

**Every separable Fréchet space contains
a non stable dense subspace**

by

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Abstract. Every separable Fréchet space contains a non stable dense subspace.

A topological vector space X over the field K of real or complex numbers is *stable* if it is isomorphic to each of its closed hyperplanes (kernel of a non zero continuous linear functional). It is well known that all closed hyperplanes of a given space are isomorphic, so X is stable if and only if it is isomorphic to one of its closed hyperplanes. Also it is obvious that X is stable if and only if X is isomorphic to $X \times K$.

It is still not known if every Banach space is stable. In 1961, Bessaga, Pełczyński and Rolewicz [1] gave an example of a nuclear Fréchet space which is not stable. Recently, Rolewicz [2] gave an example of a dense subspace of a separable Hilbert space which is not stable. In this note we use an entirely different method to extend Rolewicz's result and prove the theorem stated in the title of this paper. Thus it is seen that a positive answer to the stability problem for Banach spaces would have to use both the norm and the completeness.

Proof of theorem. First we review the example of [1]. Let $a^k = (a_n^k)_n$, $k = 1, 2, \dots$ be a sequence of sequences with $a_n^k = e^{k^2 2^n}$. Let

$$E = \bigcap_{k=1}^{\infty} \frac{1}{a^k} l_{\infty} = \{ \xi = (\xi_n) : (a_n^k \xi_n)_n \in l_{\infty} \text{ for } k = 1, 2, \dots \}.$$

Here l_{∞} denotes the Banach space of bounded sequences. Let the topology on E be given by the seminorms $(p_k)_k$ where $p_k(\xi) = \sup_n |a_n^k \xi_n|$.

Let $E_0 = \{ \xi = (\xi_n) \in E : \xi_1 = 0 \}$. Then it is easy to see that E, E_0 are Fréchet spaces and E_0 is a closed hyperplane in E . Moreover it is shown in [1] that E is not isomorphic to E_0 . (In [1] there is a typographical error: The quantity $e^{k^2 2^n}$ must be replaced by $e^{k^2 2^{2^n}}$).

Our theorem is obvious for finite dimensional Fréchet spaces (although in this case the dense subspace cannot be chosen to be proper) so we may assume that X is an infinite dimensional separable Fréchet space.

Since X is separable, we can find complete biorthogonal sequence (x_n, f_n) . That is, (x_n) is a total sequence in X and (f_n) is a sequence in X' with the property that $f_i(x_j) = \delta_{ij}$. Moreover, since X is metrisable, we can choose (x_n) to be a bounded sequence, that is, $\sup_n p(x_n) < \infty$ for any continuous seminorm p on X .

Define the map $A: E \rightarrow X$ by $A(\xi) = \sum_{n=1}^{\infty} \xi_n x_n e^{-n^2 2^n}$. Using the fact that (x_n) is bounded, we can compute, for any seminorm p on X

$$\begin{aligned} p(A(\xi)) &\leq \sum_{n=1}^{\infty} |\xi_n| p(x_n) \leq \sup_n p(x_n) \sum_{n=1}^{\infty} \frac{e^{2^n} |\xi_n|}{e^{2^{2n}}} \\ &\leq \left(\sup_n p(x_n) \sum_{n=1}^{\infty} e^{-2^{2n}} \right) p_1(\xi). \end{aligned}$$

Since X is complete it follows that A is defined and continuous. It is clearly linear and if $A(\xi) = 0$, then for each n , $0 = f_n A(\xi) = \xi_n e^{-n^2 2^n}$, so $\xi = 0$ and A is thus 1-1. Finally it is obvious that $A(E)$ is dense in X .

We complete the proof by showing that $A(E)$ is not stable. Let $A_0: E_0 \rightarrow X$ be the restriction of A to E_0 . Then clearly A_0 is again continuous and 1-1. Let $\xi \in E_0$. Then $f_1 A_0(\xi) = f_1 A(\xi) = 0$ so $A_0(\xi) \in \ker(f_1|A(E))$. Conversely, if $\xi \in E$ and $f_1 A(\xi) = 0$, then $\xi_1 = 0$ so $\xi \in E_0$. Thus we have shown that $A_0(E_0) = \ker(f_1|A(E))$. Since $f_1|A(E)$ is continuous and non zero (since $A(E)$ is dense in X) it follows that $A_0(E_0)$ is a closed hyperplane in $A(E)$.

Finally, suppose that $T: A(E) \rightarrow A_0(E_0)$ is an isomorphism. Then we have $A_0(E_0) = TA(E)$ so $A_0^{-1}TA$ is an algebraic isomorphism from E onto E_0 . If (ξ^v) is a sequence in E with $\lim \xi^v = \xi$ and $\lim A_0^{-1}TA(\xi^v) = \eta$, then $A_0(\eta) = \lim TA(\xi^v) = TA(\xi)$ so $\eta = A_0^{-1}TA(\xi)$. It follows from the closed graph theorem that $A_0^{-1}TA$ is continuous and from the open mapping theorem that it is an isomorphism. This contradicts the fact that E is not isomorphic to E_0 , so the proof is completed.

We remark that in the infinite dimensional case, the dense subspace can always be chosen to be proper. In fact, if we set $x = \sum_{n=1}^{\infty} e^{-n^2 2^n} x_n$, then it follows that $x \in X$, but $x \notin A(E)$ so $A(E)$ is proper in X .

References

- [1] C. Bessaga, A. Pełczyński, S. Rolewicz, *On diametral approximative dimension and linear homogeneity of F-spaces*, Bull. Acad. Pol. Sc. 9 (1961), pp. 677-683.
- [2] S. Rolewicz, *An example of a normed space non-isomorphic to its product by the real line*, Stud. Math. this volume, pp. 71-75

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(248)