(b) In the case of a real space the proof is simpler. The set $\Omega_r = \{\zeta \in R: \|x+\zeta y\| \geqslant \|x\|\}$ (R is the real field) contains at least one of the rays $[0, +\infty)$, $(-\infty, 0]$ and by assumption ω is negative. Then, $\omega^k \in \Omega$ for some positive integer k.

Remark. It can be easily seen, by taking the supremum norm on C^2, for example, that there may be no positive integer k such that $\|x+\omega^k y\| > \|x\|$.

PROPOSITION 2. *Let X be a real or complex normed vector space and let A be a linear operator from X into itself such that $\|Ax\| \leqslant \|x\|$ for all $x \in X$ (i. e. A is a contraction). Let λ, μ be eigenvalues of A such that $|\lambda| = 1$ and $\lambda \neq \mu$. If u and v are eigenvectors of A corresponding to λ and μ, respectively, then u is orthogonal to v.*

Proof. Let a be an arbitrary scalar. We have for all positive integers k,

$$\|u+av\| \geqslant \|A^k(u+av)\| = \|\lambda^k u + a\mu^k v\|.$$

whence, denoting $\omega = \mu/\lambda$,

$$(*) \qquad \|u+\omega^k(av)\| \leqslant \|u+av\|.$$

If $|\mu| < 1$, then $|\omega| < 1$ and letting in $(*)$ $k \to +\infty$, we obtain $\|u\| \leqslant \|u+av\|$. If $|\mu| = 1$, then we have $0 < \arg\omega < 2\pi$ (since $\omega \neq 1$) and making use of Proposition 1 we obtain from $(*)$ that $\|u\| \leqslant \|u+av\|$.

Acknowledgement. The author wishes to express his thanks to Dr. Christoph Zenger for helpful comments and conversations during the Fifth Gatlinburg Symposium on Numerical Algebra, held in Los Alamos, New Mexico, June 5–10, 1972.

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Received June 25, 1972 (522)

STUDIA MATHEMATICA, T. XLVIII. (1973)

Some examples in harmonic analysis

by

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Abstract. The paper consists essentially of five examples as follows.

(1) A Segal algebra on a commutative group which is not *closed.

(2) A Wiener-like Segal algebra which is not *closed.

(3) A group algebra such that the ideal of functions with Haar integral zero does not have an unbounded approximate unit.

(4) A group Γ with a closed normal subgroup G and a G-invariant subspace E of $L^1(\Gamma)$ such that TE is not closed where T is the canonical map of $L^1(\Gamma)$ onto $L^1(\Gamma/G)$.

(5) A compact group G such that the kernel of the convolution product map from $L^\infty(G)\hat{\otimes}L^\infty(G)$ is not the closed linear span of the tensors $\varphi*a\otimes\psi - \varphi\otimes a*\psi$, $a \in L^1(G)$, $\varphi, \psi \in L^\infty(G)$.

In this paper we give a number of examples arising in various parts of harmonic analysis. The first four are connected with the work of H. Reiter.

1. Symmetry and *symmetry in Segal algebras. A Segal algebra S ([5], p. 16) is a dense left translation invariant subset of $L^1(G)$, G a locally compact group, which is a Banach space under some left translation invariant norm $\| \ \|_S$ dominating the L^1 norm and such that the left regular representation of G on S is strongly continuous. S is symmetric if in addition $\| \ \|_S$ is right invariant and the right regular representation is strongly continuous. If G is abelian every Segal algebra is symmetric.

The Segal algebra S is *symmetric if it is stable under the hermitian involution* on $L^1(G)$. We shall construct an example with $G = \boldsymbol{R}$ of a (necessarily symmetric) Segal algebra which is not *symmetric.

Let $f \in L^1(\boldsymbol{R})$. Define

$$S_f = \{g;\ g \in L^1(\boldsymbol{R}), f*g \in C_0(\boldsymbol{R})\},$$

$$\|g\|_S = \|g\|_1 + \|f*g\|_\infty$$

where $f*g \in C_0(G)$ means $f*g$ differs from a C_0 function on a set of measure zero and $\| \ \|_p$ is the L^p norm. As S_f contains all continuous functions with compact support, S_f is dense in $L^1(\boldsymbol{R})$ and it is easy to check that S_f is a Segal algebra. Consider the case

$$f(x) = (x|\log x|)^{-1/2} \qquad 0 < x < \tfrac{1}{2},$$

$$= 0 \qquad x \leqslant 0 \text{ or } x \geqslant \tfrac{1}{2}.$$

$f*f$ is zero outside $(0,1)$ and, for any $\varepsilon>0$, f is the sum of an L^1 function on $(0,\varepsilon)$ and an L^∞ function on $[\varepsilon,\frac{1}{2})$ so that $f*f$ is continuous outside $[0,2\varepsilon]$. Thus $f*f$ is continuous on $\boldsymbol{R}\setminus\{0\}$ and zero on $(-\infty,0]$. If $0<t<\frac{1}{2}$ then

$$(f*f)(t)=\int_0^t |x\log x\,(t-x)\log(t-x)|^{\frac{1}{2}}\,dx$$

$$=\int_0^1 \xi^{-\frac{1}{2}}(1-\xi)^{-\frac{1}{2}}\,|\log\xi t\log(1-\xi)t|^{-\frac{1}{2}}\,d\xi$$

$$\to 0 \quad \text{as } t\to 0^+$$

because the integrand is dominated by $(\log 2)^{-1}\xi^{-\frac{1}{2}}(1-\xi)^{-\frac{1}{2}}$. Thus $f*f$ is also continuous at 0 and $f\in S_f$.

In a similar way we see $f*f^*$ is zero outside $(-\frac{1}{2},\frac{1}{2})$ and continuous on $\boldsymbol{R}\setminus\{0\}$. As f is monotonic on $(0,\frac{1}{2})$ we have, if $0<t<\frac{1}{2}$

$$(f*f^*)(t)=\int_t^{\frac{1}{2}} f(x)f(x-t)\,dx$$

$$\geqslant\int_t^{\frac{1}{2}} f(x)^2\,dx$$

$$\to\int_0^{\frac{1}{2}} f(x)^2\,dx=\infty \quad \text{as } t\to 0^+$$

so that $f*f^*$ is not continuous and $f^*\notin S_f$.

2. Assymetry of a class of Wiener-like Segal algebras. Let G be a locally compact group and Γ a discrete subgroup such that the left coset space G/Γ is compact. On $\mathscr{K}(G)$, the set of continous complex valued functions on G with compact support, we define the norm

$$\|f\|_S=\sup_{x\in G}\sum_{\gamma\in\Gamma}|f(x\gamma)|.$$

The completion of $\mathscr{K}(G)$ in this norm is the Segal algebra S with which we are concerned ([5], p. 23).

Let H be a compact group, Γ a group of automorphisms of H and let $h_0\in H$ with $\Gamma h_0=\{\gamma h_0;\ \gamma\in\Gamma\}$ infinite. Let G be the semi-direct product of Γ and H, that is the product space $\Gamma\times H$ (Γ has the discrete topology) with multiplication $(\gamma,h)(\gamma',h')=(\gamma\gamma',\gamma(h')h)$. Identifying Γ with $\{(\gamma,e);\ \gamma\in\Gamma\}$ and H with $\{(e,h);\ h\in H\}$, we see that the pair G, Γ satisfy the conditions of the preceeding paragraph and we shall show that in this case the Segal algebra is not symmetric.

If $f\in\mathscr{K}(G)$ then

$$\|f\|_S=\sup_{h\in H}\sum_{\gamma\in\Gamma}|f(\gamma,h)|$$

and, because G is unimodular, if $z=(e,h_0)$ then

$$\|R_{z^{-1}}f\|_S=\sup_{x}\sum_{\gamma\in\Gamma}|f(x\gamma z)|$$

$$\geqslant\sum_{\gamma}|f(\gamma z)|$$

$$=\sum_{\gamma}|f(\gamma,\gamma h_0)|.$$

Let $E_1,E_2,\ldots$ be disjoint open subsets of H and $\gamma_1,\gamma_2,\ldots$ elements of Γ with $\gamma_i h_0\in E_i$ for all i. Clearly if $i\neq j$ then $\gamma_i\neq\gamma_j$. Let v_i be a continuous function $G\to[0,1]$ with support contained in $\{\gamma_i\}\times E_i$ and $v_i(\gamma_i,\gamma_i h_0)=1$. Then $u=\Sigma i^{-1}v_i$ converges in S because

$$\sum_{\gamma}\Big|\sum_{j\geqslant i}j^{-1}v_j(\gamma,h)\Big|=k^{-1}|v_k(\gamma_k,h)|\leqslant j^{-1} \quad \text{if } h\in E_k \text{ for some } k\geqslant i,$$

$$=0 \quad \text{otherwise.}$$

However

$$\|R_{z^{-1}}u\|_S\geqslant\sum_{i}|u(\gamma_i,\gamma_i h_0)|=\sum i^{-1}=\infty$$

so that S is not symmetric.

Examples of pairs H, Γ satisfying the above conditions abound. We could take $H=\boldsymbol{Z}_2^\infty$, considered as a space of bilateral sequences of 0's and 1's, Γ as the powers of the shift and h_0 the characteristic function of $\boldsymbol{Z}^+$.

3. Approximate units in the augmentation ideal of a group algebra. The augmentation ideal $L_0(G)$ of $L^1(G)$ is $\{f;\ f\in L^1(G),\ \int f\,d\lambda=0\}$ where λ is a left Haar measure on the locally compact group G. We shall say that a Banach algebra A has a bounded right approximate unit if there is $C>0$ such that if $a\in A$, $\varepsilon>0$ then there is $e\in A$ with $\|e\|\leqslant C$ and $\|a-ae\|<\varepsilon$. A has a right approximate unit if for all $a\in A$, $\varepsilon>0$ there is $e\in A$ with $\|a-ae\|<\varepsilon$, that is each element of A lies in the closed right ideal it generates. By the Cohen factorisation theorem ([1], p. 199), if A has a bounded right approximate unit then every element of A is the product of two others. In the convolution algebra $L^2(\boldsymbol{T})$ a product is a continuous function so this algebra cannot have a bounded approximate unit but the usual construction for a bounded approximate unit in $L^1(\boldsymbol{T})$ gives an unbounded approximate unit in $L^2(\boldsymbol{T})$. Thus some Banach algebras have unbounded but no bounded right approximate units. In [3] it was shown that G is amenable if and only if $L_0(G)$ has a bounded approximate right unit. In this example we will show that if $G = F_2$, the free group on two generators a, b then $L_0(G)$ does not even have an unbounded approximate right unit.

It will be convenient to represent elements of $l^1(F_2)$ as linear combinations of group elements. Let $r = e+a-b-ba \in L_0(F_2)$. We shall define a function $\sigma\colon F_2 \to \{0, 1\}$ such that for all g in G

$$\sigma(g)+\sigma(ag)-\sigma(bg)-\sigma(bag) = 1.$$

If $\varphi \in L_0(G)^*$ is defined by

$$\varphi(\Sigma a_g g) = \Sigma a_g \sigma(g) \quad \text{where } a_g \in \boldsymbol{C}$$

and Σ means $\Sigma(g \in G)$, then $\varphi(r) = 1$ and

$$\begin{aligned}\varphi(r * \Sigma a_g g) &= \varphi\big(\Sigma(a_g + a_{a^{-1}g} - a_{b^{-1}g} - a_{(ba)^{-1}g})g\big)\\ &= \Sigma(a_g + a_{a^{-1}g} - a_{b^{-1}g} - a_{(ba)^{-1}g})\sigma(g)\\ &= \Sigma a_g\big(\sigma(g)+\sigma(ag)-\sigma(bg)-\sigma(bag)\big)\\ &= \Sigma a_g = 0.\end{aligned}$$

Thus r does not lie in the closed right ideal it generates.

To define σ, for $g \in F_2$ let $|g|$ denote the sum of the absolute values of the exponents of a and b when g is written in reduced from, so that $|e| = 0$, $|a^2ba^{-1}| = 4$ and so on. We define σ by induction on $|g|$ taking $\sigma(g) = 1$ if $|g| \leqslant 1$ and when $|g| = n+1$, $n \geqslant 1$ we take $\sigma(g) = 1$ if

(i) $g = ah$, $a^{-1}h$, or $b^{-1}h$, $|h| = n$,

(ii) $g = bah$, $|h| = n-1$, $\sigma(h) = \sigma(ah) = 1$, $\sigma(bh) = 0$ or $g = ba^{-1}h$, $|h| = n-1$, $\sigma(h) = \sigma(a^{-1}h) = 1$, $\sigma(bh) = 0$

and $\sigma(g) = 0$ otherwise.

Note that for all $g \in F_2$ we have $\sigma(bg) \leqslant \sigma(g)$; this follows by (i) if $|bg| < |g|$, is obvious if $\sigma(bg) = 0$ and by (ii) in the remaining case. Put

$$\tau(g) = \sigma(g)+\sigma(ag)-\sigma(bg)-\sigma(bag).$$

Suppose $|g| < |ag|$. Then $\sigma(ag) = 1$ by (i) (or by specific definition if $g = e$) and by the above remark $\big(\sigma(g), \sigma(bg)\big) = (1, 0)$, $(0, 0)$ or $(1, 1)$. In the first of these cases $\sigma(bag) = 1$ by (ii) and so $\tau(g) = 1$ whereas in the second and third $\sigma(bag) = 0$ and again $\tau(g) = 1$. A similar argument applies if $|g| > |ag|$.

4. Translation invariant subspaces which are not mapped onto closed subspaces by the canonical quotient map. If Γ is a locally compact group and G a closed normal subgroup with a left Haar measure λ then $T\colon L^1(\Gamma) \to L^1(\Gamma/G)$ is defined by $(Tf)(xG) = \int f(xg)\,d\lambda(g)$.

If G is amenable then the image under T of a closed G invariant subspace of $L^1(\Gamma)$ is a closed subspace of $L^1(\Gamma/G)$ ([4], p. 177).

THEOREM. *If $L_0(G)$ does not contain an approximate right unit then $L^1(\Gamma)$, where $\Gamma = \boldsymbol{Z}\times G$, contains a closed G invariant subspace E such that TE is not closed.*

Proof. We shall consider $G = \{(0, g);\ g \in G\} \subset \Gamma$ and $M(G)$ as embedded in $M(\Gamma)$ in this way. Choose $\varphi \in L_0(G)$ with $\|\varphi\| = 1$, $\|*\varphi'-\varphi\| \geqslant \delta > 0$ for all $\varphi' \in L_0(G)$ and $\psi \in L^1(G)$ with $\|\psi\| = \int \psi\,d\lambda = 1$. For $n \in \boldsymbol{Z}^+$ let $f_n \in L^1(\Gamma)$ be defined by

$$\begin{aligned}&f_n(n, g) = \varphi(g),\\ &f_n(-n, g) = n^{-1}\psi(g),\\ &f_n(m, g) = 0 \qquad |m| \neq n\end{aligned}$$

and let E be the closed right G invariant subspace generated by $\{f_n;\ n \in \boldsymbol{Z}^+\}$.

We shall show that

$$\big(f_n * L_0(G)\big)^- = \big(f_n * L^1(G)\big)^- \cap \operatorname{Ker} T.$$

Clearly the first set is contained in the second. If F lies in the second set then $TF = 0$ and there is a sequence $\{a_m\}$ from $L^1(G)$ with $f_n * a_m \to F$. Thus $T(f_n * a_m) \to TF = 0$. As

$$\begin{aligned}T(f_n * a_m)(-n) &= \int\int n^{-1}\psi(hg)\,a_m(g^{-1})\,d\lambda(g)\,d\lambda(h)\\ &= n^{-1}\int \psi\,d\lambda \int a_m\,d\lambda\\ &= n^{-1}\int a_m\,d\lambda,\end{aligned}$$

this implies $\int a_m\,d\lambda \to 0$ as $m \to \infty$ so that $\lim\limits_{m} f_n*(a_m - \psi\int a_m\,d\lambda) = f$ and $f_n*(a_m-\psi\int a_m\,d\lambda)$ is in $f_n * L_0(G)$.

As a Banach space $L^1(\Gamma)$ is the direct sum of a sequence of copies of $L^1(G)$, $\operatorname{Ker} T$ is a direct sum of its intersections with these copies and E is a direct sum of the $\big(f_n * L^1(G)\big)^-$ (by a direct sum of Banach spaces $\mathscr{X}_1, \mathscr{X}_2, \ldots$ we mean $\{x\colon x \in \mathscr{X}_1 \times \mathscr{X}_2 \times \ldots,\ \Sigma\|x_i\| < \infty\}$ with norm $\|x\| = \Sigma\|x_i\|$) we see

$$\begin{aligned}d(f_n, E \cap \operatorname{Ker} T) &= d\big(f_n, \big(f_n * L^1(G)\big)^- \cap \operatorname{Ker} T\big)\\ &= d\big(f_n, f_n * L_0(G)\big)\\ &\geqslant \inf_{\varphi' \in L_0(G)} \|\varphi * \varphi' - \varphi\|\\ &\geqslant \delta,\end{aligned}$$

where d denotes the distance from the point to the subspace.

T gives a one to one map τ of the Banach space $E/E \cap \operatorname{Ker} T$ onto

TE. If TE is closed then τ is a homeomorphism. However $\|f_n+E\cap \operatorname{Ker}T\| \geqslant \delta$ whereas

$$T(f_n)(-q) = \tau(f_n+E\cap \operatorname{Ker}\pi)(-q) = q^{-1} \quad \text{if } n = q > 0,$$
$$= 0 \quad \text{otherwise}$$

so that $\|\tau(f_n+E\cap \operatorname{Ker}T)\| = n^{-1}\to 0$ as $n\to\infty$. Hence τ is not a homeomorphism and so TE is not closed.

5. Failure of the multiplier result in $L^\infty \hat{\otimes} L^\infty$. The theory of multiplier spaces introduced by Figà–Talamanca is now well developed. The basis of the theory as given by Rieffel [6] is that, with the usual identification of the operator space $\mathscr{L}(E, F^*)$ with the space $(E\hat{\otimes}F)^*$, the operators from $L^p(G)$ into $L^{q'}(G)$ which commute with right translations correspond to those elements of $\big(L^p(G)\hat{\otimes}L^q(G)\big)^*$ which are zero on tensors of the form $\varphi g\otimes\psi-\varphi\otimes g\psi$ where $\varphi\in L^p(G)$, $\psi\in L^q(G)$, $g\in G$ and φg and $g\psi$ denote the right and left translates of φ and ψ. Thus these operators can be identified with elements of $\big(L^p(G)\hat{\otimes}_G L^q(G)\big)^*$ (for convenience we are assuming G is unimodular and $L^q(G)$ and $L^{q'}(G)$ are paired by the bilinear form $\int a(g)b(g^{-1})dg$). The most difficult part of this theory is to show that if π is defined on $L^p(G)\hat{\otimes}L^q(G)$ by $\pi(\varphi\otimes\psi) = \varphi*\psi$ then $\operatorname{Ker}\pi$ is the closed linear span of the tensors of the form $\varphi g\otimes\psi-\varphi\otimes g\psi$ so that $L^p(G)\hat{\otimes}_G L^q(G)$ can be identified with $\operatorname{Im}\pi$. In a similar way operators from $L^p(G)$ into $L^{q'}(G)$ commuting with convolution by $L^1(G)$ functions can be identified with the dual of the quotient of $L^p(G)\hat{\otimes}L^q(G)$ by the closed linear span of the tensors $\varphi*a\otimes\psi-\varphi\otimes a*\psi$ $\big(a\in L^1(G)\big)$. However in most cases an operator commutes with translation if and only if it commutes with convolution with $L^1(G)$ functions.

Certain cases in which p and q are infinite are not covered by [6] and some of the results are false in these cases.

LEMMA. *There is a non zero functional in $L^\infty(\boldsymbol{T})^*$ which* is translation invariant and zero on $C(\boldsymbol{T})$.

Proof. Let $r_1, r_2, \ldots$ be an enumeration of the rationals and $E_0 = \bigcup_{j\geqslant 1}\{e^{ix};\ x\in \boldsymbol{R},\ |r_j-x| < 2^{-j}\pi\}$. If I is any non void open arc in $\boldsymbol{T}$ and $w_1, \ldots, w_n\in\boldsymbol{T}$ then I contains a point of $\{w_1e^{iq};\ q\in\boldsymbol{Q}\}$ and hence some non void open subarc I_1 of w_1E_0. I_1 in turn contains a non void open subarc I_2 of w_2E_0 and so on. Eventually we find a non void open arc I_n contained in $I\cap w_1E_0\ldots\cap w_nE_0$. Let χ be the characteristic function of E_0 and f a convex combination of the translates of χ by the w_j. Then $f = 1$ on I_n so if $\lambda\in\boldsymbol{C}$ with $\|f-\lambda 1\|_\infty < \frac{1}{4}$ then $|1-\lambda| < \frac{1}{4}$. However if $k = \int_0^{2\pi}\chi(e^{ix})\,dx$ then $k\leqslant\pi$ so that

$$\int_0^{2\pi}|f(e^{ix})|\,dx = \int_0^{2\pi} f(e^{ix})\,dx = k\leqslant\pi$$

and $\|f-\lambda 1\|_\infty < \frac{1}{4}$ implies $2\pi|\lambda|\leqslant k+\frac{1}{2}\pi\leqslant\frac{3}{2}\pi$ and hence $|\lambda|\leqslant\frac{3}{4}$, which contradicts $|1-\lambda| < \frac{1}{4}$. Thus in $L^\infty(\boldsymbol{T})$ the closed convex hull K of the translates of χ is distance at least 1/4 from the constant functions. Hence, by the Hahn–Banach theorem there is β in $L^\infty(\boldsymbol{T})^*$ with $\beta(1) = 0$, $\operatorname{Re}\beta(x)\geqslant 1$ for all x in K. Let M be a translation invariant mean on $l^\infty(\boldsymbol{T})$, the space of all bounded functions on $\boldsymbol{T}$ ([2], Theorem 17.5), and define

$$a(f) = M(Sf)$$

where $(Sf)(w) = \beta(wf)$, $f\in L^\infty(\boldsymbol{T})$, $w\in\boldsymbol{T}$.

Clearly $a\in L^\infty(\boldsymbol{T})^*$, a is translation invariant, $a(1) = 0$ and $\operatorname{Re} a\geqslant 1$ on K so that $a\neq 0$. $a\mid C(\boldsymbol{T})$ is thus, by uniqueness, a multiple of Lebesgue measure and so is zero because $a(1)$ is zero.

When the multiplier result $\operatorname{Ker}\pi = \operatorname{Span}\{\varphi*a\otimes\psi-\varphi\otimes a*\psi;\ \varphi\in L^p(G),\ \psi\in L^q(G),\ a\in L^1(G)\}$ holds, T is an operator from $L^p(G)$ into $L^q(G)^*$ which commutes with convolution by $L^1(G)$ functions and $\varphi\in L^p(G)$, $\psi\in L^q(G)$ with $\varphi*\psi = 0$ then $\big(T(\varphi), \psi\big) = 0$. A similar remark applies to operators commuting with translation. When $p = q = \infty$ we define the convolution $a*\gamma$, $a\in L^1(G)$, $\gamma\in L^\infty(G)^*$ by $(a*\gamma, F) = (\gamma, F\tilde{*}a)$, $F\in L^\infty(G)$ and the translation $g\gamma$ by $(g\gamma, F) = (\gamma, Fg)$. In particular if γ is translation invariant then $g\gamma = \gamma$ and if γ is zero on $C(G)$ then $a*\gamma = 0$.

Let $r, s\in L^\infty(\boldsymbol{T})\diagdown C(\boldsymbol{T})$ such that $r*s = 0$ (we could choose r, s such that $r(w) = r(-w)$, $s(w) = -s(-w)$ for all $w\in\boldsymbol{T}$) and let $\sigma, \tau\in L^\infty(\boldsymbol{T})^*$ with $\sigma, \tau = 0$ on $C(\boldsymbol{T})$, $\sigma(r)\neq 0$, $\tau(s)\neq 0$. Define T by $T(F) = \sigma(F)\tau$ so that T is an operator from $L^\infty(\boldsymbol{T})$ into $L^\infty(\boldsymbol{T})^*$. Because σ is zero on $C(\boldsymbol{T})$ we see that $T(a*F) = 0$ and because $a*\tau = 0$ we see that $a*T(F) = 0$ for all a in $L^1(\boldsymbol{T})$, F in $L^\infty(\boldsymbol{T})$, and so T commutes with convolution by $L^1(\boldsymbol{T})$ functions. However $r*s = 0$ and $(Tr, s) = \sigma(r)\tau(s)\neq 0$ so $\operatorname{Ker}\pi$ is not the closed span of the tensors $\varphi*a\otimes\psi-\varphi\otimes a*\psi$, $\varphi, \psi\in L^\infty(\boldsymbol{T})$, $a\in L^1(\boldsymbol{T})$.

In a similar way defining $T\colon L^\infty(\boldsymbol{T})\to L^\infty(\boldsymbol{T})^*$ by $T(F) = a(F)\lambda$ where a is the functional in the Lemma and λ the Lebesgue integral $\lambda(F) = \int_0^{2\pi}F(e^{ix})\,dx$, we have $gT(F) = a(F)g\lambda = a(F)\lambda$ and $T(gF) = a(gF)\lambda = a(F)\lambda$ so that T commutes with translations. However if $\varphi\in L^\infty(\boldsymbol{T})$ with $a(\varphi)\neq 0$, $\lambda(\varphi) = 0$ then $\varphi*1 = 0$ and $\big(T(\varphi), 1\big) = a(\varphi)\lambda(1)\neq 0$ so that $\operatorname{Ker}\pi$ is not the closed linear span of the tensors $\varphi g\otimes\psi-\varphi\otimes g\psi$, $\varphi, \psi\in L^\infty(\boldsymbol{T})$, $g\in\boldsymbol{T}$.

Note that in the first of these examples if r' is a translate of r and $\sigma(r') = 0$ then $T(r') = 0$, $T(r)\neq 0$ so that T does not commute with translations. In the second example if $a\in L^1(\boldsymbol{T})$, $F\in L^\infty(\boldsymbol{T})$, $\lambda(1)\neq 0$, $a(F)\neq 0$ then, because $a*F$ is continuous, we have $T(a*F) = 0$ whereas $a*T(F) = a(F)\lambda(a)\neq 0$ so that T does not commute with convolution by a.

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Received September 20, 1972 (485)

STUDIA MATHEMATICA, T. XLVIII. (1973)

On commutative approximate identities and cyclic vectors of induced representations

by

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Abstract. It is shown that every locally compact group has a commutative approximate identity for $L_1(G)$ which consists of continuous positive functions which decrease very rapidly at infinity. This is applied to a construction of a cyclic vector for a representation of a locally compact first countable group induced by a cyclic representation.

The aim of this paper is twofold. To show that every locally compact group has a commutative approximate identity for $L_1(G)$ which consists of continuous positive functions which decrease very rapidly at infinity and apply this to a construction of a cyclic vector for a representation of a locally compact first countable group induced by a cyclic representation.

A construction of commutative approximate identity for a C^*-algebra was given by J. F. Aarnes and R. V. Kadison [1]. Their method uses C^*-algebras technique and does not apply to the group algebras. It would be interesting to know whether there exists an approximate identity for $L_1(G)$ consisting of commuting continuous functions with compact support.

The fact that for a first countable group representations induced by cyclic representations are cyclic was first proved by F. Greenleaf and M. Moskowitz [5] and [6] and a construction of a cyclic vector for such representations was claimed by the authors [7]. Unfortunately [7] makes use of a statement in [2], p. 49, which is false, as it has been recently discovered by R. Goodman. The construction presented here avoids this difficulty and (for induced representations) improves the construction given in [7].

Very briefly the idea is the following. For a Lie group G the fundamental solution $u(g, t) = p^t(g)$ of the heat equation is a one-parameter semi-group of non-negative functions p^t, that is $p^s * p^t = p^{s+t}$ for all positive real s, t. Moreover $p^t * f$ tends to f as t tends to zero, and for a fixed t the function p^t decreases faster than exponentially at infinity. In short, functions p^t, $t \in \boldsymbol{R}^+$, form an approximate identity for $L_1(G)$ consisting of commuting rapidly decreasing functions.