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A characterization of analytic functions of n real variables

by

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1. The main purpose of this note is to prove the following

THEOREM 1. *Assume that*

1° $f\in\mathscr{C}^\infty(D)$, *where D is a domain in R^n*,

2° *for every $x\in D$ there exists an $r>0$ such that for every $a\in R^n$, $\|a\|=1$, the function $f(x+ta)$ is analytic with respect to $t\in(-r,r)$. The number r may depend on x and a.*

Then the function f is analytic in D.

Example. The function $f(x_1,x_2)=x_1^2x_2^2(x_1^2+x_2^4)^{-1}$, $f(0,0)=0$, is continuous in R^2, analytic on every line $x=x_0+ta$, $t\in R$ ($a\in R^2$), but f is not analytic at $(0,0)$ as a function of two real variables. Moreover, given any integer p ($0<p<+\infty$), one may easily define a function $f\in\mathscr{C}^p(R^2)$ which is analytic on each line but is not analytic in R^2.

As a consequence of Theorem 1 and of the classical Weierstrass preparation theorem we shall get

THEOREM 2. *If $H(x,y)=H(x_1,\ldots,x_n,y)\not\equiv 0$ is analytic in a domain $G\subset R^{n+1}$ and $H(x,f(x))=0$ for $x\in D$, where D is a domain in R^n and $f\in\mathscr{C}^\infty(D)$, then f is analytic in D.*

Theorems 1 and 2 have been proved by Bochnak [1] also for functions in Banach spaces. Another proof of Theorem 2 (and also of a more general theorem) was earlier presented in [3]. Still another proof of Theorem 2, based on the theory of semianalytic sets, was given by S. Łojasiewicz.

2. Theorem 1 will easily follow from the following

LEMMA. *Let*

$$g(x)=\sum_{l=0}^{\infty}P_l(x),\quad x\in R^n, \tag{1}$$

be a series of homogeneous polynomials in n variables of respective degrees l. Put $S=\{a\in R^n\colon \|a\|=1\}$ and assume that there exists an open subset Ω of S, $\Omega\neq\emptyset$, such that for every $a\in\Omega$ one can find $\varrho=\varrho_a>0$ such that series (1) is convergent at $x=\varrho a$.

Then there exists $r > 0$ such that

(2) $$|P_l(z)| \leqslant 2^{-l}, \quad z \in C^n, \quad \|z\| \leqslant r, \quad l \geqslant 1,$$

i.e. the function

$$\tilde{g}(z) = \sum_{l=0}^{\infty} P_l(z)$$

is holomorphic in the ball $\|z\| < r$, $z \in C^n$.

Proof. Given an $a \in \Omega$, we may choose a $\varrho_a > 0$ so small that

$$|P_l(ta)| \leqslant 1, \quad |t| \leqslant \varrho_a, \quad l \geqslant 1.$$

For every $k = 1, 2, \ldots$ the set

$$E_k = \{a \in \Omega\colon |P_l(ta)| \leqslant 1, |t| \leqslant 1/k, l \geqslant 1\}$$

is closed in Ω, $E_k \subset E_{k+1}$ and $\Omega = \bigcup_{k=1}^{\infty} E_k$. By the Baire theorem there exists an open set $\omega \subset \Omega$ such that $\omega \subset E_k$ if k is sufficiently large, say $k \geqslant k_0$. Therefore

(3) $$|P_l(ta)| \leqslant 1, \quad |t| \leqslant r_0 = 1/k_0, \quad a \in \omega, l \geqslant 1.$$

The set $G = \{ta \in R^n\colon a \in \omega, 0 < t < r_0\}$ is open in R^n and it contains a Cartesian product $K = [a_1, b_1] \times \ldots \times [a_n, b_n]$ of n linear segments $[a_j, b_j]$ $(a_j < b_j)$, $j = 1, \ldots, n$. It is obvious that $|P_l(z)| \leqslant 1$ in K, $l \geqslant 1$.

We may treat $[a_j, b_j]$ as a subset of the real line in the complex z_j-plane. Let $f_j\colon C - [a_j, b_j] \to C$ be a conformal mapping of $C - [a_j, b_j]$ onto $\{w \in C\colon |w| > 1\}$ such that $f_j(\infty) = \infty$. Using the well-known Bernstein inequality for polynomials in one complex variable and the induction with respect to n, we get the inequality

(4) $$|P_l(z)| \leqslant |f_1(z_1) \ldots f_n(z_n)|^l, \quad z \in C^n, \quad l \geqslant 1,$$

where $|f_j(z_j)|$ is considered as continuously extended on the whole z_j-plane.

Put $M = \sup\{|f_1(z_1) \ldots f_n(z_n)|\colon \|z\| \leqslant 1, z \in C^n\}$. Then by (4) and by the homogeneity of P_l we get (2) with $r = 1/2M$. The proof of the Lemma is concluded.

3. Proof of Theorem 1. Let x_0, be a fixed point of D. Given an $a \in S$, the function $f(x_0 + ta)$ is analytic at $t = 0$, so there exists a $\varrho_a > 0$ such that

$$f(x_0 + ta) = \sum_{l=0}^{\infty} P_l(a) t^l \quad \text{for } t \in (-\varrho_a, \varrho_a),$$

where

$$P_l(a) = \sum_{|\mu|=l} \frac{D^\mu f(x_0)}{\mu!} a^\mu, \quad a^\mu = a_1^{\mu_1} \ldots a_n^{\mu_n}, \ |\mu| = \mu_1 + \ldots + \mu_n,$$
$$\mu! = \mu_1! \ldots \mu_n!.$$

By Lemma, the series $\sum_0^\infty P_l(z)$ is convergent uniformly in a ball $\|z\| < r$, $z \in C^n$ and its sum $\tilde{f}(z)$ is a holomorphic function there. But $\tilde{f}(x_0 + ta) = f(x_0 + ta)$, $t \in (-\varrho_a, \varrho_a)$, $a \in S$. By the identity property of analytic functions

$$\tilde{f}(x_0 + ta) = f(x_0 + ta), \quad |t| < \varrho = \min\big(r, \operatorname{dist}(x_0, \partial D)\big).$$

Therefore $\tilde{f}(x_0 + x) = f(x_0 + x)$, $\|x\| < \varrho$, $x \in R^n$. The proof is concluded.

4. Proof of Theorem 2. We want to prove that f is analytic at every point $x_0 \in D$. Without loss of generality we may assume that $x_0 = 0$ and $f(x_0) = 0$. We shall use induction with respect to n.

1° $n = 1$. Let us write H in the form

$$H = H_1 \ldots H_p,$$

where H_j is an analytic function irreducible in a neighborhood of $0 \in R^2$. The function f satisfies the functional equation $H\big(x, f(x)\big) = 0$ $(f(0) = 0)$. By the Weierstrass preparation theorem we may assume that

$$H_j(x, y) = y^{s_j} + a_{j1}(x) y^{s_j - 1} + \ldots + a_{js_j}(x),$$

where a_{jk} are analytic in a neighborhood of $x = 0$ and $a_{jk}(0) = 0$. At first let us observe that f is analytic in $(0, r)$, where $r > 0$ is sufficiently small. Indeed, H_j being irreducible, the discriminant $D_j(x)$ of H_j and $\partial H_j / \partial y$ does not vanish identically. So there exists an $r > 0$ such that $D_j(x) \neq 0$ $(j = 1, \ldots, p)$ for $x \in (0, r)$, because D_j is analytic and its zeros are isolated. For every $x_0 \in (0, r)$ there exists a j such that $H_j\big(x_0, f(x_0)\big) = 0$. But

$$\frac{\partial}{\partial y} H_j\big(x_0, f(x_0)\big) \neq 0,$$

because $D_j(x_0) \neq 0$. Therefore, by the implicit function theorem, the graph of f restricted to a sufficiently small neighborhood of $\big(x_0, f(x_0)\big)$ is contained in a finite union of graphs of functions analytic in a neighborhood of x_0. Consequently, f being $\mathscr{C}^\infty$ must be analytic in a neighborhood of x_0. Therefore $H_j\big(x, f(x)\big)$ are analytic in $(0, r)$. As

$$\prod_{j=1}^{p} H_j\big(x, f(x)\big) = 0$$

in $(0, r)$, there exists k such that $H_k\big(x, f(x)\big) = 0$ in $(0, r)$.

It is known ([2], p. 89) that there exists a function

$$g(z) = \sum_{l=0}^{\infty} c_l z^l$$

holomorphic in a neighborhood of $z = 0$ such that

(5) $$f(x) = g(x^{1/s}), \quad 0 < x < r_0 \ (0 < r_0 \leqslant r),$$

where $s = s_k$ and the value of $x^{1/s}$ is suitably chosen at each point of $(0, r)$.

Let m be the smallest integer such that $c_m \neq 0$ and m is not divisible by s. Then

$$f_0(x) = f(x) - \sum_{l=0}^{m-1} c_l x^{l/s}$$

is of class $\mathscr{C}^\infty$ and $f_0(x) = x^{m/s} g_0(x)$, $g_0(0) = c_m \neq 0$. In particular,

(6) $$\lim_{x\downarrow 0} |f_0(x) x^{-m/s}| = |c_m| \neq 0.$$

But, as the function f_0 is $\mathscr{C}^\infty$, so either it may be written in the form $f_0(x) = x^q f_1(x), f_1(0) \neq 0$, for a fixed positive integer q, or $\lim_{x\downarrow 0} f_0(x) x^{-q} = 0$ for every real positive q. Both cases lead to the contradiction with (6). Consequently, $c_l = 0$ if l is not divisible by s. Thus, by (5), the function f has an analytic extension from $(0, r)$ on a neighborhood of 0.

Analogously, f may be analyticaly extended from an interval $(-r, 0)$ on a neighborhood of 0. Since the function f is $\mathscr{C}^\infty$, these two extensions must coincide, and therefore f is analytic at 0.

2° Let now n be an arbitrary positive integer. The set

$$E = \{a \in S\colon H(ta, y) = 0, -r \leqslant t \leqslant r, -r \leqslant y \leqslant r\},$$

where $r > 0$ is sufficiently small, is closed and nowhere dense in S, because $H \neq 0$. Thus, there is an open set Ω in S such that for every $a \in \Omega$ we have $H(ta, y) \not\equiv 0$ $(-r < t < r, -r < y < r)$. By the assumption, $H\big(ta, f(ta)\big) = 0$ in a neighborhood of $t = 0$. By 1° the function $f(ta)$ is analytic with respect to t. It follows from the Lemma that the Taylor series of f at $0 \in R^n$ is convergent in a neighborhood of 0 to an analytic function $\tilde{f}$ and, moreover,

$$\tilde{f}(ta) = f(ta), \quad a \in S - E, \quad |t| < \varrho,$$

ϱ being a positive constant. Since the set E is nowhere dense in S, we have $\tilde{f} = f$ in a full neighborhood of $0 \in R^n$, because f and $\tilde{f}$ are continuous. The proof is ended.

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