STUDIA MATHEMATICA, T. XXXIV. (1970)

On the spectral radius in group algebras

by

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Let G be a discrete group and $l_1(G)$ the group algebra of G. For every x in $l_1(G)$ the spectral radius of it is defined as

$$\nu(x) = \max\{|\lambda|:\ \lambda \in \mathrm{Sp}(x)\} = \lim_{n\to\infty} \sqrt[n]{\|x^n\|}.$$

If G is Abelian, then $l_1(G)$ is a commutative semi-simple Banach algebra and

$$\nu(x) = \max\{|\hat{x}(\chi)|:\ \chi \in \hat{G}\}.$$

Then, of course, ν is a norm on $l_1(G)$, i.e.

$$(*)\qquad \nu(x+y) \leqslant \nu(x)+\nu(y)$$

and

$$(**)\qquad \nu(x) = 0 \text{ if and only if } x = 0.$$

For non-Abelian G relation $(*)$ remains true for all pairs $x, y \in l_1(G)$ such that $xy = yx$ (cf. [2], p. 10). If x is hermitian, i.e. if $x = x^*$ (x^* being defined by $x^*(g) = \overline{x(g^{-1})}$), then $(**)$ holds for any G.

If $l_1(G)$ is symmetric (i.e. for each x in $l_1(G)$, $\mathrm{Sp}(x^*x) \geqslant 0$), then, by Raikov's theorem (cf. [2], p. 238), for any hermitian element in $l_1(G)$

$$\nu(x) = \sup \|T_x\|,$$

where the least upper bound on the right-hand side is taken over all $*$-representations of $l_1(G)$ into the algebra of bounded operators in a Hilbert space. Therefore if $l_1(G)$ is symmetric, then $(*)$ holds for all x and y in the real subspace of hermitian elements of $l_1(G)$.

The aim of this note is to show that $(*)$ does not generally hold even for hermitian elements $x, y \in l_1(G)$ for a solvable group G. Thus, by Raikov's theorem, we obtain an alternative proof of a recent result by Jenkins [1] that solvable groups need not have symmetric group algebras.

1. The group. Let $A(\boldsymbol{R})$ be the affine group of the real line, i.e. the group of pairs of real numbers (ξ, η), with $\xi \neq 0$ and multiplication defined by the formula $(\xi, \eta)(\xi', \eta') = (\xi\xi', \xi'\eta+\eta')$. Let a be a transcendental real number and let G be the subgroup of $A(\boldsymbol{R})$ generated by the pairs $a = (\alpha, 0)$, $b = (1, 1)$. Then for any integers m, n we have $a^m b^n = (\alpha^m, n)$ and consequently

(1.1)
$$a^{m_1}b^{n_1}\ldots a^{m_k}b^{n_k} = (\alpha^{m_1+\ldots+m_k}, n_1\alpha^{m_2+\ldots+m_k}+n_2\alpha^{m_3+\ldots+m_k}+\ldots+n_k).$$

Lemma 1. *Suppose $l \leqslant k$ and $m_i, n_i > 0$, $i = 1, \ldots, k$. If*

$$a^{m_1}b^{n_1}\ldots a^{m_k}b^{n_k} = a^{m'_1}b^{n'_1}\ldots a^{m'_l}b^{n'_l} \quad \textit{and} \tag{1.2}$$
$$m'_2, \ldots, m'_l \neq 0, \; n'_1, \ldots, n'_{l-1} \neq 0,$$

then $m'_1 \neq 0$, $n'_l \neq 0$ and $k = l$.

Moreover, if $m'_1, \ldots, m'_l, n'_1, \ldots, n'_l$ are all different from zero and $l \leqslant k$, then

$$a^{m_1}b^{n_1}\ldots a^{m_k}b^{n_k} \neq b^{n'_1}a^{m'_1}\ldots b^{n'_l}a^{m'_l}.$$

Proof. By (1.1), equality (1.2) implies

$$m_1+\ldots+m_k = m'_1+\ldots+m'_l \tag{1.3}$$

and

$$n_1\alpha^{m_2+\ldots+m_k}+n_2\alpha^{m_3+\ldots+m_k}+\ldots+n_k \tag{1.4}$$
$$= n'_1\alpha^{m'_2+\ldots+m'_l}+n'_2\alpha^{m'_3+\ldots+m'_l}+\ldots+n'_l.$$

Since α is transcendental and the exponents $m_2+\ldots+m_k$, $m_3+\ldots+m_k, \ldots, 0$ are all different (and $l \leqslant k$), they must be equal to the exponents $m'_2+\ldots+m'_l$, $m'_3+\ldots+m'_l, \ldots, 0$ in some order and similarly $n_1, \ldots, n_k$ must be equal to $n'_1, \ldots, n'_l$ in some order. Hence $l = k$ and $n'_l \neq 0$. If $m'_1 = 0$, then, by (1.3), $m_1+\ldots+m_k = m'_2+\ldots+m'_k$ which is equal to one of the numbers $m_2+\ldots+m_k$, $m_3+\ldots+m_k, \ldots, 0$, but this is impossible since $m_1, \ldots, m_k$ are positive.

To prove the latter part of the lemma, we note that

$$b^{n'_1}a^{m'_1}\ldots b^{n'_l}a^{m'_l} = (\alpha^{m'_1+\ldots+m'_l}, n'_1\alpha^{m'_1+\ldots+m'_l}+n'_2\alpha^{m'_2+\ldots+m'_l}+\ldots+n'_l\alpha^{m'_l}).$$

Then the equality

$$a^{m_1}b^{n_1}\ldots a^{m_k}b^{n_k} = b^{n'_1}a^{m'_1}\ldots b^{n'_l}a^{m'_l}$$

and $l \leqslant k$ imply, as above, that $n'_1, \ldots, n'_l$ are equal to $n_1, \ldots, n_k$ in some order, $m_1+\ldots+m_k = m'_1+\ldots+m'_l$ and $m'_1+\ldots+m'_l$, $m'_2+\ldots+m'_l, \ldots$ $\ldots, m'_l$ are equal to $m_2+\ldots+m_k$, $m_3+\ldots+m_k, \ldots, 0$ in some order, whence $m_1+\ldots+m_k$ is equal to one of the numbers $m_2+\ldots+m_k$, m_3+ $+\ldots+m_k, \ldots, 0$, which is impossible, since $m_j > 0$, $j = 1, \ldots, k$.

Corollary 1. *Let*

$$g = a^{m_1}b^{n_1}\ldots a^{m_k}b^{n_k} \quad \textit{with } m_j, n_j > 0, \; j = 1, \ldots, k,$$

and suppose that g is expressible as a product

$$g = g_1g_2\ldots g_t, \quad t \leqslant 2k$$

such that the g_j belong either to $A = gp\{a\}$ or to $B = gp\{b\}$ and if $g_j \in A$, then $g_{j+1} \in B$ and if $g_j \in B$, then $g_{j+1} \in A$. Then all g_j are different from the unit of G, $t = 2k$, $g_1 \in A$ and $g_t \in B$.

Lemma 2. *Suppose that for a positive integer N we have $n_j \neq 0$,*

$$N \leqslant m_j < 2N \quad \textit{and} \quad N \leqslant |m'_j| < 2N, \quad j = 1, \ldots, k. \tag{1.5}$$

Then the equality

$$a^{m_1}b^{n_1}\ldots a^{m_k}b^{n_k} = a^{m'_1}b^{n'_1}\ldots a^{m'_k}b^{n'_k} \tag{1.6}$$

implies $m_j = m'_j$ and $n_j = n'_j$ for all $j = 1, \ldots, k$.

Proof. As before we infer that (1.6) implies the existence of a permutation σ of the indices $\{1, \ldots, k-1\}$ such that

$$n'_1 = n_{\sigma(1)}, \ldots, n'_{k-1} = n_{\sigma(k-1)}, \quad n'_k = n_k,$$

and

$$m'_1+\ldots+m'_k = m_1+\ldots+m_k$$
$$m'_2+\ldots+m'_k = m_{\sigma(1)+1}+\ldots+m_k, \ldots, m'_k = m_{\sigma(k-1)+1}+\ldots+m_k. \tag{1.7}$$

But, in virtue of (1.5), m'_k cannot be a non-trivial sum of the m_j's, and, consequently, by (1.7), $\sigma(k-1) = k-1$ and $m'_k = m_k$. Hence, by (1.7) again, $m'_{k-1} = m_{\sigma(k-2)+1}+\ldots+m_{k-1}$ and, similarly, $m'_{k-1} = m_{k-1}$ and $\sigma(k-2) = k-2$. Proceeding in this way we complete the proof of the lemma.

2. A trigonometric polynomial.

Lemma 3. *For every $\varepsilon > 0$ there exists a positive integer N and a trigonometric polynomial*

$$T(t) = \sum_{n=N}^{2N-1}(c_ne^{int}+\overline{c}_ne^{-int}) \tag{2.1}$$

such that

$$\max_t |T(t)| < \frac{\varepsilon}{4}\sum_{n=N}^{2N-1}|c_n|.$$

In fact, let

$$f(t) = \sum_{-M}^{M} d_ne^{int}$$

be such that

$$\max_t |f(t)| < \frac{\varepsilon}{8}\sum_{-M}^{M} |d_n|.$$

(For the existence of f see e.g. [3]). Then

$$g(t) = f(t)\exp[i(3M+1)t] = \sum_{n=N}^{2N-1} c_n e^{int},$$

where $N = 2M+1$, $c_n = d_{n-3M-1}$. Clearly

$$\max_t |g(t)| < \frac{\varepsilon}{8}\sum_{n=N}^{2N-1} |c_n|$$

and it suffices to put $T(t) = g(t)+\overline{g(t)}$.

3. The Theorem. Let A and B be the infinite cyclic subgroups of G generated by a and b, respectively, c_n, $N \leqslant n < 2N$, the coefficients in (2.1). Let $x, y \in l_1(G)$ be defined as follows:

$$x(g) = \begin{cases} c_n & \text{if } g = a^n, \quad N \leqslant n < 2N, \\ \overline{c_n} & \text{if } g = a^{-n}, N \leqslant n < 2N, \\ 0 & \text{elsewhere}, \end{cases}$$

$$y(g) = \begin{cases} c_n & \text{if } g = b^n, \quad N \leqslant n < 2N, \\ \overline{c_n} & \text{if } g = b^{-n}, N \leqslant n < 2N, \\ 0 & \text{elsewhere}. \end{cases}$$

Then, clearly,

$$\|x\| = \|y\| \quad \text{and} \quad \nu(x) = \nu(y) < \frac{\varepsilon}{4}\|x\|. \tag{3.1}$$

We have

$$(x+y)^{2k} = \sum x^{u_1}y^{v_1}\dots x^{u_l}y^{v_l} + \sum x^{u_1}y^{v_1}\dots y^{v_{l-1}}x^{u_l} + \sum y^{v_1}x^{u_2}\dots x^{u_l}y^{v_l} + \sum y^{v_1}x^{u_1}\dots y^{v_l}x^{u_l},$$

where the summation extends over all positive integers

$$\begin{cases} u_1, \dots, u_l, v_1, \dots, v_l & \text{with } u_1+\dots+u_l+v_1+\dots+v_l = 2k, \\ u_1, \dots, u_l, v_1, \dots, v_{l-1} & \text{with } u_1+\dots+u_l+v_1+\dots+v_{l-1} = 2k, \\ u_2, \dots, u_l, v_1, \dots, v_l & \text{with } u_2+\dots+u_l+v_1+\dots+v_l = 2k, \\ u_1, \dots, u_l, v_1, \dots, v_l & \text{with } u_1+\dots+u_l+v_1+\dots+v_l = 2k, \end{cases} \tag{3.2}$$

respectively, and $l \leqslant k$.

Let

$$C_N = \{a_1b_1\dots a_kb_k:\ a_j = a^{m_j},\ b_j = b^{n_j},\ N \leqslant m_j, n_j < 2N\}.$$

Now we evaluate $(x+y)^{2k}$ on C_N. Let $g \in C_N$. We then have

(3.3)

(a) $$x^{u_1}y^{v_1}\dots x^{u_l}y^{v_l}(g) = \sum_{a_1'b_1'\dots a_l'b_l'=g} x^{u_1}(a_1')y^{v_1}(b_1')\dots x^{u_l}(a_l')y^{v_l}(b_l'),$$

(b) $$x^{u_1}y^{v_1}\dots y^{v_{l-1}}x^{u_l}(g) = \sum_{a_1'b_1'\dots b_{l-1}'a_l'=g} x^{u_1}(a_1')y^{v_1}(b_1')\dots y^{v_{l-1}}(b_{l-1}')x^{u_l}(a_l'),$$

(c) $$y^{v_1}x^{u_2}\dots x^{u_l}y^{v_l}(g) = \sum_{b_1'a_2'\dots a_l'b_l'=g} y^{v_1}(b_1')x^{u_2}(a_2')\dots x^{u_l}(a_l')y^{v_l}(b_l'),$$

(d) $$y^{v_1}x^{u_1}\dots y^{v_l}x^{u_l}(g) = \sum_{b_1'a_1'\dots a_l'b_l'=g} y^{v_1}(b_1')x^{u_1}(a_1')\dots y^{v_l}(b_l')x^{u_l}(a_l').$$

Since $x^u(a') = 0$ for $a' \notin A$ and $y^v(b') = 0$ for $b' \notin B$, by corollary 1, we see that only (a) in (3.3) can be different from zero and in this case, by corollary 1 again, $l = k$, whence, by (3.2), $u_1 = \dots = u_k = v_1 = \dots = v_k = 1$. Consequently,

$$(x+y)^{2k}(g) = \sum_{a_1'b_1'\dots a_k'b_k'=g} x(a_1')y(b_1')\dots x(a_k')y(b_k').$$

We have

$$g = a^{m_1}b^{n_1}\dots a^{m_k}b^{n_k} \quad \text{with } N \leqslant m_j, n_j < 2N, \tag{3.4}$$

and $x(a_j') \neq 0$, $y(b_j') \neq 0$ only if

$$a_j' = a^{m_j'},\ b_j' = b^{n_j'} \quad \text{with } N \leqslant |m_j'|, |n_j'| < 2N. \tag{3.5}$$

By lemma 2, we infer that g is expressible in the form

$$a_1'b_1'\dots a_k'b_k' = g,$$

where a_j' and b_j' $(j = 1, \dots, k)$ satisfy (3.5), only if $m_1' = m_1, \dots, m_k' = m_k$, $n_1' = n_1, \dots, n_k' = n_k$. Thus we obtain the equality

$$(x+y)^{2k}(g) = x(a^{m_1})y(b^{n_1})\dots x(a^{m_k})y(b^{n_k})$$

for all g of the form (3.4).

Now we have

$$\|(x+y)^{2k}\| \geqslant \sum_{g\in C_N} |(x+y)^{2k}(g)|$$

$$= \sum_{m_1=N}^{2N-1} |x(a^{m_1})| \sum_{n_1=N}^{2N-1} |y(b^{n_1})| \dots \sum_{m_k=N}^{2N-1} |x(a^{m_k})| \sum_{n_k=N}^{2N-1} |y(b^{n_k})|$$

$$= \left(\frac{1}{2}\right)^{2k} \|x\|^k \|y\|^k = \left(\frac{\|x\|}{2}\right)^{2k}.$$

Hence, by (3.1),

$$\nu(x+y) = \lim_{k\to\infty} \|(x+y)^{2k}\|^{1/2k} \geqslant \frac{1}{2}\|x\| > \frac{1}{\varepsilon}\big(\nu(x)+\nu(y)\big).$$

We summarize the obtained result in the following

THEOREM. *If G is the discrete subgroup of the affine group of the real line as defined in section 1, ε a positive number and x and y the hermitian elements in $l_1(G)$ defined in section 3, then*

$$\varepsilon\nu(x+y) > \nu(x)+\nu(y).$$

References

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Reçu par la Rédaction le 4. 3. 1969

STUDIA MATHEMATICA, T. XXXIV. (1970)

Restrictions and extensions of Fourier multipliers*

by

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Introduction. In this paper we derive certain relations between spaces of Fourier multipliers defined on R^N, Z^N, T^N (definitions and notation are given in section 1). The main result, Theorem (3.7), is for $N=1$: if $1<p<\infty$ and $\{m_n\}$ is a multiplier sequence of type (p,p), then the piecewise constant function $m(x) = m_k$ (k is the greatest integer $\leqslant x+\frac{1}{2}$) is a multiplier of type (p,p) for Fourier transforms. In the case $1\leqslant p\leqslant\infty$, the piecewise linear continuous extension of a sequence of type (p,p) is a function of type (p,p) (see (3.6)).

Sections 2 and 4 contain mostly known results, for which we offer alternate proofs. With one exception the results are due to de Leeuw [3]. Theorem (4.3) is due to Igari [2]. The relations between $M_p^p(R^N)$ and $M_p^p(T^N)$ are given in section 2, and restrictions to Z^N and R^M of elements of $M_p^p(R^N)$ are treated in section 4.

Among the applications of these results are

(i) the Marcinkiewicz multiplier theorem for the line follows from the sequential version (section 4),

(ii) a function m defined on R^N, continuous except at 0, and homogeneous of degree 0 $\big(m(\lambda x) = m(x)$ for $\lambda>0\big)$ is in $M_p^p(R^N)$ if and only if its restriction to Z^N is a sequence of type (p,p) (section 4).

Questions raised by Professor R. Coifman and Mr. David Shreve led to this work, which has also profited by a comment of Professor Calderón.

1. Preliminaries. We first set down for reference some conventional notation. R^N denotes real N-space, x, y denote points of R^N, with coordinates $x_1,\ldots,x_N, y_1,\ldots,y_N$. $|x| = (x_1^2+\ldots+x_N^2)^{1/2}$, $x\cdot y = x_1y_1+\ldots+x_Ny_N$. $Z^N\subseteq R^N$ is the set of points n with integer coordinates. If $S\subset R^N$, $a\in R$, then $aS = \{as: s\in S\}$, and if $x\in R^N$, then $x+S = \{x+s: s\in S\}$. T^N, the Cartesian product of N copies of the unit circle in the complex

* During the preparation of this paper the author was partially supported by the National Science Foundation under NSF grant GP 8855.