STUDIA MATHEMATICA, T. XLVI. (1973)

A representation theorem for the second dual of $C[0,1]$

by

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Abstract. Assuming the continuum hypothesis is true and the cardinality of $\mathrm{ca}(S, \Sigma)$ is $2^{\aleph_0}$, (as is the case in $C^*[0,1]$), an integral representation of the functionals, T, of the dual of $\mathrm{ca}(S,\Sigma)$ is given: $T(\mu) = \int_S \psi d\mu$. Here, ψ is a real-valued function defined on Σ and the approximating sums are of the form $\Sigma\psi(E)\mu(E)$, where the sum is over all sets E of some partition of the space S. The integral is the limit of the approximating sums over the directed set of partitions.

Let $\mathfrak{M}[0,1]$ denote the space of all real-valued, countably additive, regular set functions defined on the σ-algebra, $\mathfrak{B}$, of all Borel subsets of the closed interval $[0,1]$, with the norm of a function, μ, being the total variation of μ. Let $C[0,1]$ denote the space of all real-valued continuous functions on $[0,1]$, with the norm of a function f being the least upper bound of $|f|$ on $[0,1]$. The space $\mathfrak{M}[0,1]$ is isometrically isomorphic to $C^*[0,1]$, the first dual of $C[0,1]$, ([2], p. 252).

Kakutani has shown that there is a compact Hausdorff space K such that $\mathfrak{M}^*[0,1]$ is isometric and lattice isomorphic to $C(K)$ [3]. Yu Sreider has shown [7] that each functional T in $\mathfrak{M}^*[0,1]$ can be represented as follows:

$$T(\mu) = \int_{[0,1]} f_\mu(t)\, d\mu(t),$$

where $f_\mu(t)$ is a "generalized function" meaning a function of points t in $[0,1]$ and of measures μ in $\mathfrak{M}[0,1]$.

A. P. Artemenko [1] proved that if $\{\mu_a\}_{a\in I} \subset \mathfrak{M}[0,1]$ is a maximal set of mutually singular measures (all measures of the form "a value at a point" belong to which), then

1) For each measure $\mu \in \mathfrak{M}[0,1]$ there exist measures $\nu_i \in \mathfrak{M}[0,1]$, $\nu_i \ll \mu_{a_i}$ such that $\mu = \sum_{i=1}^{\infty} \nu_i$.

2) For any functional $T \in \mathfrak{M}^*[0,1]$ there exist functions $f_a \in L_\infty(\mu_a)$ such that

$$T\mu = \sum_{i=1}^{\infty} \int_0^1 f_{a_i} d\mu \quad \text{where } \mu = \sum_{i=1}^{\infty} \nu_i \text{ and } \nu_i \ll \mu_{a_i}.$$

The purpose of this paper is to show that assuming the continuum hypothesis is true, each functional T in $\mathfrak{M}^*[0,1]$ can be represented as:

$$T(\mu) = \int_{[0,1]} \psi d\mu,$$

where ψ is a bounded real-valued function defined on the Borel subsets of $[0,1]$ and the integral is the limit of approximating sums on the directed set of subdivisions or partitions on $[0,1]$.

Remark. The techniques employed here can be extended to give an integral representation of the same type of the bounded linear functions on the space $\mathrm{ca}(S, \Sigma)$ of all real-valued countably additive set functions defined on a σ-algebra, Σ, of subsets of a set S, provided that the cardinality of $\mathrm{ca}(S, \Sigma)$ is $2^{\aleph_0}$.

DEFINITIONS. "*D is a subdivision of* $[0,1]$" means that D is a finite collection of disjoint Borel sets filling up the interval $[0,1]$ and "*D' refines D*" means D' is a subdivision of $[0,1]$ and each set in D' is a subset of some set in D[ ]. If ψ and μ are real-valued functions on B, then "*the number w is the integral of ψ with respect to μ*" means that if $\varepsilon > 0$, then there is a subdivision D of $[0,1]$ such that if D' refines D, then

$$\Big|\sum_{\text{all } B \text{ in } D'} \psi(B)\mu(B) - w\Big| < \varepsilon.$$

The integral of ψ with respect to μ is denoted by $\int_0^1 \psi d\mu$. This is an integral of the Kolmogorov–Burkhill type [6]. This integral is linear in both variables.

The main result of this paper is the following theorem.

THEOREM. *Suppose $2^{\aleph_0} = \aleph_1$. Then T is a bounded linear functional on $\mathfrak{M}[0,1]$ if and only if there is a bounded, real-valued function ψ defined on $\mathfrak{B}$ such that for each μ in $\mathfrak{M}[0,1]$, ψ is μ-integrable on $[0,1]$ and*

$$T(\mu) = \int_0^1 \psi d\mu. \tag{1}$$

Remark. If a functional T on $\mathfrak{M}[0,1]$ is defined by equation (1), where ψ is a bounded real-valued function on $\mathfrak{B}$, then T is linear and it is bounded, since

$$|T(\mu)| = \Big|\int_0^1 \psi d\mu\Big| \leqslant (\text{l. u. b}_{B \text{ in } \mathfrak{B}} |\psi(B)|)\cdot\|\mu\|.$$

In order to prove the converse, let $\{\mu_a\}_{a\in I} \subset \mathfrak{M}[0,1]$ be a maximal set of mutually singular measures. We can assume that the measures are positive. Since card $I = 2^{\aleph_0}$ and the continuum hypothesis is assumed, the index set I can be ordered into type Ω. Let $F = \{\mu = \sum_{i=1}^{n} \nu_i \colon \nu_i \ll \mu_{a_i}\}$.

Of course, F is dense in $\mathfrak{M}[0,1]$. Let $T \in \mathfrak{M}^*[0,1]$ be a non-negative functional, and $(f_a)_{a\in I}$ a sequence of functions defined by T (by Artemenko's characterization). Obviously, $f_a \geqslant 0$.

For each γ and α, $1 \leqslant \gamma < \alpha < \Omega$, let $B_{\gamma a}$ be a Borel set such that $\mu_\gamma(B_{\gamma a}) = 0$ and $\mu_a(B'_{\gamma a}) = 0$, where $B'_{\gamma a}$ denotes the complement of $B_{\gamma a}$. For each α, $1 < \alpha < \Omega$, let $B_a = \bigcap_{\gamma<a} B_{\gamma a}$; $\mu_\gamma(B_a) = 0$, if $\gamma < \alpha$ and $\mu_a(B'_a) = 0$.

If B is a Borel set and there is some α, $1 < \alpha < \Omega$ such that $B \subseteq B_a$ and $\mu_a(B) > 0$, then B does not have these properties with respect to any other ordinal number γ, $1 < \gamma < \Omega$ and $\mu_1(B) = 0$. It follows that the following function is well-defined for each Borel set B:

$$\psi(B) = \begin{cases} \text{g. l. b. } f_1(B), & \text{if } \mu_1(B) > 0, \\ \text{g. l. b. } f_a(B), & \text{if } B \subseteq B_a \text{ and } \mu_a(B) > 0 \\ & \text{for some } \alpha, 1 < \alpha < \Omega, \\ 0, & \text{otherwise}. \end{cases}$$

The function ψ is a nonnegative, real-valued function defined on B and $\psi(B) \leqslant |T|$, for each Borel set B.

Suppose ν is a nonnegative measure and $\nu \ll \mu_a$, for some α, $1 < \alpha < \Omega$. Let $\varepsilon > 0$ and let D be a subdivision of $[0,1]$ which is a refinement of the subdivision $\{B_a, B'_a\}$ and such that if D' refines D, then

$$\varepsilon > T(\nu) - \sum_{D'} (\text{g. l. b. } f_a(B))\nu(B).$$

Suppose D' refines D. If $\nu(B) > 0$, then $\mu_a(B) > 0$ and $B \subseteq B_a$. Hence, $\sum_{D'} (\text{g. l. b. } f_a(B))\nu(B) = \sum_{D'} \psi(B)\nu(B)$. Thus,

$$\varepsilon > \Big|T(\nu) - \sum_{D'} \psi(B)\nu(B)\Big|.$$

Using linearity arguments, it follows that ψ is integrable for all $\mu \in F$ and using convergence arguments, ψ is integrable for all $\mu \in \mathfrak{M}[0,1]$. Let $T'(\mu) = \int_0^1 \psi d\mu$ for $\mu \in \mathfrak{M}[0,1]$. Since $T' \in M^*[0,1]$ and $T(\mu) = T'(\mu)$ for $\mu \in F$, we have $T(\mu) = T(\mu') = \int_0^1 \psi d\mu$ for $\mu \in \mathfrak{M}[0,1]$.

The general representation theorem follows from the facts that every bounded linear functional on $\mathfrak{M}[0,1]$ is the difference of two nonnegative bounded linear functionals on $\mathfrak{M}[0,1]$ [4], and that the integral is linear in the first variable.

The author would like to thank the referee for his comments and suggestions.

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Received July 13, 1971 (515)

STUDIA MATHEMATICA, T. XLVI. (1973)

On the isomorphism of cartesian products of locally convex spaces

by

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Abstract. The following relation $\mathfrak{R}$ between topological linear spaces is studied: $(X, Y) \in \mathfrak{R}$ iff every continuous linear operator $T: X \to Y$ is compact. The results concerning the relation $\mathfrak{R}$ are applied to give conditions which guarantee that the isomorphism of certain product spaces $\mathop{\times}\limits_{i=1}^{\infty} X_i$ and $\mathop{\times}\limits_{i=1}^{\infty} Y_i$ implies near isomorphisms $X_i \approx Y_i$ (i.e. the existence of Fredholm operators from X_i onto Y_i) for $i = 1, 2, \ldots$, and to establish some criteria of quasi-equivalence of all bases is product spaces $X \times Y$.

§ 1. Let X and Y be locally convex spaces (lcs's) (1). A linear operator $T: X \to Y$ will be called a *near-isomorphism* (почти изоморфизм) if the following conditions are satisfied:

a) $T(X)$ is closed in Y and T is an open map from X onto $T(X)$,

b) $\alpha(T) = \dim \operatorname{Ker} T < \infty$,

c) $\beta(T) = \operatorname{codim} T(X) = \dim Y/T(X) < \infty$ (cf. [24])(2). The lcs's X and Y are said to be *nearly isomorphic* (почти изоморфным) ($X \approx Y$)(3) if there exists a near-isomorphism T from X onto Y.

In this paper we give some general conditions under which from (near) isomorphism cartesian products of lcs's $X_1 \times X_2$ and $Y_1 \times Y_2$ there follows that the factors are (near) isomorphic (Section II). The binary relation $(X, Y) \in \mathfrak{R}$ defined on the set of pairs of lcs's by the condition "every continuous linear operator from X to Y is compact" plays a very important role here. The greater part of this paper, Sections I, III is an examination of this relation. Our methods lead effectively to an answer to the question of the isomorphism of a wide class of spaces which are not distinguishable by their diametral dimension: $\Gamma(X_1 \times X_2) = \Gamma(Y_1 \times Y_2) = \Gamma(X_1)$, cf. [2], [17], [21]. In particular, we give a complete isomorphic classification of spaces of the form $X_1 \times X_2$, where X_i are finite or infinite centers of Riesz scales which are Montel spaces (§ 13).

(1) We consider only Hausdorff locally convex spaces.

(2) In [24] T is called an σ-map; one says also that T is a Fredholm operator or Φ-operator.

(3) If X and Y are isomorphic we shall write $X \simeq Y$.