

Limit and infinite integral of a Mikusiński operator

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Introduction. It is well known that the convolution quotients of Mikusiński have a double character. We may regard convolution quotients as operators and thus the theory of Mikusiński provides an algebraic foundation of Heaviside's operational calculus. On the other, convolution quotients may be regarded as generalizations of functions ([2]). In [9] is shown that every distribution with left-sided bounded carrier is an operator, but not every operator is a distribution. We may thus consider Mikusiński's convolution quotients as extensions not only of functions, but also of distributions with left-sided bounded carriers. In this paper we deal mainly with the function character of the convolution quotients. Therefore, the convolution quotients will be referred to here mostly as generalized functions and rarely as operators. We shall denote generalized functions (that is convolution quotients) by symbols $\{f(t)\}$, $\{g(t)\}$, etc. It should be remarked that this notation is purely symbolic and, in general, it is not allowed to substitute numbers for the variable t . For example, the Dirac delta function is $\{\delta(t)\} = 1$, where 1 is the unit element of the field of convolution quotients (i. e. of Mikusiński operators).

By the derivative of order n of the generalized function $\{f(t)\}$ we understand the generalized function $s^n\{f(t)\}$ where s is the differential operator, i. e. the convolution quotient $\{1\}/\{t\}$. In this case we write

$$(1) \quad \{f^{(n)}(t)\} = s^n\{f(t)\}.$$

The "indefinite integral" $\left\{ \int_{-\infty}^t f(\tau) d\tau \right\}$ of the generalized function $\{f(t)\}$ is defined by the equation

$$(2) \quad \left\{ \int_{-\infty}^t f(\tau) d\tau \right\} = \frac{1}{s} \{f(t)\}.$$

Thus, for example, $\left\{ \int_{-\infty}^t \delta(\tau) d\tau \right\} = \{H(t)\}$, where $H(t) = 0$ if $t < 0$ and $H(t) = 1$ if $t \geq 0$.

The "ordinary" product $\{-tf(t)\}$ of the generalized function $\{f(t)\}$ by the function $-t$ is defined as follows:

$$(3) \quad \{-tf(t)\} = D\{f(t)\},$$

where $D\{f(t)\}$ is the algebraic derivative of $\{f(t)\}$. For example: $\{-t\delta(t)\} = D1 = \{0\}$. The "ordinary" product $\{\varphi(t)f(t)\}$ of the generalized function $\{f(t)\}$ by a function $\varphi(t)$ can be defined similarly by means of an operator transformation commutable with D ([3]). However, we need here only (3).

The purpose of this paper is to introduce the notions of the limit and the definite integral for generalized functions (i. e. for Mikusiński operators), which are well known in the theory of distributions (see [4], [7]). In Section 1 we introduce the concept of the limit at ∞ for generalized functions. The limit at ∞ will be used in order to define an improper integral of generalized functions (Section 2).

The last section contains Tauberian theorems in which conclusions about the existence of the Lebesgue integral of a locally integrable function are drawn from the existence of the generalized integral with certain auxiliary conditions.

§ 1. The limit of a generalized function as $t \rightarrow \infty$. Let \mathcal{L} be the set of all in general complex-valued functions, vanished in $(-\infty, 0)$ and integrable in the Lebesgue sense in each finite segment $0 \leq t \leq T$. Let \mathcal{M} be the field of Mikusiński operators. $\{\varphi(t)\}_+$ denotes a function which is $\varphi(t)$ if $t \geq 0$ and 0 if $t < 0$.

THEOREM 1. *If $f(t)$ belongs to \mathcal{L} and if*

$$(1.1) \quad \lim_{t \rightarrow \infty} f(t) = a$$

exists, then the sequence of functions $\{f(nt)\}$ ($n = 1, 2, \dots$) is convergent in \mathcal{M} to the function $\{a\}_+ = \frac{a}{s}$. Moreover, the sequence of continuous functions

$$(1.2) \quad \{g_n(t)\} = \frac{1}{s}\{f(nt)\} = \left\{ \int_0^t f(n\tau) d\tau \right\}$$

is convergent to the limit

$$(1.3) \quad \{g(t)\} = \frac{a}{s^2} = \{at\}_+$$

uniformly in any finite interval $[0, T]$ ⁽¹⁾.

⁽¹⁾ An analogous theorem is proved by Marjanović [4].

Proof. Fix $\varepsilon > 0$ and $T > 0$. Then there is an integer N_1 so that

$$(1.4) \quad |f(t) - a| < \frac{\varepsilon}{2T}$$

whenever $t > N_1$.

Since $\{|f(t) - a|\}_+ \in \mathcal{L}$, there exist functions $\varphi_1(t)$ and $\varphi_2(t)$ such that (see [6], p. 45)

$$(i) \quad |f(t) - a| = \varphi_1(t) + \varphi_2(t) \quad \text{for} \quad 0 \leq t \leq N_1,$$

$$(ii) \quad \varphi_1(t) \text{ is bounded and integrable in } [0, N_1],$$

$$(iii) \quad \int_0^{N_1} |\varphi_2(\tau)| d\tau < \varepsilon.$$

We extend these functions to (N_1, ∞) such that $\varphi_2(t) = 0$ for $t > N_1$ and $\varphi_1(t) = |f(t) - a|$ for all $t > N_1$. Thus, in view of (1.4) and (ii), there is a constant $K > 0$ such that

$$(1.5) \quad |\varphi_1(t)| < K$$

for every $t \in [0, \infty)$. Let $0 \leq t \leq T$. If $0 \leq t \leq \varepsilon/4K$, then, by (i), (iii) and (1.5),

$$(1.6) \quad |g_n(t) - g(t)| = \left| \int_0^t [f(n\tau) - a] d\tau \right| \leq \int_0^t |f(n\tau) - a| d\tau \\ \leq \int_0^{\varepsilon/4K} |f(n\tau) - a| d\tau \leq \int_0^{\varepsilon/4K} |\varphi_1(n\tau)| d\tau + \int_0^{\varepsilon/4K} |\varphi_2(n\tau)| d\tau \\ \leq \int_0^{\varepsilon/4K} |\varphi_1(n\tau)| d\tau + \frac{1}{n} \int_0^{n\varepsilon/4K} |\varphi_2(\sigma)| d\sigma \\ \leq \frac{\varepsilon}{4K} K + \frac{1}{n} \int_0^{N_1} |\varphi_2(\sigma)| d\sigma < \frac{\varepsilon}{4} + \frac{\varepsilon}{n}.$$

Hence

$$(1.7) \quad |g_n(t) - g(t)| \leq \int_0^{\varepsilon/4K} |f(n\tau) - a| d\tau < \varepsilon/2 < \varepsilon$$

for all $n > 4$ and $t \in [0, \varepsilon/4K]$. If $T \geq t \geq \varepsilon/4K$, then

$$n\tau \geq \frac{4KN_1}{\varepsilon} \tau \geq \frac{4KN_1}{\varepsilon} \frac{\varepsilon}{4K} = N_1$$

for $n > 4KN_1/\varepsilon = N_2$. Thus, it follows from (1.4) and (1.7)

$$|g_n(t) - g(t)| \leq \int_0^t |f(n\tau) - a| d\tau = \int_0^{t/4K} |f(n\tau) - a| d\tau + \int_{t/4K}^t |f(n\tau) - a| d\tau$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2T} \left(t - \frac{\varepsilon}{4K} \right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2T} T = \varepsilon$$

for $n > \max(4, N_2) = N$. This proves the theorem.

Remark. If $\{f(nt)\} \rightarrow \{a\}_+$, then

$$s\{f(nt)\} \rightarrow s\{a\}_+ = s \frac{a}{s} = a.$$

Thus, in virtue of Theorem 1, we have the following statement: If $\lim_{t \rightarrow \infty} f(t)$ exists, then

$$(1.8) \quad \lim_{t \rightarrow \infty} f(t) = \lim_{n \rightarrow \infty} s\{f(nt)\}.$$

In connection with (1.8) we observe that the existence of the limit on the right, where the limit is understood in the operator sense, does not imply the existence of the ordinary limit on the left. Just this situation suggests a generalization of the ordinary concept of the limit.

Before we give the definition of the generalized limit, we make use of the operator transformation U_n . We shall need here certain properties of this transformation. The proofs of these are given in [3] and [5].

(I) The transformation U_n , defined for $n > 0$ as follows:

$$U_n(f) = \{nf(nt)\} \text{ if } f \in \mathcal{C}, \quad U_n\left(\frac{f}{g}\right) = \frac{U_n(f)}{U_n(g)} \quad \text{if } \frac{f}{g} \in \mathcal{M}, f, g \in \mathcal{C}.$$

(II) $U_n(xy) = U_n(x)U_n(y)$ for all $x, y \in \mathcal{M}$. (Multiplicativity.)

$$(III) \quad U_n[R(s)] = R\left(\frac{s}{n}\right) \quad \text{for} \quad R(s) = \frac{\alpha_0 + \dots + \alpha_k s^k}{\beta_0 + \dots + \beta_m s^m}$$

($\alpha_0, \dots, \alpha_k, \beta_0, \dots, \beta_m$ are numbers).

(IV) If, for any $y \in \mathcal{M}$,

$$\lim_{n \rightarrow \infty} U_n(y) = a$$

exists, then the limit a is always a number.

(V) $U_n D = n D U_n$.

We now turn to the definition of the generalized limit. Let $f \in \mathcal{C}$. Then, by (II) and (III), we have

$$s\{f(nt)\} = \frac{s}{n} \{nf(nt)\} = U_n(s) U_n(f) = U_n(sf).$$

Suppose that (1.8) holds, then we may write

$$(1.9) \quad \lim_{t \rightarrow \infty} f(t) = \lim_{n \rightarrow \infty} U_n(sf).$$

The expression on the right of (1.9) may have a reason even if f is not a function or else if $\lim_{t \rightarrow \infty} f(t)$ does not exist.

Definition 1. Let $\{f(t)\} = f$ be any generalized function such that the limit $\lim_{n \rightarrow \infty} U_n(sf) = a$ exists. The number a will be called the *limit of the generalized function* $f(t)$ as $t \rightarrow \infty$ and will be denoted by

$$(1.10) \quad \text{Lim} f(t) = \lim_{n \rightarrow \infty} U_n(sf).$$

The principal properties of the generalized limit are summarized in the following theorem in which $\{f(t)\}, \{g(t)\} \in \mathcal{M}$.

THEOREM 2. (i) $\lim_{t \rightarrow \infty} f(t)$, if it exists, is unique.

(ii) If $f(t) \in \mathcal{L}$ and $\lim_{t \rightarrow \infty} f(t) = a$ then $\text{Lim} f(t) = a$.

(iii) If $\text{Lim} f(t)$ and $\text{Lim} g(t)$ exist, then

$$\text{Lim}[f(t) + g(t)] = \text{Lim} f(t) + \text{Lim} g(t).$$

(iv) If $\lim_{t \rightarrow \infty} f(t)$ exists, then

$$\text{Lim} \lambda f(t) = \lambda \text{Lim} f(t)$$

for each number λ .

The proof of (ii) follows immediately from Theorem 1. The proof of the remaining properties is very simple and will be omitted here.

Several examples will be given to illustrate some further properties of the generalized limit.

EXAMPLE 1. If $\omega \neq 0$ (ω is a number), $\lim_{t \rightarrow \infty} \sin \omega t$ does not exist in the ordinary sense. However $\text{Lim} \sin \omega t = 0$. In fact,

$$\text{Lim} \sin \omega t = \lim_{n \rightarrow \infty} U_n[s\{\sin \omega t\}_+] = \lim_{n \rightarrow \infty} \frac{s}{n} \{n \sin \omega t\}_+ = 0$$

in consequence of

$$\frac{1}{s} \{\sin \omega t\}_+ = \left\{ \int_0^t \sin \omega \tau d\tau \right\}_+ = \left\{ \frac{1 - \cos \omega t}{\omega} \right\}_+ \neq 0.$$

EXAMPLE 2. $\lim_{t \rightarrow \infty} \cos \omega t = 0$ ($\omega \neq 0$); indeed, in view of (III),

$$\lim_{n \rightarrow \infty} U_n[s\{\cos \omega t\}_+] = \lim_{n \rightarrow \infty} U_n\left(\frac{s^2}{s^2 + \omega^2}\right) = \lim_{n \rightarrow \infty} \frac{\left(\frac{s}{n}\right)^2}{\left(\frac{s}{n}\right)^2 + \omega^2} = 0.$$

EXAMPLE 3. $\lim_{t \rightarrow \infty} e^t = 0$, indeed,

$$\lim_{t \rightarrow \infty} e^t = \lim_{n \rightarrow \infty} U_n[s\{e^t\}_+] = \lim_{n \rightarrow \infty} U_n\left(\frac{s}{s-1}\right) = \lim_{n \rightarrow \infty} \frac{\frac{s}{n}}{\frac{s}{n} - 1} = 0.$$

EXAMPLE 4. $\lim t$ fails to exist. Indeed,

$$U_n[s\{t\}_+] = U_n[\{1\}_+] = \{n\}_+$$

does not converge in the operator sense.

EXAMPLE 5. The sequence of the operators

$$U_n[s\{t^\alpha\}_+] = U_n\left[s\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}\right] = \Gamma(\alpha+1) U_n\left(\frac{1}{s^\alpha}\right) = \Gamma(\alpha+1) \frac{n^\alpha}{s^\alpha}$$

diverges obviously provided $\alpha > 0$. Therefore $\lim_{t \rightarrow \infty} t^\alpha$ does not exist for $\alpha > 0$.

EXAMPLE 6. To show that $\lim_{t \rightarrow \infty} \ln(t+1)$ fails to exist. Suppose, on the contrary, that

$$(1.11) \quad U_n[s\{\ln(t+1)\}_+] = U_n\left[\left\{\frac{1}{t+1}\right\}_+\right] = \left\{\frac{1}{t+1/n}\right\}_+ \rightarrow a$$

as $n \rightarrow \infty$, where a is necessarily a number. Hence, by the continuity of the algebraic derivation,

$$D\left\{\frac{1}{t+1/n}\right\}_+ \rightarrow D(a) = 0.$$

In order to obtain a contradiction observe that

$$\frac{1}{n} \left\{\frac{1}{t+1/n}\right\}_+ \rightarrow 0 \cdot a = 0,$$

in consequence of (1.11), and thus

$$D\left\{\frac{1}{t+1/n}\right\}_+ = \left\{\frac{-t}{t+1/n}\right\}_+ = \{-1\}_+ + \frac{1}{n} \left\{\frac{1}{t+1/n}\right\}_+ \rightarrow \{-1\}_+$$

as $n \rightarrow \infty$. This proves the statement.

§ 2. Infinite integral of a generalized function.

DEFINITION 2. Let $\{f(t)\}$ be any generalized function. If $\lim_{t \rightarrow \infty} \int_{-\infty}^t f(\tau) d\tau$ exists, then we say that the infinite integral $\int_{-\infty}^{\infty} f(\tau) d\tau$ exists and

$$(2.1) \quad \int_{-\infty}^{\infty} f(\tau) d\tau = \lim_{t \rightarrow \infty} \int_{-\infty}^t f(\tau) d\tau.$$

Applying (2) and (1.10), we may write (2.1) in the form

$$(2.2) \quad \int_{-\infty}^{\infty} f(\tau) d\tau = \lim_{n \rightarrow \infty} U_n(f).$$

As an immediate consequence of the Definition 2 we have

$$\int_{-\infty}^{\infty} [f(\tau) + g(\tau)] d\tau = \int_{-\infty}^{\infty} f(\tau) d\tau + \int_{-\infty}^{\infty} g(\tau) d\tau,$$

$$\int_{-\infty}^{\infty} \lambda f(\tau) d\tau = \lambda \int_{-\infty}^{\infty} f(\tau) d\tau$$

for each complex number λ provided that the integrals in sense of Definition 2 exist.

Since $e^{\lambda s}\{u(t)\} = \{u(t+\lambda)\}$ holds for all $u(t) \in U$ (see [5]) it is natural the following definition:

DEFINITION 3. By the shifted generalized function $\{f(t+\lambda)\}$ of the generalized function $\{f(t)\}$ we understand the generalized function

$$\{f(t+\lambda)\} = e^{\lambda s}\{f(t)\}.$$

THEOREM 3. Let $\{f(t)\}$ be any generalized function. If $\int_{-\infty}^{\infty} f(\tau) d\tau$ exists, then

$$\int_{-\infty}^{\infty} f(\tau+\lambda) d\tau = \int_{-\infty}^{\infty} f(\tau) d\tau$$

for each $\lambda \in (-\infty, \infty)$.

Proof. Since $\lim_{n \rightarrow \infty} e^{\frac{\lambda}{n}s} = 1$, it follows from the multiplicativity of U_n that

$$\begin{aligned} \int_{-\infty}^{\infty} f(\tau + \lambda) d\tau &= \lim_{n \rightarrow \infty} U_n(e^{\lambda s} f) = \lim_{n \rightarrow \infty} [U_n(e^{\lambda s}) U_n(f)] \\ &= \lim_{n \rightarrow \infty} e^{\frac{\lambda}{n}s} \lim_{n \rightarrow \infty} U_n(f) = \lim_{n \rightarrow \infty} U_n(f) = \int_{-\infty}^{\infty} f(\tau) d\tau, \end{aligned}$$

which proves the theorem.

Let \mathcal{M}_0 denote the set of all generalized functions $f = p/q \in \mathcal{M}$, where $p, q \in \mathcal{C}$ and q does not vanish identically in any right neighbourhood of 0. It can be verified that every generalized function $\{f(t)\} \in \mathcal{M}$ has a shifted generalized function $\{f(t - \lambda)\}$ by sufficient large $\lambda \geq 0$ such that $\{f(t - \lambda)\} \in \mathcal{M}_0$. Thus, in consequence of Theorem 3, we may restrict ourselves to the integrals of generalized functions of the set \mathcal{M}_0 .

Let $\{f(t)\}$ be a generalized function of \mathcal{M}_0 which is integrable in the sense of Definition 2. We shall use, in this case, the notation

$$\int_0^{\infty} f(\tau) d\tau = \int_{-\infty}^{\infty} f(\tau) d\tau.$$

THEOREM 4. If $f(t)$ belongs to \mathcal{L} and if the improper Lebesgue integral $\int_0^{\infty} f(\tau) d\tau$ exists, then

$$(2.3) \quad \int_0^{\infty} f(\tau) d\tau = \lim_{t \rightarrow \infty} \int_0^t f(\tau) d\tau.$$

This theorem follows immediately from Theorem 2 (ii).

The following examples show that $\int_0^{\infty} f(\tau) d\tau$ may exist in the sense of Definition 2 even though it fails to converge in the ordinary sense.

EXAMPLE 1. $\int_0^{\infty} \sin \omega \tau d\tau = \frac{1}{\omega}$ ($\omega \neq 0$). For,

$$\frac{1}{s^2} U_n \{\sin \omega t\}_+ = \frac{1}{s^2} \{n \sin \omega n t\}_+ = \left\{ \frac{t}{\omega} - \frac{\sin \omega n t}{\omega^2 n} \right\}_+ \Rightarrow \left\{ \frac{t}{\omega} \right\}_+ = \frac{1}{\omega} \frac{1}{s^2}$$

as $n \rightarrow \infty$.

EXAMPLE 2. $\int_0^{\infty} \cos \omega \tau d\tau = 0$ ($\omega \neq 0$). Indeed,

$$\frac{1}{s^2} U_n \{\cos \omega t\}_+ = \left\{ \frac{1 - \cos \omega n t}{\omega^2 n} \right\}_+ \Rightarrow 0$$

as $n \rightarrow \infty$.

EXAMPLE 3. $\int_0^{\infty} 1 d\tau$ fails to exist. In fact, $U_n(\{1\}_+) = \{n\}_+$ diverges.

EXAMPLE 4. The integral of the delta function exists and

$$\int_0^{\infty} \delta(\tau) d\tau = 1.$$

Since the unit element of \mathcal{M} is the delta function and $1 \in \mathcal{M}_0$ we have

$$\int_0^{\infty} \delta(\tau) d\tau = \lim_{n \rightarrow \infty} U_n(\{\delta(t)\}) = \lim_{n \rightarrow \infty} U_n(1) = \lim_{n \rightarrow \infty} 1 = 1.$$

EXAMPLE 5. $\int_0^{\infty} e^{\tau} d\tau = -1$.

Proof.

$$\int_0^{\infty} e^{\tau} d\tau = \lim_{n \rightarrow \infty} U_n(\{e^t\}_+) = \lim_{n \rightarrow \infty} U_n\left(\frac{1}{s-1}\right) = \lim_{n \rightarrow \infty} \frac{1}{s/n-1} = -1.$$

§ 3. Tauberian theorems. In this paragraph we will prove some theorems. Several of these theorems involve "Tauberian" inferences from the integrability in the sense of Definition 2 to the existence of the Lebesgue or improper Lebesgue integral, subject to auxiliary conditions.

We shall be concerned mainly with improper Lebesgue integral defined for $f(t) \in \mathcal{L}$ by the ordinary limit

$$(3.1) \quad \lim_{t \rightarrow \infty} \int_0^t f(\tau) d\tau = \int_0^{\infty} f(\tau) d\tau.$$

In this case $f(t)$ is referred to, simply, as integrable in $[0, \infty)$. Integrability in $[0, \infty)$ in the sense of Lebesgue demands more, then (3.1). $f(t)$ is said to be, in this case, Lebesgue integrable in $[0, \infty)$. If

$$(3.2) \quad \lim_{t \rightarrow \infty} \int_0^t f(\tau) d\tau = \int_0^{\infty} f(\tau) d\tau$$

exists, then $f(t)$ is called integrable in the operator sense.

We will first introduce a necessary condition for $f(t)$ to be integrable in $[0, \infty)$.

THEOREM 5. If $f(t)$ is integrable in $[0, \infty)$, then

$$(3.3) \quad \frac{1}{s^2} U_n(f) \Rightarrow \left\{ \int_0^{\infty} f(\tau) d\tau \cdot t \right\}_+$$

as $n \rightarrow \infty$. That is, the sequence of the continuous functions

$$\int_0^t \int_0^\tau n f(n\sigma) d\sigma d\tau \quad (n = 1, 2, \dots)$$

converges uniformly in each finite interval $[0, T]$ to the function At , where

$A = \int_0^\infty f(\tau) d\tau$ is the improper Lebesgue integral of $f(t)$.

Proof. The ordinary limit $\lim_{t \rightarrow \infty} F(t) = A$ exists for the continuous function $F(t) = \int_0^t f(\tau) d\tau$. Then, in virtue of Theorem 1,

$$\frac{1}{s^2} U_n(f) = \left\{ \int_0^t \int_0^\tau n f(n\sigma) d\sigma d\tau \right\}_+ = \left\{ \int_0^t F(n\tau) d\tau \right\}_+ \rightarrow \{At\}_+ \quad (n \rightarrow \infty).$$

Remark 1. This theorem asserts somewhat more than Theorem 4. In general a sequence of operators $a_n \in \mathcal{M}$ is said to be convergent to the operator $a \in \mathcal{M}$, if there exists a sequence of continuous functions $q_n \in \mathcal{C}$ ($n = 1, 2, \dots$) such that⁽²⁾

$$(A) \quad q_n \rightarrow q \neq 0 \quad (n \rightarrow \infty),$$

$$(B) \quad q_n a_n, q a \in \mathcal{C} \quad (n = 1, 2, \dots),$$

$$(C) \quad q_n a_n \rightarrow q a \quad (n \rightarrow \infty).$$

Thus Theorem 5 states that, under assumption of the integrability, it is possible to choose $q_n = \{t\}_+ = \frac{1}{s^2}$.

Remark 2. The Examples 1 and 2 of § 2 show that (3.3) is not a sufficient condition for $f(t)$ to be integrable in $[0, \infty)$,

We shall need the following definition:

DEFINITION 4. Let $a_n \in \mathcal{M}$ ($n = 1, 2, \dots$) be a sequence of operators. We say that the convergence of the sequence a_n is provable by the sequence of continuous functions $q_n \in \mathcal{C}$ ($n = 1, 2, \dots$) if the properties (A), (B) and (C) are fulfilled.

THEOREM 6. If $f(t)$ belongs to \mathcal{L} and $f(t) \geq 0$ for all t and if

$$\frac{1}{s^2} U_n(f) \rightarrow \{At\}_+$$

as $n \rightarrow \infty$, then $f(t)$ is Lebesgue integrable in $(0, \infty)$ and holds

$$A = \int_0^\infty f(\tau) d\tau.$$

⁽²⁾ The symbol $f_n \rightarrow f$ is to be understood that the sequence of functions f_n converges to f uniformly in every finite interval $[0, T]$ as $n \rightarrow \infty$.

This result follows, in an obvious way, from the following more general theorem:

THEOREM 7. Let $f(t)$ be a non-negative function of \mathcal{L} . If the convergence of $U_n(f)$ ($n = 1, 2, \dots$) is provable by a sequence of non-negative continuous functions q_n of \mathcal{C} , then $f(t)$ is Lebesgue integrable in $[0, \infty)$ and holds

$$\lim_{n \rightarrow \infty} U_n(f) = \int_0^\infty f(\tau) d\tau.$$

Proof. Let $A = \lim_{n \rightarrow \infty} U_n(f)$. Then, necessarily, A is a number (property (IV) of U_n). Since the convergence of $U_n(f)$ is provable by $q_n \geq 0$, we have

$$(3.4) \quad q_n \rightarrow q \geq 0 \quad (q \in \mathcal{C}, q \neq 0 \text{ in } [0, \infty)),$$

$$(3.5) \quad q_n U_n(f) \rightarrow Aq \quad (n \rightarrow \infty).$$

(3.5) can be written in the following form

$$(3.6) \quad \int_0^t q_n(t-\tau) n f(n\tau) d\tau = \int_0^{nt} q_n\left(t - \frac{\sigma}{n}\right) f(\sigma) d\sigma \rightarrow Aq(t).$$

Since $q_n(t) = 0$ if $t < 0$, (3.6) can be written in the form

$$(3.7) \quad \int_0^\infty q_n\left(t - \frac{\sigma}{n}\right) f(\sigma) d\sigma \rightarrow Aq(t) \quad (n \rightarrow \infty).$$

Let $t^* > 0$ be any point such that $q(t^*) > 0$. Then it follows from (3.4) that

$$(3.8) \quad q_n\left(t^* - \frac{\sigma}{n}\right) f(\sigma) \rightarrow q(t^*) f(\sigma)$$

for every fixed $\sigma \geq 0$ as $n \rightarrow \infty$. Furthermore, by hypothesis

$$q_n\left(t^* - \frac{\sigma}{n}\right) f(\sigma) \geq 0$$

for all $\sigma \geq 0$. It follows from (3.7) that

$$(3.9) \quad \int_0^\infty q_n\left(t^* - \frac{\sigma}{n}\right) f(\sigma) d\sigma \rightarrow Aq(t^*) \quad (n \rightarrow \infty).$$

Thus the assumptions of the Fatou lemma are satisfied ([6]). Consequently, $q(t^*) f(\sigma)$ is Lebesgue integrable in $[0, \infty)$.

Thus, considering that $q(t^*) \neq 0$, $f(\sigma)$ is Lebesgue integrable too. Then, in virtue of Theorem 4, $A = \lim_{n \rightarrow \infty} U_n(f) = \int_0^\infty f(\tau) d\tau$, and the proof of the theorem is completed.

It is an elementary fact that $\lim_{t \rightarrow \infty} t^a f(t) = 0$ ($f \in \mathcal{L}$), is sufficient for $f(t)$ to be integrable in $[0, \infty)$ provided $a > 1$. Obviously, the condition

$$(3.10) \quad \lim_{t \rightarrow \infty} t f(t) = 0 \quad (f \in \mathcal{L})$$

already does not imply the integrability of $f(t)$ in $[0, \infty)$.

However the following well known Tauberian theorem holds ([1], [8]):

THEOREM T. If $f(t)$ belongs to \mathcal{L} and if the integral

$$F(p) = \int_0^\infty f(\tau) e^{-p\tau} d\tau$$

converges for $p > 0$, then the conditions

$$\lim_{p \rightarrow 0+} F(p) = A, \quad \lim_{t \rightarrow \infty} t f(t) = 0$$

imply

$$F(0+) = \int_0^\infty f(\tau) d\tau = A.$$

We prove now a slightly similar result.

THEOREM 8. If $f(t)$ belongs to \mathcal{L} , then the conditions

$$(3.11) \quad \frac{1}{s^2} U_n(f) \rightarrow At \quad (n \rightarrow \infty),$$

$$\lim_{t \rightarrow \infty} t f(t) = 0,$$

imply

$$(3.12) \quad A = \lim_{t \rightarrow \infty} \int_0^t f(\tau) d\tau = \int_0^\infty f(\tau) d\tau.$$

Proof. Integration by parts gives

$$\frac{1}{s^2} U_n(f) = \int_0^t \int_0^{nr} f(\sigma) d\sigma d\tau = t \int_0^{nt} f(\sigma) d\sigma - \frac{1}{n} \int_0^{nt} \sigma f(\sigma) d\sigma.$$

Thus, by hypothesis,

$$(3.13) \quad t \int_0^{nt} f(\sigma) d\sigma - \frac{1}{n} \int_0^{nt} \sigma f(\sigma) d\sigma \rightarrow At \quad (n \rightarrow \infty).$$

Consequently, for each $\varepsilon > 0$ and each $T > 0$ there is an integer

$N = N(\varepsilon, T)$ so that

$$(3.14) \quad \left| t \int_0^{nt} f(\sigma) d\sigma - \frac{1}{n} \int_0^{nt} \sigma f(\sigma) d\sigma - At \right| < \varepsilon$$

whenever $n > N$ and $t \in [0, T]$.

Let $T \geq 2$ and let $1 \leq t \leq T$. Then

$$(3.15) \quad \left| \int_0^{nt} f(\sigma) d\sigma - \frac{1}{nt} \int_0^{nt} \sigma f(\sigma) d\sigma - A \right| < \frac{\varepsilon}{t} \leq \varepsilon$$

for $n > N$ and $1 \leq t \leq T$. Let $x > N+1$. If $n = [x]$, where $[x]$ is the greatest integer $\leq x$, then, obviously, $n = [x] > N$. If $t = x/[x]$, then

$$1 \leq \frac{x}{[x]} < \frac{[x]+1}{[x]} \leq 2 \leq T.$$

Thus, from (3.15), we get

$$(3.16) \quad \left| \int_0^x f(\sigma) d\sigma - \frac{1}{x} \int_0^x \sigma f(\sigma) d\sigma - A \right| < \varepsilon$$

for $x > N+1 = N_1$. It follows from (3.10) that

$$\lim_{x \rightarrow \infty} \frac{1}{x} \int_0^x \sigma f(\sigma) d\sigma = 0.$$

Thus, it follows from (3.16) that $\lim_{x \rightarrow \infty} \int_0^x f(\sigma) d\sigma = A$, so that the result is established.

We wish to obtain in the sequel the same conclusion as far as possible by weaker additional conditions. The condition (3.10) is not necessary for $f(t)$ to be integrable in $[0, \infty)$. Namely $\int_0^\infty f(\tau) d\tau$ may exist in the Lebesgue sense even if $\lim_{t \rightarrow \infty} t f(t)$ does not exist. The next theorem shows that the situation is different in the case of the generalized limit and generalized integral.

THEOREM 9. The necessary condition for the generalized function $f(t)$ to be integrable in $(-\infty, \infty)$ in the operator sense is that

$$(3.17) \quad \lim_{t \rightarrow \infty} t f(t) = 0.$$

Proof. Since $\int_{-\infty}^\infty f(\tau) d\tau = \lim_{n \rightarrow \infty} U_n(f) = a$ is a number, it follows, by the continuity of the algebraic derivation, that

$$D U_n(f) \rightarrow D(a) = 0 \quad (n \rightarrow \infty).$$

Hence, by the property (V) of U_n , using the definition (3) of the ordinary product, we get

$$\begin{aligned}\lim_{t \rightarrow \infty} tf(t) &= \lim_{n \rightarrow \infty} U_n[s\{tf(t)\}] = \lim_{n \rightarrow \infty} \frac{s}{n} U_n[-D(f)] \\ &= -\lim_{n \rightarrow \infty} \frac{s}{n} n D U_n(f) = -s \lim_{n \rightarrow \infty} D U_n(f) = 0,\end{aligned}$$

and the theorem is proved.

We see, by virtue of this theorem, that we cannot obtain a generalization of Theorem 8 supposing (3.17) instead of (3.10). The condition (3.17) is too weak, since (3.17) is not any restriction under the hypothesis (3.11).

The condition (3.17) is equivalent to

$$(3.18) \quad \lim_{n \rightarrow \infty} U_n[s\{tf(t)\}] = 0.$$

In the following theorem we shall suppose

$$(3.19) \quad \frac{1}{s^2} U_n[s\{tf(t)\}] \rightarrow 0 \quad (n \rightarrow \infty)$$

instead of (3.10). The condition (3.19) is weaker than (3.10), but stronger than (3.18). In this case also the condition (3.11) may be relaxed somewhat. We note that (3.19) is expressible in the form

$$\frac{1}{s^2} U_n[s\{tf(t)\}] = \frac{1}{s} \{ntf(nt)\} = \left\{ \frac{1}{n} \int_0^{nt} \sigma f(\sigma) d\sigma \right\} \rightarrow 0.$$

We introduce first the following definition:

DEFINITION 5. Let

$$P(t) = a_0 + a_1 t + \dots + a_m t^m$$

be any polynomial, where the coefficients a_k are numbers. Let $T > 0$. The real number $|a_0| + |a_1|T + \dots + |a_m|T^m$ is called the *norm* of $P(t)$ in the interval $[0, T]$ and will be denoted by

$$\|P(t)\|_T = |a_0| + |a_1|T + \dots + |a_m|T^m.$$

THEOREM 10. Let $f(t)$ be a function of \mathcal{L} such that

$$(3.20) \quad \frac{1}{n} \int_0^{nt} \sigma f(\sigma) d\sigma \rightarrow 0 \quad (n \rightarrow \infty).$$

Let the convergence of $U_n(f) = \{nf(nt)\}$ ($n = 1, 2, \dots$), be provable by a sequence of polynomials $P_n(t)$. If for each $T > 0$ there is a positive num-

ber $K(T)$ depending only on T so that

$$(3.21) \quad \left\| \frac{dP_n(t)}{dt} \right\|_T < K(T)$$

for each integer $n > 0$, then $f(t)$ is integrable in $[0, \infty)$ and

$$\lim_{t \rightarrow \infty} \int_0^t f(\tau) d\tau = \lim_{n \rightarrow \infty} U_n(f).$$

We need the following lemma:

LEMMA 1. Let the positive constant M be such that

$$(3.22) \quad \left| \frac{1}{n} \int_0^{nt} \sigma f(\sigma) d\sigma \right| \leq M$$

for some $n > 0$ and for all $0 \leq t \leq T$. Then

$$(3.23) \quad \left| \int_0^{nt} P\left(t - \frac{\sigma}{n}\right) f(\sigma) d\sigma - P(t) \int_0^t f(\sigma) d\sigma \right| \leq M \left\| \frac{dP(t)}{dt} \right\|_T$$

for any polynomial $P(t)$ whenever $t \in [0, T]$.

Proof. Suppose first that $P(t) = t^k$, where k is a non-negative integer. Since

$$\left\{ \int_0^{nt} \left(t - \frac{\sigma}{n}\right)^k f(\sigma) d\sigma \right\} = \left\{ \int_0^t (t - \tau)^k n f(n\tau) d\tau \right\} = \{t^k\}_+ U_n(f) = \frac{k!}{s^{k+1}} U_n(f)$$

and

$$\left\{ t^k \int_0^{nt} f(\sigma) d\sigma \right\} = (-1)^k D^k \left[\frac{1}{s} U_n(f) \right],$$

we obtain (3.23) for $P(t) = t^k$ from

$$(3.24) \quad \left| \frac{k!}{s^{k+1}} U_n(f) - (-1)^k D^k \left[\frac{1}{s} U_n(f) \right] \right| \leq M k t^{k-1} \leq M k T^{k-1}.$$

We prove (3.24) by induction. Obviously, (3.24) holds for $k = 0$. Suppose that (3.24) holds for $k \geq 0$. Since $t \geq 0$, it follows from (3.24) that

$$(3.25) \quad \left| \frac{(k+1)!}{s^{k+2}} U_n(f) - \frac{k!}{s^k} \frac{1}{s} D U_n(f) - (-1)^{k+1} D^{k+1} \left[\frac{1}{s} U_n(f) \right] \right| \leq M k t^k.$$

For the sake of brevity, set

$$\varphi_n = \frac{1}{s} D U_n(f) = \left\{ -\frac{1}{n} \int_0^{nt} \sigma f(\sigma) d\sigma \right\}.$$

Thus, we get from (3.25) and (3.22), that

$$\left| \frac{(k+1)!}{s^{k+2}} U_n(f) - (-1)^{k+1} D^{k+1} \left[\frac{1}{s} U_n(f) \right] \right| \leq Mkt^k + \left| \int_0^t q_n(t-\tau) k\tau^{k-1} d\tau \right| \leq Mkt^k + Mt^k = M(k+1)t^k \leq M(k+1)T^k.$$

Thus (3.24) is proved. Multiplied both sides of (3.24) with $|a_k|$ and summarized with respect to k , we obtain (3.23) and the lemma is proved.

We now pass to the proof of Theorem 10. Let $P_n(t)$ ($n = 1, 2, \dots$) be a sequence of polynomials having the property (3.21) so that the convergence of $U_n(f)$ is provable by the sequence of P_n . Then there exists a function g of \mathcal{C} such that $g \neq 0$ and

$$(3.26) \quad P_n(t) \rightarrow g(t) \quad (n \rightarrow \infty).$$

Furthermore

$$(3.27) \quad \{P_n(t)\}_+ U_n(f) \rightarrow \{ag(t)\}_+ \quad (n \rightarrow \infty),$$

where $a = \lim_{n \rightarrow \infty} U_n(f)$.

We need to show that for each $\varepsilon > 0$ there is a number $N = N(\varepsilon)$ so that

$$(*) \quad \left| a - \int_0^x f(\sigma) d\sigma \right| < \varepsilon$$

whenever $x > N$.

Since g is continuous in $[0, \infty)$ and $g(t) \neq 0$, there is an interval $0 < t_1 \leq t \leq t_2$ such that $g(t) \neq 0$ whenever $t \in [t_1, t_2]$. We may assume, without restriction, that $g(t) > 0$ for $t \in [t_1, t_2]$. Else set $-P_n$ instead of P_n . Let m be a positive number such that

$$(3.28) \quad 0 < m < g(t)$$

for $t_1 \leq t \leq t_2$. Let $\varepsilon^* > 0$ be any number such that

$$(3.29) \quad 0 < \varepsilon^* \frac{1 + |a|}{m - \varepsilon^*} < \varepsilon.$$

Choose a $T > t_2$ and let $0 \leq t \leq T$. In view of (3.20) there is a number N_1 so that

$$\left| \frac{1}{n} \int_0^{nt} af(\sigma) d\sigma \right| < \frac{\varepsilon^*}{2K(T)}$$

whenever $n > N_1$ and $0 \leq t \leq T$. Then, in virtue of Lemma 1, according to (3.21), we have

$$(3.30) \quad \left| \int_0^{nt} P_n \left(t - \frac{\sigma}{n} \right) f(\sigma) d\sigma - P_n(t) \int_0^{nt} f(\sigma) d\sigma \right| \leq \frac{\varepsilon^*}{2K(T)} \left\| \frac{dP_n(t)}{dt} \right\|_T < \frac{\varepsilon^*}{2}.$$

In view of (3.27) there is a number N_2 so that

$$(3.31) \quad \left| ag(t) - \int_0^{nt} P_n \left(t - \frac{\sigma}{n} \right) f(\sigma) d\sigma \right| < \frac{\varepsilon^*}{2}$$

whenever $n > N_2$ and $0 \leq t \leq T$. Let $n > \max(N_1, N_2) = N_3$. Then, by (3.30) and (3.31), we have

$$(3.32) \quad \left| ag(t) - P_n(t) \int_0^{nt} f(\sigma) d\sigma \right| < \varepsilon^*$$

for all $n > N_3$ and $t \in [0, T]$.

Let N_4 be a number so that

$$(3.33) \quad |P_n(t) - g(t)| < \varepsilon^*$$

for all $n > N_4$ and $t \in [0, T]$. The existence of such number follows from (3.26).

Consequently, by (3.28) and (3.29), we have

$$(3.34) \quad 0 < m - \varepsilon^* < g(t) - \varepsilon^* < P_n(t)$$

for all $n > N_4$ and $t_1 \leq t \leq t_2$. It follows from (3.33) that

$$(3.35) \quad |aP_n(t) - ag(t)| < |a| \varepsilon^*$$

for each $n > N_4$ and $t \in [0, T]$. Let $n > \max(N_3, N_4) = N_5$. Then (3.32) and (3.35) imply that

$$(3.36) \quad \left| aP_n(t) - P_n(t) \int_0^{nt} f(\sigma) d\sigma \right| < \varepsilon^* + |a| \varepsilon^*$$

for all $n > N_5$ and $t \in [0, T]$. If we let $t_1 \leq t \leq t_2$, we obtain from (3.34), (3.36) and (3.29) that

$$(3.37) \quad \left| a - \int_0^{nt} f(\sigma) d\sigma \right| < \frac{\varepsilon^* + |a| \varepsilon^*}{P_n(t)} < \frac{\varepsilon^* + |a| \varepsilon^*}{m - \varepsilon^*} < \varepsilon$$

for all $n > N_5$ and $t_1 \leq t \leq t_2$. Let

$$(3.38) \quad N = \max \left(t_2 N_5, \frac{t_1 t_2}{t_2 - t_1} \right) \quad \text{and } x > N.$$

Since, in view of (3.38),

$$\frac{x}{t_1} - \frac{x}{t_2} = x \frac{t_2 - t_1}{t_1 t_2} > N \frac{t_2 - t_1}{t_1 t_2} \geq 1,$$

there is an integer n_0 such that

$$(3.39) \quad \frac{x}{t_2} \leq n_0 \leq \frac{x}{t_1}.$$

Thus, by (3.38), we have

$$(3.40) \quad n_0 \geq \frac{x}{t_2} > \frac{N}{t_2} \geq N_5.$$

It follows from (3.39) that $t_0 = x/n_0 \in [t_1, t_2]$. Consequently (3.37) holds for $n = n_0$ and $t = t_0$, that is (*) holds for all $x > N$. The theorem is proved.

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Sequence spaces and interpolation problems for analytic functions

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§ 1. INTRODUCTION

1.1. DEFINITION. Let $w = \{z_n\}$ be a sequence of distinct points in the disk $D = \{z: |z| < 1\}$ with $|z_n| \rightarrow 1$. For $1 \leq p \leq \infty$ let H^p be the usual Hardy class of analytic functions on D with boundary values in L^p . Let $H^p(w) = \{f(z_n): f \in H^p\}$.

The purpose of the present work is three-fold. First, an examination of the sequence space structure of $H^p(w)$ is given. Then in the context of general FK spaces some results, many of which were suggested by properties of $H^p(w)$, are considered. In particular the conull property of FK spaces is examined. (See [6] and [10] for previous work on the conull property. J. Sember in [4] studied the conull property in its relation to variation matrices.) Finally, it is shown that there exists a sequence w such that $H^2(w)$ contains all bounded sequences and $H^\infty(w)$ does not, answering a natural question on interpolation by analytic functions.

In § 3 it is shown that $H^p(w)$ is a BK space. If $p < \infty$, then $H^p(w)$ has the AD property. If $1 < p < \infty$, then the coordinate projections are fundamental in $H^p(w)^*$, but $H^1(w)^*$ is not separable.

In § 4 $H^p(w)$ is considered in the context of the conull, conservative, coercive, and wedge properties, and in terms of three new sequence space properties. In particular, it is shown that $H^p(w)$, for $1 < p < \infty$, is conull if and only if $H^p(w)$ contains every sequence of bounded variation. The fact that $H^p(w)$ may be coregular for $p < \infty$ shows that Theorem 6 of [6] fails in the context of non-conservative spaces. (Recently, J. Sember has announced an essentially different example of this failure. See 4.12 for an outline of some of his results.)

Let δ^n denote the sequence $\delta_k^n = 0$ for $k \neq n$, $\delta_n^n = 1$. It is shown in § 5 that $\{\delta^n\}$ is a basis for $H^p(w)$, $p < \infty$, if and only if $w = \{z_n\}$ is an interpolating sequence.