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Optimal control by means of switchings

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Abstract. A stochastic control model is considered. A general theorem about the optimal strategy and the maximal reward is proved. In two special cases the optimal solutions are found effectively.

Introduction. Let $\{X^d\}_{d\in D}$ be a finite family of Markov processes and let non-negative functions f, c_d $(d\in D)$, be defined on the state space E. At time $t=0$, when being in a state $x\in E$ we choose a process X^{d_1}. The cost arising from this choice is equal to $c_{d_1}(x)$. We observe the process X^{d_1} and at the stopping time τ_1 we choose a process X^{d_2}. Our reward at any time $t<\tau_1$ is equal to

$$\int_0^t f(x_s^{d_1})\,ds - c_{d_1}(x).$$

Next we observe the process X^{d_2} and at the stopping time $\tau_2\geqslant\tau_1$ we choose a process X^{d_3}. At time $t\in[\tau_1,\tau_2)$ our reward is equal to

$$\Big(\int_0^{\tau_1} f(x_s^{d_1})\,ds - c_{d_1}(x)\Big) + \Big(\int_{\tau_1}^{t} f(x_s^{d_2})\,ds - c_{d_2}(x_{\tau_1}^{d_1})\Big).$$

Suppose that we can repeat these selections N times. What is the maximal total expected reward? Which strategies should be chosen to maximize the total expected reward starting in some state $x\in E$?

In this paper we prove a theorem which gives an answer to these questions. In two special cases we find effectively the maximal reward and optimal strategies.

The author of this paper wishes to thank Professor E. B. Dynkin whose questions, posed on his Seminar in Moscow, suggested the subject of this note.

The optimality theorem. To precise the above problems we shall formulate the described situation in terms of controlled Markov chains. To do this we assume that the Markov processes X^d are equal to $(\Omega, \boldsymbol{M}, \boldsymbol{M}_t, X_t, \Theta_t, P_x^d)$ where only P_x^d depends on d (see [1], p. 20).

If $d\in D$ and τ is a stopping time (with respect to $\{M_t\}$) we define transition probabilities:

$P^{(d,\tau)}(x, A) = P_x^d(x_\tau \in A)$, $x\in E$, A a Borel subset of E,

$P^{(d,\tau)}(\Delta, \{\Delta\}) = 1$, Δ an arbitrary element not in E,

and the reward function

$$f^{(d,\tau)}(x) = E_x^d\Big(\int_0^\tau f(x_s)\,ds\Big) - c_d(x),$$

$$f^{(d,\tau)}(\Delta) = 0.$$

A strategy π is a sequence

$$\pi = \big((\delta_n, \tau_n)\big) \stackrel{\mathrm{df}}{=} \big((\delta_1, \tau_1), (\delta_2, \tau_2), \ldots\big)$$

where $\delta_1, \delta_2, \ldots$ are Borel measurable functions from $E\cup\{\Delta\}$ into D, τ_1, $\tau_2, \ldots$ are stopping times with respect to $\{M_t\}$. We define also the operators $P^{(\delta,\tau)}$ by

$$P^{(\delta,\tau)}f(x) = \int_E P^{(\delta(x),\tau)}(x, dy)f(y), \qquad x\in E.$$

For any strategy π and any number $N < +\infty$ we have the total expected reward:

$$v_N^\pi = f^{(\delta_1,\tau_1)} + (P^{(\delta_1,\tau_1)})f^{(\delta_2,\tau_2)} + (P^{(\delta_{N-1},\tau_{N-1})}\circ \ldots \circ P^{(\delta_1,\tau_1)})f^{(\delta_N,\tau_N)}.$$

If the limit: $\lim_N v_N^\pi$ exists, we define

$$V_\infty^\pi = \lim_N v_N^\pi.$$

Finally we define

$$v_N = \sup_\pi v_N^\pi, \qquad v_\infty = \sup_\pi v_\infty^\pi.$$

We assume in the sequel that X^d $(d\in D)$ are standard processes with the same fine topology for which the excessive functions are Borel measurable.

For the notions of standard processes, fine topology, excessive functions and so on refer to see [1] or [2]. We shall need the following special case of a theorem of Dynkin (see [4]).

Theorem (Dynkin [3]). *Let X be a standard Markov process and let g be a bounded, finely continuous function on the state space E. Then v, where*

$$v(x) = \sup_\tau E_x\big(g(x_\tau)\big), \qquad x\in E,\ \tau \text{ stopping time},$$

is a finely continuous function and for every $\varepsilon > 0$ and for $\tau_\varepsilon = \inf\{t\geqslant 0;\ v(x_t)\leqslant g(x_t)+\varepsilon\}$ we have $v(x)-\varepsilon \leqslant E_x\big(g(x_{\tau_\varepsilon})\big)$, $x\in E$.

If $g\geqslant 0$ then v is the smallest excessive function for which $v\geqslant g$.

Now we shall prove the following theorem:

Theorem 1. *Let g and c_d $(d\in D)$ be non-negative Borel functions on $E\cup\{\Delta\}$ for which*

$$\sup_{x\in E, d\in D}\Big[E_x^d\Big(\int_0^{+\infty} g(x_s)\,ds\Big) + c_d(x)\Big] < +\infty,$$

holds, moreover let c_d $(d\in D)$ be finely continuous. Then

1° *v_N are bounded, finely continuous Borel functions,*

$$v_{N+1}\geqslant v_N \quad \text{for } N = 1, 2, \ldots,$$

and

$$\lim_N v_N = V_\infty.$$

2° $v_{N+1}(x) = \sup_{d,\tau}\big(f^{(n,\tau)}(x) + P^{(d,\tau)}v_N(x)\big)$, $x\in E$,

3° $v_\infty(x) = \sup_{d,\tau}\big(f^{(d,\tau)}(x) + P^{(d,\tau)}v_\infty(x)\big)$, $x\in E$.

Proof. We first prove by induction in N that v_N are bounded, finely continuous Borel functions, and that for any $\varepsilon > 0$ and N there exists a strategy $\bar\pi = \big((\bar\delta_n, \bar\tau_n)\big)$ such that

$$v_N^{\bar\pi}(x)\geqslant v_N(x)-\varepsilon \quad \text{for each } x\in E.$$

This is true when $N = 1$ because

$$v_1(x) = \max_d\big(G^d g(x) - c_d(x)\big), \quad \text{where} \quad G^d g(x) = E_x^d\Big(\int_0^{+\infty} g(x_s)\,ds\Big).$$

Suppose that the above statement is true for some N. For every strategy $\pi = \big((\delta_n, \tau_n)\big)$ we have

$$v_{N+1}(x) = f^{(\delta_1(x),\tau_1)}(x) + (P^{(\delta_1,\tau_1)})v_N^{\pi'}(x), \qquad x\in E$$

where

$$\pi' = \big((\delta_2, \tau_2), (\delta_3, \tau_3), \ldots\big).$$

Hence

$$v_{N+1}^\pi(x)\leqslant f^{(\delta_1(x),\tau_1)}(x) + P^{(\delta_1,\tau_1)}v_N(x)\leqslant \sup_{d,\tau}\big(f^{(d,\tau)}(x) + P^{(d,\tau)}v_N(x)\big)$$

and therefore

$$v_{N+1}\leqslant \sup_{d,\tau}\big(f^{(d,\tau)} + P^{(d,\tau)}v_N\big).$$

Let

$$\bar v_{N+1} = \sup_{d,\tau}\big(f^{(d,\tau)} + P^{(d,\tau)}v_N\big).$$

Since

$$\bar v_{N+1}(x) = \max_d\Big[\sup_\tau\big(E_x^d(v_N - G^d f)(x_\tau)\big) + G^d f(x) - c_d(x)\Big]$$

and in virtue of our supposition and Dynkin's theorem the function

$$\sup_{\tau}[E^{d}_{(.)}(v_N - G^d f)(x_\tau)]$$

is bounded, finely continuous and Borel measurable, thus also $\bar{v}_{N+1}$ is a bounded, finely continuous and Borel measurable function.

Dynkin's theorem implies again that we can find a stopping time $\tilde{\tau}$ and a Borel measurable function δ such that

$$f^{(\delta(x),\tilde{\tau})}(x) + P^{(\delta(x),\tau)}v_N(x) \geqslant \bar{v}_{N+1}(x) - \varepsilon, \qquad x \in E.$$

Thus

$$\bar{v}_{N+1} \leqslant v_{N+1}^{((\delta,\tilde{\tau}),(\delta,\tau_1),\ldots)} + 2\varepsilon \leqslant v_{N+1} + 2\varepsilon$$

and consequently $\bar{v}_{N+1} = v_{N+1}$. Induction is completed. Also 2° is thereby proved.

For

$$\pi = ((\delta_n, \tau_n)), \qquad \pi' = ((\delta'_n, \tau'_n))$$

where $(\delta'_n, \tau'_n) = (\delta_n, \tau_n)$ for $n \neq N$ and $\delta'_N = \delta_N$, $\tau'_N = +\infty$, we have

$$v_N^{\pi} \leqslant v_N^{\pi'} = v_{N+1}^{\pi'}$$

and so

$$v_N \leqslant v_{N+1}, \qquad N = 1, 2, \ldots$$

From the last inequality and the definitions we obtain

$$\lim_N v_N = v_\infty.$$

To prove 3° we notice that

$$v_\infty \geqslant v_{N+1} \geqslant f^{(d,\tau)} + P^{(d,\tau)}v_N, \qquad N = 1, 2, \ldots$$

This implies

$$v_\infty \geqslant f^{(d,\tau)} + P^{(d,\tau)}v_\infty, \qquad \text{for any } d, \tau,$$

hence

$$v_\infty \geqslant \sup_{d,\tau}[f^{(d,\tau)} + P^{(d,\tau)}v_\infty].$$

On the other hand

$$v_{N+1} = \sup_{d,\tau}[f^{(d,\tau)} + P^{(d,\tau)}v_N] \leqslant \sup_{d,\tau}[f^{(d,\tau)} + P^{(d,\tau)}v_\infty]$$

and hence

$$v_\infty \leqslant \sup_{d,\tau}(f^{(d,\tau)} + P^{(d,\tau)}v_\infty).$$

Thus 3° holds and the proof of Theorem 1 is complete.

Remarks. Theorem 1 is also true if the reward function is defined by

$$f^{d,\tau}(x) = E^d_x(A_\tau) - c_d(x),$$

where $A = (A_t)$ is an additive functional (see [1], p. 148) of X^d with bounded potentials $G^d(A_\infty)$:

$$G^d(A_\infty)(x) = E^d_x(A_\infty) \qquad \text{for each } d \in D,\ x \in E.$$

Dynkin's theorem and Theorem 1 give an algorithm to calculate $v_1, v_2, \ldots$ consecutively.

Two special cases. In the sequel we shall use the notations

$$E = [-\alpha, \alpha], \quad \alpha > 0; \qquad D = \{-1, 1\}.$$

X^{-1}, X^1 are Markov processes which are solutions of the equations

$$dX_t^{-1} = -dt + dw_t, \qquad dX_t^1 = dt + dw_t \qquad \text{in } (-\alpha, \alpha).$$

The states $-\alpha, \alpha$ are stopping points for X^{-1}, X^1.

$$c_{-1}(x) = c_1(x) = \begin{cases} c & \text{for } |x| < \alpha, \quad c > 0, \\ 0 & \text{for } |x| = \alpha. \end{cases}$$

For any sequence (γ_n), $0 \leqslant \gamma_n \leqslant \alpha$, $n = 1, 2, \ldots$ we define the strategy $\pi(\gamma_n)$:

$$\pi(\gamma_n) = ((\delta, \tau_{\delta\cdot\gamma_1}), (\delta, \tau_{\delta\cdot\gamma_2}), \ldots),$$

where

$$\delta(x) = \begin{cases} 1 & \text{for } x \in [-\alpha, 0], \\ -1 & \text{for } x \in (0, \alpha] \end{cases}$$

and

$$\tau_a = \inf\{t \geqslant 0;\ x_t = a,\ a \in [-\alpha, \alpha]\}.$$

When $\gamma_n = \gamma$ for $n = 1, 2, \ldots$ we put $\pi(\gamma) = \pi(\gamma_n)$.

Theorem 2 deals with the case when the reward is equal to the sejourn time in the interval $(-\alpha, \alpha)$ and Theorem 3 with the case when, roughly speaking

$$g(x) = \begin{cases} 0, & x \neq 0, \\ +\infty, & x = 0. \end{cases}$$

THEOREM 2. *Assume that $g(x) = 1$ for $|x| < \alpha$, $g(x) = 0$ for $|x| = \alpha$ and that $\bar{\gamma}$ is a positive solution of the equation*

$$\frac{e^{2\gamma} - e^{-2\gamma}}{e^{2\gamma} + e^{-2\gamma}} = 2\gamma - c.$$

1° *If $\bar{\gamma} < \alpha$ and $(e^{2\bar{\gamma}} + e^{-2\bar{\gamma}})^{-1} > 2\alpha(e^{2\alpha} - e^{-2\alpha})^{-1}$ then*

$$v_\infty(x) = v_\infty^{\pi(\bar{\gamma})}(x),$$

$$v_\infty(x) = |x| - (\alpha + c) + (e^{2\bar{\gamma}} + e^{-2\bar{\gamma}})^{-1}(e^{2\alpha} - e^{2|x|}) \qquad \textit{for } |x| < \alpha.$$

2° *If* $\bar{\gamma} \geq a$ *or* $\bar{\gamma} < a$ *and* $(e^{2\bar{\gamma}} + e^{-2\bar{\gamma}})^{-1} \leq 2a(e^{2a} - e^{-2a})^{-1}$ *then*

$$v_\infty(x) = v_\infty^{\pi(a)}(x),$$

$$v_\infty(x) = |x| - (a+c) + 2a\frac{e^{2a} - e^{2|x|}}{e^{2a} - e^{-2a}} \quad \textit{for } |x| < a.$$

Proof. In the first case it is sufficient to prove that the function $v_\infty^{\pi(\bar{\gamma})}(\cdot)$ satisfies equation 3° from Theorem 1 and in the second case that $v_\infty^{\pi(a)}(\cdot)$ satisfies the same equation. Indeed, it is easy to show by induction that then these functions are greater or equal to v_N $(N = 1, 2, \ldots)$ and therefore are greater or equal to v_∞.

To prove 1° we notice that for $\gamma \in (0, a)$ and $x \in (-a, 0)$ we have

$$v_\infty^{\pi(\gamma)}(x) = [m^1_{(-a,\gamma)}(x) - c] + P^1_{(-a,\gamma)}(x) v_\infty^{\pi(\gamma)}(\gamma),$$

where

$$\begin{aligned} m^d_{(a,b)}(x) &= E^d_x(\min(\tau_a, \tau_b)), \\ P^d_{(a,b)}(x) &= P^d_x(\tau_b < \tau_a) \end{aligned} \qquad \text{for } d \in D,\ x \in (a, b),\ -a \leq a < b \leq a.$$

But $v_\infty^{\pi(\gamma)}(\gamma) = v_\infty^{\pi(\gamma)}(-\gamma)$, therefore

$$v_\infty^{\pi(\gamma)}(-\gamma) = \left(m^1_{(-a,\gamma)}(-\gamma) - c\right) + P^1_{(-a,\gamma)}(-\gamma) v_\infty^{\pi(\gamma)}(-\gamma).$$

Thus

$$v_\infty^{\pi(\gamma)}(-\gamma) = \frac{m^1_{(-a,\gamma)}(-\gamma) - c}{1 - P^1_{(-a,\gamma)}(-\gamma)}$$

and for $x \in (-a, 0]$

$$v_\infty^{\pi(\gamma)}(x) = \left(m^1_{(-a,\gamma)}(x) - c\right) + P^1_{(-a,\gamma)}(x)\frac{m^1_{(-a,\gamma)}(-\gamma) - c}{1 - P^1_{(-a,\gamma)}(-\gamma)}.$$

Since for $x \in (a, b)$

$$m^1_{(a,b)}(x) = (b-a)p^1_{(a,b)}(x) - (x-a), \qquad p^1_{(a,b)}(x) = \frac{e^{-2a} - e^{-2x}}{e^{-2a} - e^{-2b}}$$

so

$$\begin{aligned} v_\infty^{\pi(\gamma)}(x) &= -x - (a+c) + p^1_{(-a,\gamma)}(x)\left(\gamma + a + \frac{\left(p^1_{(-a,\gamma)}(-\gamma)\right)(\gamma+a) + \gamma - a - c}{1 - p^1_{(-a,\gamma)}(-\gamma)}\right) \\ &= -x - (a+c) + p^1_{(-a,\gamma)}(x)\frac{2\gamma - c}{1 - p^1_{(-a,\gamma)}(-\gamma)} \\ &= -x - (a+c) + \frac{e^{2a} - e^{-2x}}{e^{2\gamma} - e^{-2\gamma}}(2\gamma - c). \end{aligned}$$

When x is fixed, this function takes on its maximum at the point $\bar{\gamma}$ (independently of x).

We shall prove in turn that for the function $v \overset{\mathrm{df}}{=} v_\infty^{\pi(\bar{\gamma})}$ we have

$$\text{(1)} \qquad \sup_\tau \left[E^1_x\left(\int_0^\tau g(x_s)\,ds + v(x_\tau)\right)\right] - c = v(x), \qquad x(-a, 0],$$

$$\text{(2)} \qquad \sup_\tau \left[E^{-1}_x\left(\int_0^\tau g(x_s)\,ds + v(x_\tau)\right)\right] - c \leq v(x), \qquad x \in (-a, 0].$$

According to Dynkin's theorem there exists a stopping time τ', which in fact is the first hitting time of some closed subset of $[-a, a]$, for which

$$\begin{aligned} E^1_x\left(\int_0^{\tau'} g(x_s)\,ds + v(x_{\tau'})\right) - c &= \sup_\tau \left[E^1_x\left(\int_0^\tau g(x_s)\,ds + v(x_\tau)\right)\right] - c \\ &= \sup_\tau [E^1_x(v - m^1_{(-a,a)})(x_\tau)] + m^1_{(-a,a)}(x) - c. \end{aligned}$$

Simple considerations show that τ' has the form $\tau' = \tau_{\gamma'}$, where $\gamma' \in [0, a]$. This is equivalent to

$$v_\infty^{\pi(\gamma', \bar{\gamma}, \bar{\gamma}, \bar{\gamma}, \ldots)} \geq v_\infty^{\pi(\bar{\gamma}, \bar{\gamma}, \bar{\gamma}, \ldots)}$$

and, by induction, to

$$v_\infty^{\pi(\overbrace{\gamma', \ldots, \gamma'}^{n\text{-times}}, \bar{\gamma}, \ldots)} \geq v_\infty^{\pi(\bar{\gamma}, \bar{\gamma}, \bar{\gamma}, \ldots)}.$$

Hence

$$v_\infty^{\pi(\gamma')} = \lim_n v_\infty^{\pi(\overbrace{\gamma', \ldots, \gamma'}^{n\text{-times}}, \bar{\gamma}, \ldots)} \geq v_\infty^{\pi(\bar{\gamma}, \bar{\gamma}, \ldots)} = v_\infty^{\pi(\bar{\gamma})}.$$

This gives $\gamma' = \bar{\gamma}$, hence (1) holds.

In view of (1), inequality (2) holds for $x \in [0, a)$ and the hitting time $\tau_{-\bar{\gamma}}$ is optimal for $x \in (-\bar{\gamma}, a)$.

Note that

$$\left(\frac{1}{2} \cdot \frac{d^2}{dx^2} + \frac{d}{dx}\right) v = -1 \qquad \text{for } x \in (-a, 0),$$

and

$$\left(\frac{1}{2} \cdot \frac{d^2}{dx^2} - \frac{d}{dx}\right) m^{-1}_{(-a,a)} = -1 \qquad \text{for } x \in (-a, a);$$

hence

$$\left(\frac{1}{2} \cdot \frac{d^2}{dx^2} - \frac{d}{dx}\right)(v - m^{-1}_{(-a,a)}) = -2\frac{d}{dx}v \qquad \text{for } x \in (-a, 0).$$

Therefore the function $v - m^{-1}_{(-a,a)}$ is superharmonic for X^{-1} (see [2], p. 513) in the interval $(-a, \tilde{\gamma})$ in which $\frac{d}{dx}v \geq 0$.

Since $2e^{-2\tilde{\gamma}} = e^{2\tilde{\gamma}} + e^{-2\bar{\gamma}}$ hence $-\bar{\gamma} < \tilde{\gamma}$.

But $\lim_{x\downarrow -a}(v-m^{-1}_{(-a,a)}+c)=0$ and consequently in $[-a,-\bar\gamma]$ the function $v-m^{-1}_{(-a,a)}+c$ is greater than $v-m^{-1}_{(-a,a)}$. This implies that

$$\sup_\tau\Big[E_x^{-1}\Big(\int_0^\tau g(x_s)\,ds+v(x_\tau)\Big)\Big]=\sup_\tau\,[E_x^{-1}(v-m^{-1}_{(-a,a)})(x_\tau)]+m^1_{(-a,a)}(x)$$
$$\leqslant\big(v(x)-m^{-1}_{(-a,a)}(x)+c\big)+m^{-1}_{(-a,a)}(x)\leqslant v(x)+c$$

and (2) holds in $[-a,-\bar\gamma]$.

To complete the proof of (2) it is sufficient to consider $x\in(-\bar\gamma,0)$. For these x we have

$$v(x)-\sup_\tau\Big[E_x^{-1}\Big(\int_0^\tau f(x_s)\,ds+v(x_\tau)\Big)\Big]+c$$
$$=v(x)-E_x^{-1}\Big(\int_0^{\tau_{-\bar\gamma}}f(x_s)\,ds+v(x_{\tau_{-\bar\gamma}})\Big)+c$$
$$=\big(p^1_{(-a,\bar\gamma)}(x)v(\bar\gamma)+m^1_{(-a,\bar\gamma)}(x)\big)-\big(p^1_{(-a,\bar\gamma)}(-x)v(\bar\gamma)+m^1_{(-a,\bar\gamma)}(-x)\big)$$
$$=\frac{v(\bar\gamma)+(\bar\gamma+a)}{e^{2a}-e^{-2\bar\gamma}}(e^{2x}-e^{-2x})-2x.$$

We need the inequality

$$\frac{v(\bar\gamma)+(\bar\gamma+a)}{e^{2a}-e^{-2\bar\gamma}}(e^{2x}-e^{-2x})-2x>0\quad\text{ for } x\in(-\bar\gamma,0].$$

The function $\dfrac{e^{-2x}-e^{2x}}{-2x}$ is decreasing for $x<0$ and therefore it suffices to show that

$$\lim_{x\downarrow-\bar\gamma}\frac{v(\bar\gamma)+(\bar\gamma+a)}{e^{2a}-e^{-2\bar\gamma}}\cdot\frac{e^{-2x}-e^{2x}}{-2x}<1.$$

The last inequality is true, since

$$\lim_{x\downarrow-\bar\gamma}\frac{v(\bar\gamma)+(\bar\gamma+a)}{e^{2a}-e^{-2\bar\gamma}}-(e^{2x}-e^{-2x})-2x=c>0.$$

We now assume that $\bar\gamma\geqslant0$ or $\bar\gamma<0$ and

$$(e^{2\bar\gamma}+e^{-2\bar\gamma})^{-1}\leqslant 2a(e^{2a}-e^{-2a})^{-1}.$$

In this case the function $v_\infty^{\pi(\cdot)}(x)$, for fixed x, takes on its maximum at $\gamma=a$. Assume that for $x\in(-a,a)$

$$v(x)\overset{\text{df}}{=}v_\infty^{\pi(a)}(x)=m^1_{(-a,a)}(-|x|)-c$$
$$=|x|-(a+c)+2a\frac{e^{2a}-e^{2|x|}}{e^{2a}-e^{-2a}}.$$

In the same way as above we prove that the function v satisfies equality (1).

Inequality (2) is also true, since

$$\big(m^1_{(-a,a)}(x)-c\big)-\big(m^1_{(-a,a)}(-x)-c\big)>0\quad\text{ for } x\in(-a,0).$$

This completes the proof.

THEOREM 3. *Assume that $(A_t)_{t\geqslant0}$ is a local time at 0 (see [1], p. 212).*

1° *If $c\geqslant1$ then*

$$v_\infty(x)=v_\infty^{\pi(a)}(x)\quad\textit{for } |x|<a.$$

Let $c<1$ and $\bar\gamma$ be a positive solution of the equation

$$\frac{1}{1-c}=\frac{e^{2\gamma}+e^{-2\gamma}}{2}.$$

2° *If $\bar\gamma<a$ and*

$$\frac{(1-e^{-2\bar\gamma})-c}{e^{2\bar\gamma}-e^{-2\bar\gamma}}>\frac{1-e^{-2a}}{e^{2a}-e^{-2a}}$$

then

$$v_\infty(x)=v_\infty^{\pi(a)}(x),$$
$$v_\infty(x)=(e^{2a}-e^{2|x|})\frac{(1-e^{-2\bar\gamma})}{e^{2\bar\gamma}-e^{-2\bar\gamma}}-c.$$

3° *If $\bar\gamma\geqslant a$ or*

$$\frac{(1-e^{-2\gamma})-c}{e^{2\bar\gamma}-e^{-2\bar\gamma}}\leqslant\frac{1-e^{-2a}}{e^{2a}-e^{-2a}}$$

then

$$v_\infty(x)=v_\infty^{\pi(a)}(x),$$
$$v_\infty(x)=(e^{2a}-e^{2|x|})\frac{1-e^{-2a}}{e^{2a}-e^{-2a}}\quad\textit{for } |x|<a.$$

Proof. The potential kernel (see [1], p. 69) of the process X^1 in the interval (a,b), where a, b are stopping points for X^1, has the form

$$G^1_{(a,b)}(x,y)=\begin{cases}p^1_{(a,b)}(x)G(y) & \text{for } a<x\leqslant y<b,\\ \big(1-p^1_{(y,b)}(x)\big)G(y) & \text{for } a<y\leqslant x<b,\end{cases}$$

where

$$G(y) = \frac{(e^{-2y}-e^{-2b})(e^{-2a}-e^{-2y})}{e^{-2y}(e^{-2a}-e^{-2b})}.$$

Moreover, if $0\in(a,b)$, then (see [1], p. 279)

$$E_x^1(A_\infty) = G^1_{(a,b)}(x,0) \quad \text{for } x\in(a,b).$$

Consequently, for $\gamma\in(0,a)$, $x\in(-a,0)$ it holds

$$v_\infty^{\pi(\gamma)}(x) = G^1_{(-a,\gamma)}(x,0)+p^1_{(-a,\gamma)}(x)\frac{G^1_{(-a,\gamma)}(-\gamma,0)-c}{1-p^1_{(-a,\gamma)}(-\gamma)}-c$$
$$= p^1_{(-a,0)}(x)\,[p^1_{(-a,\gamma)}(0)\,v_\infty^{\pi(\gamma)}(\gamma)+G^1_{(-a,\gamma)}(0,0)]-c.$$

Since

$$v_\infty^{\pi(\gamma)}(\gamma) = \frac{e^{2a}-e^{-2\gamma}}{e^{2\gamma}-e^{-2\gamma}}\left[\frac{e^{2a}-e^{2\gamma}}{e^{2a}-e^{-2\gamma}}(1-e^{-2\gamma})-c\right],$$

we have

$$v_\infty^{\pi(\gamma)}(x) = p^1_{(-a,0)}(x)\left[\frac{e^{2a}-1}{e^{2\gamma}-e^{-2\gamma}}\cdot\frac{e^{2a}-e^{2\gamma}}{e^{2a}-e^{-2\gamma}}(1-e^{-2\gamma})-\right.$$
$$\left.-\frac{c(e^{2a}-1)}{e^{2\gamma}-e^{-2\gamma}}+\frac{(1-e^{-2\gamma})(e^{2a}-1)}{e^{2a}-e^{-2\gamma}}\right]-c$$
$$= p^1_{(-a,0)}(x)\,(e^{2a}-1)\left[-\frac{c}{e^{2a}-e^{-2\gamma}}+\frac{(1-e^{-2\gamma})}{e^{2a}-e^{-2\gamma}}\left(\frac{e^{2a}-e^{2\gamma}}{e^{2\gamma}-e^{-2\gamma}}+1\right)\right]-c$$
$$= (e^{2a}-e^{-2x})\left[\frac{(1-e^{-2\gamma})-c}{e^{2\gamma}-e^{-2\gamma}}\right]-c \quad \text{for } x\in(-a,0).$$

We now assume $\bar\gamma<a$ and

$$\frac{(1-e^{-2\bar\gamma})-c}{e^{2\bar\gamma}-e^{-2\bar\gamma}} > \frac{1-e^{-2a}}{e^{2a}-e^{-2a}};$$

let

$$v \overset{\mathrm{df}}{=} v_\infty^{\pi(\bar\gamma)}.$$

We shall prove that v satisfies equation (3°) in Theorem 1. To prove this it is sufficient to show that

(3) $$\sup_\tau[E_x^1(A_\tau+v(x_\tau))]-c = v(x) \quad \text{for } x\in(-a,0),$$

(4) $$\sup_\tau[E_x^{-1}(A_\tau+v(x_\tau))]-c \leqslant v(x) \quad \text{for } x\in(-a,0).$$

The proof of equality (3) is analogous to that of equality (1) in Theorem 2 (the following equality is helpful in the proof:

$$\sup_\tau[E_x^1(A_\tau+v(x_\tau))] = \sup_\tau[E_x^1(v(x_\tau)-G^1_{(-a,a)}(x_\tau,0))]+G^1_{(-a,a)}(x,0)).$$

Now we shall deal with inequality (4). We restrict attention only to $x\in(-\bar\gamma,0]$. For these x we have

$$v(x)-\sup_\tau[E_x^{-1}(A_\tau+v(x_\tau))]+c$$
$$= \{p^1_{(-a,\bar\gamma)}(x)\,[v(\bar\gamma)-G^1_{(-a,a)}(\bar\gamma,0)]+G^1_{(-a,a)}(x,0)\}-$$
$$-\{p^1_{(-a,a)}(-x)\,[v(-\bar\gamma)-G^{-1}_{(-a,a)}(-\bar\gamma,0)]+G^{-1}_{(-a,a)}(x,0)\}$$
$$= [p^1_{(-a,\bar\gamma)}(x)-p^1_{(-a,\bar\gamma)}(-x)]\,[v(\bar\gamma)-G^1_{(-a,a)}(\bar\gamma,0)]+$$
$$+G^1_{(-a,a)}(x,0)-G^1_{(-a,a)}(-x,0).$$

Therefore we need only to show that

(5) $$[p^1_{(-a,\bar\gamma)}(x)-p^1_{(-a,\bar\gamma)}(-x)]\,[v(\bar\gamma)-G^1_{(-a,a)}(\bar\gamma,0)]+$$
$$+G^1_{(-a,a)}(x,0)-G^1_{(-a,a)}(-x,0)\geqslant 0.$$

Note that

$$\frac{p^1_{(-a,\bar\gamma)}(-x)-p^1_{(-a,\bar\gamma)}(x)}{G^1_{(-a,a)}(x,0)-G^1_{(-a,a)}(-x,0)} = \frac{e^{2a}-1}{G^1_{(-a,a)}(0,0)}\cdot\frac{e^{-2x}-e^{2x}}{(e^{2a}+1)-(e^{-2x}+e^{2a}e^{2x})}.$$

Since inequality (5) is true for x sufficiently close to $-\gamma$ and the function

$$\frac{e^{-2x}-e^{2x}}{(e^{2a}+1)-(e^{-2x}+e^{2a}e^{2x})}$$

is increasing in $(-a,0]$, thus inequality (5) holds for every $x\in(-\bar\gamma,0]$. The proof of 2° is complete.

The proof of 1° is easy and the proof of 3° is analogous to that of 2° and will be omitted.

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