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Equivalent nuclear systems

by

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In this paper we introduce the notion of nuclear system as a means of constructing nuclear Fréchet spaces whose topologies are defined by a family of seminorms which are actually norms. We then show that all such spaces are obtained by this construction. The main result (Theorem 2) is an "intrinsic" characterization of when two nuclear systems are equivalent, that is when the spaces which they construct are isomorphic. This result is then applied to the basis problem for nuclear Fréchet spaces. Finally some examples and open questions are listed.

This method of constructing nuclear Fréchet spaces gives rise to examples which have not previously been discussed as well as providing a new way of studying the familiar spaces. These examples will be discussed in detail in a forthcoming paper.

Let $A_k\colon l_2 \to l_2$, $k = 1, 2, \ldots$, be a sequence of nuclear maps and define the *associated space*,

$$\hat{E} = \hat{E}\{(A_k)\} = \{(x_k)_k\colon x_k \in l_2,\ x_k = A_k x_{k+1},\ k = 1, 2, \ldots\}.$$

Thus $\hat{E}$ is a subspace of the countable product of copies of l_2, and we may equip $\hat{E}$ with the topology induced by the usual product topology. Let $P_k\colon \hat{E} \to l_2$ by $P_k\big((x_k)_k\big) = x_k$. We call $(A_k)_k$ a *nuclear system* if

(i) each A_k has dense range

(ii) each P_k is injective.

THEOREM 1. *The associated space of nuclear system is a nuclear Fréchet space with a fundamental sequence of seminorms which are norms; and, conversely, every such space is the associated space of a nuclear system (up to isomorphism).*

Proof. Clearly, $\hat{E}\{(A_k)\}$ is nothing more than the projective limit of the sequence of maps, $(A_k)_k$ and hence it is a Fréchet space. Evidently, a fundamental system of neighborhoods of 0 is given by the sets

$$V_n = \left\{(x_k) \in \hat{E}\colon \|x_k\| \leqslant \frac{1}{n},\ k = 1, 2, \ldots, n\right\}, \quad n = 1, 2, \ldots,$$

and it follows immediately from (ii) that the gage of each V_n is a norm. To see that $\hat{E}$ is nuclear, set

$$\|x\|_n = n\,\mathrm{gage}(V_n) = \max\{\|x_1\|, \ldots, \|x_n\|\}, \quad x = (x_k) \in \hat{E},\ n = 1, 2, \ldots$$

Then it suffices to show that any $\|\cdot\|_{n+1}$-summable sequence, (x^ν), in $\hat{E}$, is $\|\cdot\|_n$-absolutely summable.

Now if σ runs through the collection of finite sets of positive integers, ordered by inclusion, and $S_\sigma = \sum_{\nu\in\sigma} x^\nu$, then the net $(S_\sigma)_\sigma$ is $\|\cdot\|_{n+1}$-Cauchy. Hence for each $k \leqslant n+1$, the net $(P_k(S_\sigma))_\sigma$ is Cauchy. In other words, if $x^\nu = (x_k^\nu)_k$, then for each $k \leqslant n+1$, $(x_k^\nu)_\nu$ is a summable family in l_2 so $(A_k x_k^\nu)_\nu$ is absolutely summable for $k \leqslant n+1$, so $(x_k^\nu)_\nu$ is absolutely summable for each $k \leqslant n$, so $(x^\nu)_\nu$ is $\|\cdot\|_n$-absolutely summable.

For the converse let E be given. Since E is nuclear Fréchet, it is separable, so if V is a barreled neighborhood of 0, then E_V is also separable. Thus we can choose a decreasing fundamental sequence of neighborhoods of 0, $(V_k)_k$, such that

1° the gage of each V_k is a norm,

2° each $\hat{E}_{V_k}$ is a separable Hilbert space,

3° the canonical map $\eta_k\colon \hat{E}_{V_{k+1}} \to \hat{E}_{V_k}$ is nuclear for $k = 1, 2, \ldots$

Then if $\theta_k\colon \hat{E}_{V_k} \to l_2$ is an isometry and $A_k = \theta_k \eta_k \theta_{k+1}^{-1}$, it is easy to check that $(A_k)_k$ is a nuclear system. Finally, it is well known (and straightforward to show) from the theory of projective limits that E is isomorphic to $\hat{E}\{(A_k)\}$.

This completes the proof of the theorem.

It is easy to see that in the definition of nuclear system, condition (ii) is satisfied if each A_k is injective. If this is so, we shall say that the nuclear system is *injective*. This is not always the case, in fact we now give an example of a nuclear system in which no A_k is injective.

Let e^n, $n = 1, 2, \ldots$, be the usual basis for l_2 and define $A\colon l_2 \to l_2$ by

$$Ae^n = \begin{cases} \dfrac{1}{4} e_1 & \text{if } n = 1, \\[2ex] \dfrac{1}{(n+1)^2} e^n - \dfrac{1}{(n-1)^2} e^{n-1} & \text{if } n > 1. \end{cases}$$

It is easy to see that A is nuclear and also $e^n \in A(l_2)$ for $n = 1, 2, \ldots$, so A has dense range. Thus if we take $A_k = A$ for $k = 1, 2, \ldots$, then we need only check condition (ii). In fact, we first observe that the kernel of A is the subspace of l_2 generated by the vector $x^0 = (1/n^2)_n$. Thus, if $(x_k) \in \hat{E}\{(A_k)\}$ and $x_k = 0$ for some k, then $Ax_{k+1} = 0$ so we have a scalar

$\lambda \neq 0$ with $x_{k+1} = \lambda x^0$. Hence

$$x^0 = A\left(\frac{1}{\lambda} x_{k+2}\right) \in A(l_2).$$

But suppose that $x^0 = Ax$, $x = (x_n) \in l_2$. Then by definition of A,

$$\frac{x_n}{(n+1)^2} - \frac{x_{n+1}}{n^2} = \frac{1}{n^2}, \quad n = 1, 2, \ldots,$$

so

$$x_n - \left(\frac{n+1}{n}\right)^2 x_{n+1} = \left(\frac{n+1}{n}\right)^2, \quad n = 1, 2, \ldots$$

Passing to the limit and using the fact that $\lim_n x_n = 0$, we obtain

$$0 = \lim_n x_n = 1,$$

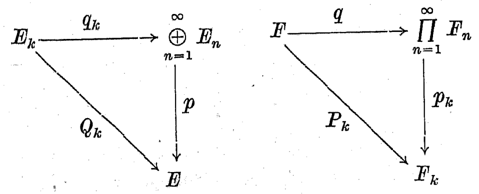
which is a contradiction. Thus each P_k is injective. On the other hand each $A_k = A$ has a non-trivial kernel and the desired example has been obtained.

It is natural to ask if every nuclear Fréchet space with a fundamental sequence of norms is the associated space of an *injective* nuclear system. To study this question we define two nuclear systems to be equivalent if their associated spaces are isomorphic. The main result of this paper will be to give an "intrinsic" characterization of equivalence and apply this to the basis problem for Fréchet nuclear spaces. To do this we must first show that the range of each P_k is dense.

LEMMA. *Let $B_k\colon E_k \to E_{k+1}$, $k = 1, 2, \ldots$, be a sequence of injective linear continuous maps of reflexive Banach spaces. Then the inductive limit of these maps is Hausdorff.*

Proof. This result is well known. For a proof see [1].

We now wish to make use of the duality between inductive and projective limits. To do this precisely, some notation is necessary. Let F_k be the strong dual of E_k and $A_k\colon F_{k+1} \to F_k$ the adjoint of B_k. Then by reflexivity, E_k is the strong dual of F_k and B_k is the adjoint of A_k. Let E be the inductive limit of (B_k) and F the projective limit of (A_k). Then the situation is schematized in the diagrams



where the injections q, q_k, Q_k and projections p, p_k, Q_k have the usual meanings. The well known fact which we wish to use is that F is the dual of E in the sense that all elements of E' are obtained uniquely by taking elements $u = (u_n)_n$ in F and defining a linear functional on elements $x = (x_n)$ in E by the relation,

$$\langle x, u\rangle = \sum_{n=1}^{\infty} \langle x_n, u_n\rangle,$$

where this sum is finite because x is finitely non-zero. With this notation, we may prove the

PROPOSITION. *Let $A_k\colon F_{k+1} \to F_k$ be a sequence of linear continuous maps of reflexive Banach spaces such that each $A_k(F_{k+1})$ is dense in F_k and let F be the projective limit. Then for each k, $P_k(F)$ is dense in F_k.*

Proof. In the notation described above, the injectivity of B_k follows from the density of $A_k(F_{k+1})$. Hence by the lemma, E is Hausdorff. Now fix k and suppose that $u \in E_k = F_k'$ such that for each $x \in F$ we have $\langle P_k x, u\rangle = 0$. Then,

$$0 = \langle P_k x, u\rangle = \langle p_k \circ q x, u\rangle = \langle x, p \circ q_k u\rangle = \langle x, Q_k u\rangle.$$

Since this is true for all $x \in F$, it follows that $Q_k u = 0$, and since Q_k is injective (because B_k is) we conclude that $u = 0$. This shows that $P_k(F)$ is dense in F_k.

COROLLARY. *If $(A_k)_k$ is a nuclear system, then each P_k has dense range.*

Proof. The conditions of the proposition are obviously satisfied.

THEOREM 2. *Two nuclear systems (A_k), $(\tilde{A}_k)$ are equivalent if and only if there is a subsequence $(n_k)_k$ of indices and continuous linear maps $f_k\colon l_2 \to l_2$, $k = 1, 2, \ldots,$ such that*

(i) $A_k f_{k+1} = f_k \tilde{A}_{n_k} \ldots \tilde{A}_{n_{k+1}-1}$, $k = 1, 2, \ldots,$

(ii) *each* $f_k \tilde{P}_{n_k}$ *is injective,*

(iii) $f_k \tilde{P}_{n_k}(\hat{E}\{(\tilde{A}_k)\}) \supset P_k(\hat{E}\{(A_k)\})$, $k = 1, 2, \ldots$

If (A_k) is injective, then (ii) *can be replaced by*

(ii)$'$. $f_1 \tilde{P}_{n_1}$ *is injective*

and (iii) *can be replaced by*

(iii)$'$ $f_1 \tilde{P}_{n_1}(\hat{E}\{(\tilde{A}_k)\}) \supset P_1(\hat{E}\{(A_k)\})$

or

(iii)$''$ $f_1\big(\bigcap_{k=1}^{\infty} \tilde{A}_{n_1} \ldots \tilde{A}_{n_k-1}(l_2)\big) \supset \bigcap_{k=1}^{\infty} A_1 \ldots A_k(l_2)$.

Proof. For convenience, we shall write $\hat{E} = \hat{E}\{(A_k)\}$ and $\hat{F} = \hat{E}\{(\tilde{A}_k)\}$. First suppose that the maps f_k exist. For each $x \in \hat{F}$ and $k = 1, 2, \ldots,$

define $y_k = f_k \tilde{P}_{n_k}(x)$. Then

$$A_k(y_{k+1}) = A_k f_{k+1} \tilde{P}_{n_{k+1}}(x) = f_k \tilde{A}_{n_k} \ldots \tilde{A}_{n_{k+1}-1} P_{n_{k+1}}(x) = f_k P_{n_k}(x) = y_k.$$

Hence $(y_k) \in \hat{E}$, so we can define $f\colon \hat{F} \to \hat{E}$ by

$$P_k f(x) = f_k \tilde{P}_{n_k}(x), \qquad k = 1, 2, \ldots$$

Clearly, f is linear and continuous. Suppose $x \in \hat{F}$ and $f(x) = 0$. Then each $f_k \tilde{P}_{n_k}(x) = 0$, so by (ii), $x = 0$. Thus f is injective. Now let $y \in \hat{E}$. For each k, it follows from (iii) that there exists $x^k \in \hat{F}$ with $P_k(y) = f_k \tilde{P}_{n_k}(x^k)$. But

$$\begin{aligned} f_k \tilde{P}_{n_k}(x^{k+1}) &= A_k f_{k+1} \tilde{P}_{n_{k+1}}(x^{k+1}) = A_k P_{k+1}(y) \\ &= P_k(y) = f_k \tilde{P}_{n_k}(x^k). \end{aligned}$$

Hence by (ii) we have $x^k = x^{k+1}$ so $f(x^1) = y$ and f is onto. Therefore by the open mapping theorem, f is an isomorphism.

Conversely, suppose we have an isomorphism $f\colon \hat{F} \to \hat{E}$. By the continuity of F and the definition of the topologies on $\hat{E}$, $\hat{F}$ we can find a sequence (n_k) of indices such that $n_k < n_{k+1}$ and

$$f\big(\{(x_n) \in \hat{F}\colon \|x_n\| \leqslant 1 \quad \text{for } n \leqslant n_k\}\big) \subset \{(y_n) \in \hat{E}\colon \|y_n\| \leqslant k \text{ for } n \leqslant k\}.$$

Now define $g_k\colon \tilde{P}_{n_k}(\hat{F}) \to l_2$ by $g_k \tilde{P}_{n_k}(x) = P_k f(x)$. This definition is unambiguous because $\tilde{P}_{n_k}$ is injective. First we check the continuity of g_k. Choose $\delta \leqslant 1$ such that

$$\delta \leqslant \min\{\|\tilde{A}_n \ldots \tilde{A}_{n_k-1}\|\colon n < n_k\}.$$

Then if $\|\tilde{P}_{n_k}(x)\| \leqslant \delta$, where $x = (x_n) \in \hat{F}$, it follows that $\|\tilde{P}_n x\| \leqslant 1$ for $n \leqslant n_k$, so $\|P_n f(x)\| \leqslant k$ for $n \leqslant k$. In particular,

$$\|g_k \tilde{P}_{n_k}(x)\| = \|P_k f(x)\| \leqslant k.$$

Thus g_k is continuous, so it may be extended to $f_k\colon l_2 \to l_2$. We check the three conditions.

$$\begin{aligned} f_k \tilde{A}_{n_k} \ldots \tilde{A}_{n_{k+1}-1} \tilde{P}_{n_{k+1}} &= f_k \tilde{P}_{n_k} = g_k \tilde{P}_{n_k} = P_k f = A_k P_{k+1} f \\ &= A_k g_{k+1} \tilde{P}_{n_{k+1}} = A_k f_{k+1} \tilde{P}_{n_{k+1}}. \end{aligned}$$

Thus (i) holds on the range of $\tilde{P}_{n_{k+1}}$ which is dense by the corollary and hence (i) holds. For condition (ii) we observe that $f_k \tilde{P}_{n_k} = g_k \tilde{P}_{n_k} = P_k f$ which is injective. For (iii) let $y \in \hat{E}$ and take $x = f^{-1}(y) \in \hat{F}$. Then

$$f_k \tilde{P}_{n_k}(x) = g_k \tilde{P}_{n_k}(x) = P_k f(x) = P_k(y),$$

and the first part of the theorem has been completed.

Next we observe that (ii) $\Rightarrow$ (ii)′ and for the converse, we have for any k that

$$f_1 P_{n_1} = A_1 \dots A_{k-1} f_k \tilde{P}_{n_k}$$

and since $f_1 P_{n_1}, A_1 \dots A_{k-1}$ are injective it follows that $f_k \tilde{P}_{n_k}$ is injective. Also it is clear that (iii) $\Rightarrow$ (iii)′ and if we have (iii)′ then for any k,

$$A_1 \dots A_{k-1} f_k \tilde{P}_k(\hat{F}) = f_1 \tilde{P}_{n_1}(\hat{F}) \supset P_1(\hat{E}) = A_1 \dots A_{k-1} P_k(\hat{E}),$$

so (iii)′ follows from the injectivity of $A_1 \dots A_{k-1}$.

Finally, (iii)″ follows from the observation that in any nuclear system (A_k) we have

$$P_1(\hat{E}) = \bigcap_{k=1}^{\infty} A_1 \dots A_k(l_2),$$

and the proof is completed.

We now apply Theorem 2 to nuclear spaces with bases. If E is a nuclear Fréchet space whose topology is determined by norms and E has a Schauder basis then it follows (see [3], Chapt. 10) that E is isomorphic to a nuclear sequence space. This means that E is isomorphic to some $\hat{E}\{(D_k)\}$, where each $D_k: l_2 \to l_2$ is a diagonal map. Furthermore each D_k has only non-zero terms on its diagonal else it would fail to have dense range. Thus (D_k) is an injective nuclear system. We see that Theorem 2 can be applied in several different ways depending on which form of the statement is used and on whether (D_k) replaces (A_k) or $(\tilde{A}_k)$. For convenience we state only one version.

COROLLARY. *Let (A_k) be an injective nuclear system. Then $\hat{E}\{(A_k)\}$ has a Schauder basis if and only if there exist diagonal nuclear maps $D_k: l_2 \to l_2$ and continuous linear maps $f_k: l_2 \to l_2$, $k = 1, 2, \dots$, such that*

(i) $A_k f_{k+1} = f_k D_k$, $k = 1, 2, \dots$,

(ii) $f_1 D_1$ *is injective,*

(iii) $f_1\left(\bigcap_{k=1}^{\infty} D_1 \dots D_k(l_2)\right) \supset \bigcap_{k=1}^{\infty} A_1 \dots A_k(l_2)$.

The corollary permits us to establish the existence of Schauder bases for many examples of nuclear Fréchet spaces. We list a few here and save the details for a forthcoming paper which will be an extensive study of examples of nuclear systems.

1° If each A_k is equal to a fixed $A: l_2 \to l_2$ which is normal, then $\hat{E}$ has a basis.

2° If each A_k is a weighted permutation of the usual basis vectors in l_2, then $\hat{E}$ has a basis.

3° If Ω is a sufficiently nice bounded open region in $\boldsymbol{R}^n$ and

$$C_0^\infty(\bar{\Omega}) = \{\varphi \in C^\infty(\Omega): D^p\varphi(\partial\Omega) = 0 \text{ for all } p\}$$

and $C_0^\infty(\bar{\Omega})$ is given the topology of uniform convergence of each derivative, then $C_0^\infty(\bar{\Omega})$ is a nuclear Fréchet space with a Schauder basis.

Finally, we list some open questions which might have a bearing on the Schauder basis problem for nuclear Fréchet spaces.

1° Under what conditions is a sequence $(B_k)_k$ of operators on l_2 of the form $B_k = A_1 \dots A_k$, where (A_k) is some nuclear system.

2° Let $A: l_2 \to l_2$ be a nuclear operator. Is the sequence space $\bigcap_k A^k(l_2)$ equal to a power series space (see [3]). Ist it even a "Stufenraum"? (see [2]).

3° Under what conditions on a nuclear system (A_k) does there exist an operator A on l_2 such that $\bigcap_k A_k(l_2) = A(l_2)$.

4° Is every nuclear system equivalent to an injective nuclear system? Since every Fréchet nuclear space with a basis is the associated space of an injective nuclear system, a negative answer to this question would imply a negative answer to the basis question.

Added in proof. The details and computations in example 3° are equivalent to those provided by H. Triebel, Math. Z. 90 (1965), p. 325-337. See also M. Zerner, C. R. Acad. Sc. Paris 268 (1969), p. 218-220.

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