\gamma\}_{\gamma\in\Gamma}$. Let $\{\gamma_n\}\subset\Gamma$ be an arbitrary sequence. Then for $N=1,2,\ldots$ the following identity holds on $\mathscr{D}_{D^N}$:*

$$I = F_{\gamma_0}+\sum_{k=1}^{N-1} R_{\gamma_0}\ldots R_{\gamma_{k-1}}D^k+R_{\gamma_0}\ldots R_{\gamma_{N-1}}D^N. \tag{2.5}$$

Proof, by induction. For $N=1$ Formula (2.4) implies $I=F_{\gamma_0}+R_{\gamma_0}D$. Suppose that Formula (2.5) is true for an arbitrary N. Then on $\mathscr{D}_{D^{N+1}}$ we have

$$\begin{aligned}R_{\gamma_0}\ldots R_{\gamma_N}D^{N+1} &= R_{\gamma_0}\ldots R_{\gamma_{N-1}}(R_{\gamma_N}D)D^N = R_{\gamma_0}\ldots R_{\gamma_{N-1}}(I-F_{\gamma_N})D^N\\ &= R_{\gamma_0}\ldots R_{\gamma_{N-1}}D^N-R_{\gamma_0}\ldots R_{\gamma_{N-1}}F_{\gamma_N}D^N\\ &= I-F_{\gamma_0}-\sum_{k=1}^{N-1}R_{\gamma_0}\ldots R_{\gamma_{k-1}}F_{\gamma_k}D^k-R_{\gamma_0}\ldots R_{\gamma_{N-1}}F_{\gamma_N}D^N\\ &= I-F_{\gamma_0}-\sum_{k=1}^{N}R_{\gamma_0}\ldots R_{\gamma_{k-1}}F_{\gamma_k}D^k,\end{aligned}$$

which was to be proved.

Putting $R_{\gamma_n}=R$ and $F_{\gamma_n}=F$ we obtain

COROLLARY 2.1 (Taylor Formula). *Let R be a right inverse of an operator $D\in\mathbf{R}(X)$. Then for $N=1,2,\ldots$*

$$I=\sum_{k=0}^{N-1}R^kFD^k+R^ND^N \quad \text{on } \mathscr{D}_{D^N}, \tag{2.6}$$

where F is the initial operator for D corresponding to R.

COROLLARY 2.2. *Suppose that the assumptions of Theorem 2.3 are satisfied. Then*

$$Z_{D^N}=\left\{z\in X:\ z=z_0+\sum_{k=1}^{N-1}R_{\gamma_0}\ldots R_{\gamma_{k-1}}z_k,\ z_k\in Z_D\right\}\quad (N=1,2,\ldots).$$

Proof. Let $z=z_0+\sum_{k=1}^{N-1}R_{\gamma_0}\ldots R_{\gamma_{k-1}}z_k$, where $z_0,\ldots,z_{N-1}\in Z_D$ are arbitrary. Then

$$D^Nz=D^Nz_0+\sum_{k=1}^{N-1}D^NR_{\gamma_0}\ldots R_{\gamma_{k-1}}z_k=\sum_{k=1}^{N-1}D^{N-k}z_k=0.$$

Hence $z\in Z_{D^N}$.

Conversely, let $z\in Z_{D^N}$. By Taylor-Gontcharov Formula (2.5), since $D^Nz=0$, we have

$$z=F_{\gamma_0}z+\sum_{k=1}^{N-1}R_{\gamma_0}\ldots R_{\gamma_{k-1}}F_{\gamma_k}D^kz.$$

Writing $z_k=F_{\gamma_k}D^kz$ ($k=0,1,\ldots,N-1$) we conclude that $z_k\in Z_D$ and that z is of the required form.

We assume, as previously that $\{R_\gamma\}_{\gamma\in\Gamma}$ is a family of right inverses of an operator $D\in\mathbf{R}(X)$ and that $\{F_\gamma\}_{\gamma\in\Gamma}$ is the corresponding family of initial operators. We have

$$F_\alpha F_\beta=F_\beta \quad \text{for all } \alpha,\beta\in\Gamma, \tag{2.7}$$

$$F_\beta R_\alpha=R_\alpha-R_\beta \quad \text{for all } \alpha,\beta\in\Gamma. \tag{2.8}$$

Indeed, since $DF_\beta=0$, hence $F_\alpha F_\beta=(I-R_\alpha D)F_\beta=F_\beta-R_\alpha DF=F_\beta$. Moreover,

$$F_\beta R_\alpha=(I-R_\beta D)R_\alpha=R_\alpha-R_\beta(DR_\alpha)=R_\alpha-R_\beta.$$

PROPOSITION 2.1. *For an arbitrary $x\in X$*

$$R_\alpha x-R_\beta x=z,\quad \textit{where } z \textit{ is a constant } (\alpha,\beta\in\Gamma).$$

Indeed, by Formula (2.8), $Dz=D(R_\alpha x-R_\beta x)=DF_\beta R_\alpha x=0$. Hence $z\in Z_D$.

PROPOSITION 2.2. *The operator $F_\beta R_\gamma-F_\alpha R_\gamma$ does not depend of the choice of R_γ ($\alpha,\beta,\gamma\in\Gamma$) and is equal to $F_\beta R_\alpha$.*

Indeed, from (2.8) we have

$$F_\beta R_\gamma-F_\alpha R_\gamma=R_\gamma-R_\beta-(R_\gamma-R_\alpha)=R_\alpha-R_\beta=F_\beta R_\alpha.$$

By Proposition 2.2 we can write

$$I_\alpha^\beta=F_\beta R_\gamma-F_\alpha R_\gamma\quad(\alpha,\beta,\gamma\in\Gamma). \tag{2.9}$$

The proof of Proposition 2.2 immediately implies

$$I_\alpha^\beta=F_\beta R_\alpha\quad(\alpha,\beta\in\Gamma). \tag{2.10}$$

PROPERTY 2.1. *For all $\alpha,\beta\in\Gamma$ and $x\in X$*

$$I_\alpha^\beta x=z,\quad \textit{where } z \textit{ is a constant.}$$

Indeed, Formula (2.10) implies that $Dz=DI_\alpha^\beta x=DF_\beta R_\alpha x=0$. Hence $z\in Z_D$.

PROPERTY 2.2. *For all α,β*

$$I_\alpha^\beta=-I_\beta^\alpha. \tag{2.11}$$

Indeed, $I_\alpha^\beta+I_\beta^\alpha=F_\beta R_\gamma-F_\alpha R_\gamma+F_\alpha R_\gamma-F_\beta R_\gamma=0$.

PROPERTY 2.3. *For all* $\alpha, \beta, \delta \in \Gamma$

$$I_\alpha^\delta + I_\delta^\beta = I_\alpha^\beta. \tag{2.12}$$

Indeed,

$$I_\alpha^\delta + I_\delta^\beta = F_\delta R_\gamma - F_\alpha R_\gamma + F_\beta R_\gamma - F_\delta R_\gamma = F_\beta R_\gamma - F_\alpha R_\gamma = I_\alpha^\beta.$$

PROPERTY 2.4. *For all* $\alpha, \beta \in \Gamma$

$$I_\alpha^\beta D = F_\beta - F_\alpha. \tag{2.13}$$

Indeed, Formulae (2.4), (2.7) and (2.10) together imply

$$I_\alpha^\beta D = F_\beta R_\alpha D = F_\beta(I - F_\alpha) = F_\beta - F_\beta F_\alpha = F_\beta - F_\alpha.$$

In view of Propositions 2.1 and 2.2 and Properties 2.1–2.4 we can call the operator I_α^β – the *definite integral*. The family $\{R_\gamma\}_{\gamma \in \Gamma}$ plays the role of an *indefinite integral*. If $x \in X$ and $\gamma \in \Gamma$, then the element $R_\gamma x$ is said to be *primitive for* x, since $D(R_\gamma x) = x$. We therefore can say that D is a *derivative* (in the papers [1]–[7] D was called an *algebraic derivative*).

THEOREM 2.4. *Suppose that we are given* $D \in \boldsymbol{R}(X)$ *and an operator* $F \in L(X)$ *such that* $F^2 = F$ *and* $FX = Z_D$. *Then* F *is an initial operator for* D *corresponding to the right inverse* $R = \hat{R} - F\hat{R}$, *where* R *is uniquely defined independently of the choice of a right inverse* $\hat{R}$ *of* D.

Proof. Since, by assumptions, $DF = 0$, we have $DR = D(\hat{R} - F\hat{R}) = D\hat{R} - DF\hat{R} = I$. Hence R is a right inverse of D. Furthermore $FR = F(\hat{R} - F\hat{R}) = F\hat{R} - F^2\hat{R} = F\hat{R} - F\hat{R} = 0$. Hence F is an initial operator for D corresponding to R. We have to show that R is uniquely determined. Suppose that $\hat{R}_1 \neq \hat{R}$ also is a right inverse of D. Write $R_1 = \hat{R}_1 - F\hat{R}_1$. In the same manner, as before, we obtain that $D\hat{R}_1 = I$ and $F\hat{R}_1 = 0$. Since F is an initial operator for D, we have $I - F = RD$ on $\mathscr{D}_D$. Hence

$$R_1 - R = \hat{R}_1 - \hat{R} - F(\hat{R}_1 - \hat{R}) = (I - F)(\hat{R}_1 - \hat{R})$$
$$= RD(\hat{R}_1 - \hat{R}) = R(D\hat{R}_1 - D\hat{R}) = R(I - I) = 0,$$

which was to be proved.

PROPOSITION 2.3. *Let* $D \in \boldsymbol{R}(X)$ *and let* R *and* R_1 *be two right inverses of* D *which are commutative*: $R_1R = RR_1$. *Then* $R_1 = R$.

Indeed, by assumption, we have $R = (DR_1)R = D(R_1R) = D(RR_1) = (DR)R_1 = R_1$.

PROPOSITION 2.4. *Let* $D \in \boldsymbol{R}(X)$ *and let* F_1, F *be two commutative initial operators for* D: $F_1F = FF_1$. *Then* $F_1 = F$.

Indeed, by (2.7) we have $F = F_1F = FF_1 = F_1$.

Observe that a convex combination (and only a convex combination) of initial operators for a $D \in \boldsymbol{R}(X)$ is again an initial operator for D. This initial operator corresponds to a right inverse of D which is the convex combination of the corresponding right inverses with the same coefficients.

3. Initial value and mixed boundary value problems for equations with right invertible operators. Let $R_0, \ldots, R_{N-1}$ be right inverses of an operator $D \in \boldsymbol{R}(X)$. As in Formula (1.1), we write $Q(D) = \sum_{k=0}^{N} Q_k D^k$, where $Q_k \in L(X)$ for $k = 0, 1, \ldots, N-1$ and $Q_N = I$. Let $F_0, \ldots, F_{N-1}$ be a system of initial operators induced by $R_0, \ldots, R_{N-1}$. The following problem will be called a *mixed boundary value problem* for $Q(D)$:

Find all solutions of the equation

$$Q(D)x = y, \quad y \in X, \tag{3.1}$$

satysfying the *mixed boundary conditions*:

$$F_k D^k x = y_k, \text{ where } y_k \in Z_D \quad (k = 0, 1, \ldots, N-1). \tag{3.2}$$

In particular, if $R_0 = \ldots = R_{N-1} = R$, we have $F_0 = \ldots = F_{N-1} = F$, and the problem (3.1)–(3.2) is said to be an *initial value problem* for $Q(D)$.

We say that a mixed boundary value problem (3.1)–(3.2) is *well-posed* if this problem has a unique solution for every $y \in X$ and $y_0, \ldots, y_{N-1} \in Z_D$. By definition, if the problem (3.1)–(3.2) is well-posed, then the corresponding homogeneous problem has only zero as a solution, i.e.

$$Q(D)x = 0 \text{ and } F_k D^k x = 0 \text{ for } k = 0, 1, \ldots, N-1 \text{ imply } x = 0. \tag{3.3}$$

An immediate consequence of Corollary 2.2 is

PROPOSITION 3.1. *The mixed boundary value problem*

$$D^N u = v, \quad x \in X, \tag{3.4}$$

$$F_k D^k u = v_k, \quad v_k \in Z_D \quad (k = 0, 1, \ldots, N-1) \tag{3.5}$$

is well-posed and its unique solution is

$$u = R_0 \ldots R_{N-1} v + v_0 + \sum_{k=1}^{N-1} R_0 \ldots R_{k-1} v_k. \tag{3.6}$$

Indeed, $D^k u = v_k$, hence $F_k D^k u = F_k v_k = v_k$ $(k = 0, 1, \ldots, N-1)$ and $D^N u = v$.

PROPOSITION 3.2. *The following identity holds on the domain of* D^N:

$$Q(D) = (I+\hat{Q})D^N + \sum_{m=0}^{N-1} \hat{Q}_m, \tag{3.7}$$

where we write

$$\hat{Q}_m = Q_m\Big[\sum_{k=m+1}^{N-1} R_m \ldots R_{k-1}F_kD^k + F_mD^m\Big] \tag{3.8}$$

for $m = 0, 1, \ldots, N-2$,

$$\hat{Q}_{N-1} = Q_{N-1}F_{N-1}D^{N-1}, \qquad \hat{Q} = \sum_{m=0}^{N-1} Q_mR_m \ldots R_{N-1}.$$

Proof. From Taylor–Gontcharov Formula (Theorem 2.3) we have

$$I = F_0 + \sum_{k=1}^{N-1} R_0 \ldots R_{k-1}F_kD^k + R_0 \ldots R_{N-1}D^N \quad \text{on } \mathcal{D}_{D^N}. \tag{3.9}$$

Since $DF_j = 0$ for $j = 0, 1, \ldots, N-1$, we obtain

$$
\begin{aligned}
Q(D) &= \\
&= Q(D)\Big[F_0 + \sum_{k=1}^{N-1} R_0 \ldots R_{k-1}F_kD^k + R_0 \ldots R_{N-1}D^N\Big] \\
&= \sum_{m=0}^{N} Q_mD^m\Big[F_0 + \sum_{k=1}^{N-1} R_0 \ldots R_{k-1}F_kD^k + R_0 \ldots R_{N-1}D^N\Big] \\
&= Q_0F_0 + \sum_{m=1}^{N} Q_m \sum_{k=1}^{N-1} D^mR_0 \ldots R_{k-1}F_kD^k + \sum_{m=0}^{N-1} Q_mD^mR_0 \ldots R_{N-1}D^N + D^N \\
&= D^N + \sum_{m=1}^{N-1} Q_mR_m \ldots R_{N-1}D^N + Q_0F_0 + \sum_{k=1}^{N-1} Q_0R_0 \ldots R_{k-1}F_kD^k + \\
&\quad + \sum_{m=1}^{N-1} Q_m\Big[\sum_{k=m+1}^{N-1} R_m \ldots R_{k-1}F_kD^k + F_mD^m\Big] \\
&= D^N + \hat{Q}D^N + \hat{Q}_0 + \sum_{m=1}^{N-1} \hat{Q}_m = (I+\hat{Q})D^N + \sum_{m=0}^{N-1} \hat{Q}_m.
\end{aligned}
$$

THEOREM 3.1. *If the operator* $I+\hat{Q} = I+\sum_{m=0}^{N-1} Q_mR_m \ldots R_{N-1}$ *is invertible, then the mixed boundary value problem* (3.1)–(3.2) *is well-posed and its unique solution is*

$$x = R_0 \ldots R_{N-1}(I+\hat{Q})^{-1}\hat{y} + y_0 + \sum_{k=1}^{N-1} R_0 \ldots R_{k-1}y_k, \tag{3.10}$$

where

$$\hat{y} = y - \sum_{m=0}^{N-2} Q_m\Big[\sum_{k=m+1}^{N-1} R_m \ldots R_{k-1}y_k + y_m\Big] - Q_{N-1}y_{N-1}. \tag{3.11}$$

Proof. Proposition 3.2 implies that equation (3.1) can be written in the form

$$(I+\hat{Q})D^Nx + \sum_{m=0}^{N-1} \hat{Q}_mx = y. \tag{3.12}$$

From conditions (3.2) we conclude that

$$
\begin{aligned}
y - \sum_{m=0}^{N-1} \hat{Q}_mx &= y - \sum_{m=0}^{N-2} Q_m\Big[\sum_{k=m+1}^{N-1} R_m \ldots R_{k-1}F_kD^kx + F_mD^mx\Big] - \\
&\qquad - Q_{N-1}F_{N-1}D^{N-1}x \\
&= y - \sum_{m=0}^{N-2} Q_m\Big[\sum_{k=m+1}^{N-1} R_m \ldots R_{k-1}y_k + y_m\Big] - Q_{N-1}y_{N-1} = \hat{y}\,.
\end{aligned}
$$

We therefore obtain an equivalent equation

$$D^Nx = (I+\hat{Q})^{-1}\hat{y}, \tag{3.13}$$

because the operator $I+\hat{Q}$ is invertible by assumptions. Proposition 3.1 and conditions (3.2) together imply that the problem (3.13)–(3.2) is well-posed and its unique solution is given by Formula (3.10). Hence the problem (3.1)–(3.2) is well-posed and its unique solution is of the required form (3.10).

COROLLARY 3.1. *Let* R *be a right inverse of an operator* $D \in \boldsymbol{R}(X)$, *and let* F *be an initial operator for* D *induced by* R. *Then the initial value problem*

$$Q(D)x = y, \qquad y \in X, \tag{3.1}$$

$$FD^kx = y_k, \qquad y_k \in Z_D \qquad (k = 0, 1, \ldots, N-1), \tag{3.44}$$

is well-posed provided that the operator $Q_0^*(R) = \sum_{k=0}^{N-1} Q_kR^{N-k}$ *is invertible. The unique solution of the problem* (3.1)–(3.14) *is of the form*

$$x = R^N[Q_0^*(R)]^{-1}\Big[y - \sum_{m=0}^{N-1} Q_my_m - \sum_{j=0}^{N-2}\Big(\sum_{m=0}^{j} Q_mR^{j-m}\Big)y_{j+m}\Big] + \sum_{k=0}^{N-1} R^ky_k. \tag{3.15}$$

This follows immediately from Theorem 3.1, if we put $R_0 = \ldots = R_{N-1} = R$, $F_0 = \ldots = F_{N-1} = F$, because in our case $I+\hat{Q} = Q_0^*(R)$.

Corollary 3.2. *Let R be a V-right inverse of an operator $D \in \boldsymbol{VR}(X)$. Let $Q_k = q_k I$, where q_k are scalars ($k = 0, 1, \ldots, N-1$). Suppose that $\lambda = 0$ is not a root of the polynomial $Q(\lambda)$. Then the initial value problem for the operator $D^M Q(D)$ is well-posed and its unique solution is*

$$x = [Q_0^*(R)]^{-1}\Big[R^{N+M}y + \sum_{j=0}^{N+M-1}\sum_{k=0}^{j} q_{N+k-j}R^j y_k\Big]. \tag{3.16}$$

Indeed, from conclusion (1) of Theorem 1.2 it follows that the operator $Q_0^*(R)$ is invertible. This, and conclusion (6) of Theorem 1.2 together imply that

$$x = [Q_0^*(R)]^{-1}\Big[R^{N+M}y + \sum_{k=0}^{N+M-1} R^k z_k\Big],$$

where $z_k \in Z_D$ are uniquely determined. It is not difficult to verify, writing

$$Q_0^*(R)x = R^N y + \sum_{k=0}^{N-1} R^k z_k$$

and acting on both sides of this equation by operators $D^0 = I, D, \ldots, D^{N-1}$, that z_k are of required form.

Up till this moment we have considered the case when the operator $I+\hat{Q}$ is invertible, i.e. when -1 is not an eigenvalue of the operator $\hat{Q}$. If -1 is an eigenvalue of the operator $\hat{Q}$, the problem (3.1)–(3.2) is not well-posed. However we have

Theorem 3.2. *If -1 is an eigenvalue of the operator $\hat{Q} = \sum_{m=0}^{N-1} Q_m R_m \ldots \ldots R_{N-1}$, then a solution of the problem (3.1)–(3.2) exists if and only if*

$$\hat{y} \in (I+\hat{Q})X, \quad \textit{where } \hat{y} \textit{ is defined by Formula } (3.11). \tag{3.17}$$

If this condition is satisfied, then the solutions are of the form

$$x = R_0 \ldots R_{N-1}(I+\hat{Q})_{-1}\hat{y} + y_0 + \sum_{k=1}^{N-1} R_0 \ldots R_{k-1} y_k + \hat{x}, \tag{3.18}$$

where $(I+\hat{Q})_{-1}\hat{y}$ denotes an element of the inverse image of $\hat{y}$ under the operator $I+\hat{Q}$ and $\hat{x}$ is an arbitrary element of the eigenspace $\hat{X}_{-1}$ of the operator $\hat{Q}$ corresponding to the eigenvalue -1.

Proof. In the same manner as in the proof of Theorem 3.1 we obtain the equation

$$(I+\hat{Q})D^N x = y. \tag{3.19}$$

This equation has a solution if and only if the condition (3.17) is satisfied. If this condition is satisfied, we obtain from (3.19) that $D^N x = (I+\hat{Q})_{-1}\hat{y} + \hat{x}$, where $(I+\hat{Q})_{-1}\hat{y}$ and $\hat{x}$ are described above. This and Proposition 3.1 together imply that solutions of our problem exist and are of the required form.

4. Examples of applications. In this section we shall indicate some applications of the results obtained above. To be short we shall omit proofs. We remark only that to prove the invertibility of the operator $I+\hat{Q}$ is exactly the same as to prove that a Volterra integral equation of the second kind has a unique solution.

Example 1. Let $X = C[0, 1]$, $D = d/dt$. Then the operators defined by means of the equality $(R_a x)(t) = \int_a^t x(s)\,ds$, where $0 \leqslant a < 1$, are V-right inverses of D. Observe that $\dim Z_D = 1$. The family of initial operators induced by R_a is defined as follows: $(F_a x)(t) = x(a)$. Hence it is a family of linear operators.

Example 2. Let $X = C([0,1]\times[0,1])$ and $(Dx)(t, s) = \dfrac{\partial x(t, s)}{\partial t}$. Then the operators R_a defined by means of the equality $(R_a x)(t, s) = \int_a^t x(u, s)\,du$, $0 \leqslant a < 1$, are V-right inverses of D. Observe that $\dim Z_D = +\infty$. The induced family of initial operators is defined as follows: $(F_a x)(t, s) = x(a, s)$.

Example 3 (Difference equations). Let X be the space of all sequences $\{x_n\}$ ($n = \ldots -2, -1, 0, 1, 2, \ldots$) and let $Dx = \{x_{n+1} - x_n\}$. Then the operators R_m defined by means of the equality $R_m x = \{\sum_{k=m}^{n-1} x_k\}$ ($m = 0, \pm 1, \pm 2, \ldots$) are V-right inverses of D. Each of the induced initial operators F_m maps any element $x = \{x_n\}$ into a constant sequence $\{x_m\}$ (i. e. $F_m x_n = x_m$ for all integers n). Using these definitions we can solve difference equations, for instance of the form $\sum_{k=0}^{N} P_k(n)x_{n+k} = y_n$ ($n = 0, \pm 1, \pm 2, \ldots$) with either initial conditions: $x_{m+j} = y_m$ (m is a fixed integer, $j = 0, 1, \ldots, N-1$) or with mixed boundary conditions: $x_{m+j} = y_m$ ($m = p_0, \ldots, p_{N-1}$, where $p_0, \ldots, p_{N-1}$ are fixed integers; $j = 0, 1, \ldots \ldots, N-1$).

Example 4[2] (The Darboux Problem for hyperbolic equations). Let $X = C([a, b]\times[a, b])$, $D = \dfrac{\partial^2}{\partial t \partial s}$. In this case $Z_D = \{x\colon\ x(t, s)$

(2) I wish to express my profound thanks to Professor Adam Bielecki, who is, in fact, the author of examples concerning hyperbolic equations.

$= \varphi(t)+\psi(s),\ \varphi, \psi \in C[a, b]\}$. The operators $R_{u,v}$ defined of the equality $(R_{u,v}x)(t, s) = \int_u^t\int_v^s x(\xi, \eta)\,d\xi\,d\eta$, $a \leqslant u, v < b$, are V-right inverses of D. The induced family $\{F_{u,v}\}$ of initial operators is given by the formula $(F_{u,v}x)(t, s) = x(u, s)+x(t, v)-x(u, v)$. Consider the hyperbolic equation

$$(4.1)\qquad \frac{\partial^2 x}{\partial t\partial s} = A(t, s)\frac{\partial x}{\partial t} + B(t, s)\frac{\partial x}{\partial s} + C(t, s)x(t, s)+y(t, s),$$

where $A, B, C, y \in X$, with the initial conditions

$$(4.2)\qquad x(u, s) = \varphi(s), \quad x(t, v) = \psi(t),$$

where $\varphi, \psi \in C[a, b]$ and $\psi(u) = \varphi(v)$. This is the Darboux problem for equation (4.1). Write

$$(Hx)(t, s) = A(t, s)\frac{\partial x}{\partial t} + B(t, s)\frac{\partial x}{\partial s} + C(t, s)x(t, s).$$

We can rewrite equation (4.1) and condition (4.2) as follows:

$$(4.3)\qquad (D-H)x = y,$$

$$(4.4)\qquad F_{u,v}x = y_{u,v}, \quad \text{where } y_{u,v}(t, s) = \varphi(t)+\psi(s)-\psi(u).$$

Since the operator $I-R_{u,v}H$ is invertible, the problem (4.3)–(4.4) is well-posed and has the unique solution $x = (I-R_{u,v}H)^{-1}(R_{u,v}y+y_{u,v})$.

EXAMPLE 5 (The Cauchy problem for hyperbolic equations). Let $X = C(\Omega)$, where $\Omega = \{(t, s)\colon 0 \leqslant t \leqslant a,\ 0 \leqslant s \leqslant b\}$ and let D be defined as above. We consider the Cauchy problem for equation (4.1), i.e. we admit the following condition

$$(4.5)\qquad x(t, g(t)) = \sigma(t), \quad \frac{\partial x}{\partial t}(t, g(t)) = \omega(t),$$

where the given function $g \in C^1[0, a]$, $g'(t) > 0$, $g(0) = 0$, $g(a) = b$, $\sigma \in C[0, a]$, $\omega \in C^1[0, a]$. The operator R defined by the equality $(Rx)(t, s) = \int_{g^{-1}(s)}^t \Big[\int_{g(\xi)}^s x(\xi, \eta)\,d\eta\Big]d\xi$ is a V-right inverse of D. The induced initial operator is $(Fx)(t, s) = x(g^{-1}(s), s)+\int_{g^{-1}(s)}^t \frac{\partial x}{\partial t}(\xi, g(\xi))\,d\xi$. Let H be defined as in Example 4. We consider equation (4.3) together with the initial condition

$$(4.6)\qquad Fx = y_0, \quad \text{where } y_0(t, s) = \sigma(g^{-1}(s), s)+\int_{g^{-1}(s)}^t \omega(\xi)\,d\xi.$$

Since the operator $I-RH$ is invertible, the problem (4.3)–(4.6) is well-posed and has the unique solution $x = (I-RH)^{-1}(Ry+y_0)$.

EXAMPLE 6 (The Picard problem for hyperbolic equations). Let X and D be defined as in Example 5. Let $g \in C^1[0, a]$, $g'(t) > 0$, $g(0) = 0$, $g(a) = b$. We consider the Picard problem for equation (4.1), i. e. we are looking for solutions of equation (4.1) satysfying the condition

$$(4.7)\qquad x(t, 0) = \varphi(t), \quad \varphi \in x(g(s), s) = \psi(s),$$

where $\varphi \in C^1[0, a]$, $\psi \in C^1[0, b]$ and $\varphi(0) = \psi(0)$. We put $(Rx)(t, s) = \int_{g(s)}^t \Big[\int_0^s x(\xi, \eta)\,d\eta\Big]d\xi$. The operator R is a V-right inverse of D. The corresponding initial operator is defined as follows: $(Fx)(t) = x(g(s), s)+ x(t, 0)-x(0, 0)$. Let H be defined as in Example 4. We consider equation (4.3) together with the initial condition

$$(4.8)\qquad Fx = y_0, \quad \text{where } y_0(t, s) = \varphi(t)+\psi(s)-\varphi(0).$$

Since the operator $I-RH$ is invertible, the problem (4.3)–(4.8) is well-posed and has the unique solution $x = (I-RH)^{-1}(Ry+y_0)$.

EXAMPLE 7 (Generalized Cauchy problem for hyperbolic equations). Let X and D be defined as in Example 5. Suppose we are given two functions $g_0, g_1 \in C^1[0, a]$ such that $g_0' > 0$, $g_1' > 0$, $g_0(0) = g_1(0) = 0$, $g_0(a) = g_1(a) = b$. Our problem is to find solutions of equation (4.1) satisfying the conditions

$$(4.9)\qquad x(t, g_0(t)) = \varphi(t); \quad x_t'(t, g_1(t)) = \psi(t),$$

where $\varphi, \psi \in C^1[0, a]$. We write $(Rx)(t, s) = \int_{g_0^{-1}(s)}^t \Big[\int_{g_1(\xi)}^s x(\xi, \eta)\,d\eta\Big]d\xi$. The operator R is a right inverse of D. The corresponding initial operator is defined by the equality $(Fx)(t, s) = x(g_0^{-1}(s), s)+\int_{g_1^{-1}(s)}^t x(\xi, g_1(\xi))\,d\xi$. Let H be defined as in Example 4. We consider equation (4.3) together with the initial condition

$$(4.10)\qquad Fx = y_0, \quad \text{where } y_0(t, s) = \varphi(g_0^{-1}(s), s)+\int_{g_1^{-1}(s)}^t \psi(\xi)\,d\xi.$$

If the operator $I-RH$ is invertible, then the problem (4.3)–(4.10) is well-posed and has the unique solution $x = (I-RH)^{-1}(Ry+y_0)$. Observe that the operator $I-RH$ is invertible if g_0-g_1 has a constant sign for $t \in (0, a)$.

We can consider similar initial and mixed boundary value problems for hyperbolic equations of higher orders in a similar way.

Some applications to functional-differential equations are given in [4]. Left invertible operators were considered in [7].

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Nonlinear integral operators on $C(S, E)$

by

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Abstract. We investigate the class of operators T on the space $C(S, E)$ of vector-valued continuous functions on a compact Hausdorff space S which are uniformly continuous on bounded sets and satisfy the algebraic relation

$$T(f+f_1+f_2) = T(f+f_1)+T(f+f_2)-Tf$$

for all $f, f_1, f_2 \in C(S, E)$ with f_1 and f_2 having disjoint support. We derive integral representations and prove universal properties of these operators. Special attention is given to the (weakly) compact operators.

Introduction. Since A. Grothendieck's celebrated treatment [22] of this subject in 1953, linear bounded transformations on spaces of continuous functions and their universal properties have been of interest for many writers. Our present study is related to the work of A. Pełczyński [24], C. C. Brown [10], E. Thomas [27] and H. H. Schaefer [26] and especially to the results obtained in connection with generalizations of the Riesz Theorem (see [7] for a detailed account) via integral representations for the operators on the basis of the well-known paper of R. Bartle, N. Dunford and J. Schwartz [1] in 1955, namely the results of C. Foiaş and I. Singer [17], P. W. Lewis [23], I. Dobrakov [14], J. K. Brooks and D. R. Lewis [9] and of the author [2] [3] [4]. In 1965 N. A. Friedman, R. V. Chacon and M. Katz began to derive representation theorems for real, so-called "additive" (not necessarily linear) functionals on spaces of real-valued continuous functions in a series of three papers [11] [18] [19] with successive improvements. It was a natural question to ask which results an integration theory could yield in the investigation of nonlinear operators on spaces of continuous functions. Let E and F be Banach spaces and $C(S, E)$ the space of continuous functions on the compact Hausdorff space S with values in E (with the uniform norm): In this note we present what we think is the adequate extension of the theory of the linear bounded transformations to the class of nonlinear transformations $T: C(S, E) \to F$ which are at all representable as integrals with respect to additive "nonlinear" set functions (hereby we understand a set function which takes its values in a space of operators from one Banach space into another which are uniformly continuous on bounded sets). This class consists of those transformations T which are uniformly