STUDIA MATHEMATICA, T. XXXII. (1969)

An isomorphic characterization of L_p and c_0-spaces*

by

L. TZAFRIRI (Minnesota)

1. Introduction. Let X be either one of the Banach spaces $L_p(\Omega, \Sigma, \mu)$; $1 \leqslant p < +\infty$ of all complex-valued measurable functions whose absolute p^{th} powers are integrable with respect to a finite measure space (Ω, Σ, μ) or c_0, the space of all the sequences of complex numbers converging to zero. In X, one can consider the Boolean algebra of projections $\mathscr{E}$ consisting of "multiplications" by characteristic functions

$$E(\sigma)f = \chi_\sigma f, \quad \sigma \in \Sigma, \quad f \in L_p(\Omega, \Sigma, \mu), \quad 1 \leqslant p < +\infty,$$

or

$$E(\delta)\{x_n\} = \{\chi_\delta(n)x_n\}, \quad \delta \subset N, \quad \{x_n\} \in c_0.$$

This Boolean algebra of projections satisfies the following conditions:

(a) $\mathscr{E}$ is σ-complete i.e. $E(\cdot)x$ is a σ-additive vector-valued measure on (Ω, Σ) for every $x \in X$.

(b) $X = \operatorname{clm}\{E(\sigma)x_0 \mid E(\sigma) \in E\}$ for some $x_0 \in X$ (which can be chosen as $x_0 \equiv 1$ for L_p and $x_0 = \{1/n\}$ for c_0).

(c) $\|x\| = \big(\sum\limits_n \|E(\sigma_n)x\|^p\big)^{1/p}$, $x \in L_p(\Omega, \Sigma, \mu)$, $1 \leqslant p < +\infty$,

or

$$\|x\| = \sup_n \|E(\sigma_n)x\|; \; x \in c_0$$

for every sequence of disjoint projections of $\mathscr{E}$, finite or infinite, whose sum is the identity I. If X is only isomorphic either to an L_p-space, $1 \leqslant p < +\infty$, or to c_0, then the images under the isomorphism of the "multiplications" by characteristic functions will form a Boolean algebra of projections, again denoted by $\mathscr{E}$, which still satisfies (a) and (b) while (c) should be replaced by the following condition:

(d) There exists a constant K such that

$$K^{-1}\big(\sum_n \|E(\sigma_n)x\|^p\big)^{1/p} \leqslant \|x\| \leqslant K\big(\sum_n \|E(\sigma_n)x\|^p\big)^{1/p}$$

* Supported by the National Science Foundation, Grant GP-7475.

for some $1 \leqslant p < \infty$ or

$$K^{-1}\sup_n \|E(\sigma_n)x\| \leqslant \|x\| \leqslant K \sup_n \|E(\sigma_n)x\|$$

for every $x \in X$ and every sequence of disjoint projections $E(\sigma_n) \in \mathscr{E}$, finite or infinite, whose sum is the identity I.

In essence, condition (d) assures the existence of a two-sided estimate of $\|\sum_n E(\sigma_n)x\|$ in terms of $\{\|E(\sigma_n)x\|\}_n$ satisfied by every $x \in X$ and every sequence of disjoint projections $\{E(\sigma_n)\}$ of $\mathscr{E}$ and this estimate is independent of the choice of $x \in X$ and $E(\sigma_n) \in \mathscr{E}$.

The purpose of the paper is to show that conditions (a), (b) and (d) characterize the spaces $L_p(\Omega, \Sigma, \mu)$, $1 \leqslant p < +\infty$, μ finite, and c_0; more precisely, the existence of a Boolean algebra of projections in a Banach space X satisfying (a) and (b) and admiting *any* two-sided estimate of the above described type (details will be given in the next sections) is possible only if X is isomorphic either to an L_p-space, $1 \leqslant p < +\infty$, on a finite measure space (Ω, Σ, μ) or to c_0. Replacing conditions (a) and (b) by other adequate conditions stated in terms of Bade's theory of multiplicity for Boolean algebras of projections (cf. [1] and [2] we obtain a characterization of L_p-spaces, $1 \leqslant p < +\infty$, on any measure space (not necessarily finite) and $c_0(\Gamma)$.

Related results have been obtained recently by Lindenstrauss and Zippin [6] who have shown that a Banach space with an unconditional basis ought to be isomorphic to either l_1, l_2 or c_0 provided there exists a two-sided estimate valid for *every* Boolean algebra of projections. They also conjectured that if the existence of an unconditional basis is replaced by a requirement guaranteeing the existence of "many" Boolean algebras of projections admiting a two-sided estimate, then the underlying space is either an $\mathscr{L}_1, \mathscr{L}_2$ or $\mathscr{L}_\infty$-space in the sense of [5]. This conjecture has been proved recently by them in [7].

Acknowledgment. The author wishes to express his thanks to M. Zippin for his very valuable comments and suggestions.

2. Preliminaries. In this section we shall bring some notation, definitions and results which will be useful in the sequel. The term "isomorphic" as used in the introduction has the following meaning: two Banach spaces X and Y are isomorphic if there exists an invertible bounded linear operator from X onto Y. The distance between X and Y (cf. [5]) will be defined as follows:

$$d(X, Y) = \inf \|\tau\| \; \|\tau^{-1}\|,$$

where the infimum is taken over all invertible bounded linear operators τ mapping X onto Y if such operators exists; $d(X, Y) = +\infty$ if X and Y are not isomorphic (d, which is not a metric is used instead of $\log d$ which is a metric.)

For any abstract set Γ, $l_p(\Gamma)$, $1 \leqslant p \leqslant +\infty$, will denote the Banach space of all functions φ defined on Γ for which

$$\|\varphi\|_p = \Big(\sum_{\gamma\in\Gamma} |\varphi(\gamma)|^p\Big)^{1/p} < +\infty, \quad 1 \leqslant p < +\infty,$$

$$\|\varphi\|_\infty = \sup_{\gamma\in\Gamma} |\varphi(\gamma)| < +\infty, \quad p = +\infty.$$

If Γ is countable or it has a finite number n of elements, we shall denote $l_p(\Gamma)$ by l_p, respectively l_p^n. Another space considered is $c_0(\Gamma)$, consisting of those $\varphi \in l_\infty(\Gamma)$ for which the set $\{\gamma \mid |\varphi(\gamma)| \geqslant \varepsilon\}$ is finite for every $\varepsilon > 0$. When Γ is countable, $c_0(\Gamma)$ is denoted as usual by c_0.

The following definition is due to Lindenstrauss and Pełczyński [5] and introduces a new class of Banach spaces $\mathscr{L}_p$ which is larger than the class of L_p-spaces.

Definition 1. A Banach space X is called an $\mathscr{L}_{p,\lambda}$-space ($1 \leqslant p \leqslant +\infty$, $1 \leqslant \lambda < +\infty$), provided that for every finite-dimensional subspace Y of X there is a finite-dimensional subspace $Z \supset Y$ such that $d(Z, l_p^n) \leqslant \lambda$, where $n = \dim Z$. A Banach space X is called an *$\mathscr{L}_p$-space* ($1 \leqslant p \leqslant +\infty$), if it is an $\mathscr{L}_{p,\lambda}$-space for some $1 \leqslant \lambda < +\infty$.

A set of vectors $\{u_n\}_{n=1}^\infty$ is called an *unconditional basis* of X (cf. Day [3]) provided every $x \in X$ can be represented uniquely as $x = \sum_{n=1}^\infty a_n u_n$ and this series converges unconditionally, i.e. $\Sigma a_{\pi(n)} u_{\pi(n)}$ converges for every permutation π of the integers. If $\|u_n\| = 1$, $n = 1, 2, \ldots$, the basis is called *normalized*. Two bases $\{u_n\}$ and $\{v_n\}$ are said to be *equivalent* if a series $\sum_{n=1}^\infty a_n u_n$ converges whenever $\sum_{n=1}^\infty a_n v_n$ does.

Finally, we shall summarize some results concerning Boolean algebras of projections which mostly are due to Bade [1] and [2].

A Boolean algebra of projections $\mathscr{E}$ will be called *complete* (*σ-complete*) if for every family (sequence) $E_a \in \mathscr{E}$ the projections $\vee E_a$ and $\wedge E_a$ exist in $\mathscr{E}$ and, moreover,

$$(\vee E_a)X = \operatorname{clm}\{E_a X\}, \quad (\wedge E_a)X = \bigcap (E_a X).$$

If $\mathscr{E}$ is σ-complete, there is a uniform bound for the norm of the projections in $\mathscr{E}$ (cf. [1], Theorem 2.2). A projection $E \in \mathscr{E}$ is called *countably decomposable* if every family of disjoint projections in $\mathscr{E}$ bounded by E is at most countable. For every $E \in \mathscr{E}$ there is a family of disjoint countable decomposable projections $E_\gamma \in \mathscr{E}$, $\gamma \in \Gamma$, such that $E = \bigvee_{\gamma\in\Gamma} E_\gamma$ (cf. [2], Lemma 3.1). If for every γ there exists $x_\gamma \in X$ such that

$$E_\gamma X = \operatorname{clm}\{Ex_\gamma \mid E \in \mathscr{E}\}, \quad \gamma \in \Gamma,$$

then $E \in \mathscr{E}$ is said *to have multiplicity one* (cf. Bade [2], Definition 3.2).

3. Boolean algebras of projections with two-sided estimate. This concept has been first considered by Lindenstrauss and Zippin [6] but a concrete definition has not been given there. The following definition, which seems to be the most general possible, appears also in [7]:

Definition 2. Let $\mathfrak{B}$ be a bounded Boolean algebra of projections in a Banach space X. We say that $\mathfrak{B}$ *has a two-sided estimate* if there exist a constant K and a function ψ defined for every sequence of complex numbers such that

$$K^{-1}\psi(\|P_1x\|, \|P_2x\|, \ldots, \|P_nx\|, \ldots) \leqslant \|x\| \leqslant K\psi(\|P_1x\|, \|P_2x\|, \ldots, \|P_nx\|, \ldots), \quad x \in X,$$

for every finite or infinite sequence of disjoint projections $P_n \in \mathfrak{B}$ whose sum is the identity I.

Remarks. 1. ψ may take infinite values.

2. ψ should not be a symmetric function although the norms $\|P_nx\|$ can be substituted in ψ in any desired order.

3. Mackey [8], Theorem 55 (see also Wermer [11]), has proved that every Boolean algebra of projections on a Hilbert space has a two-sided estimate with $\psi(a_1, \ldots, a_n, \ldots) = \big(\sum_n |a_n|^2\big)^{1/2}$ while Lindenstrauss and Pełczyński [5], Corollary 8 of Theorem 6.1, have shown that on an $\mathscr{L}_1$-space ($\mathscr{L}_\infty$-space) every Boolean algebra of projections has a two-sided estimate with $\psi(a_1, \ldots, a_n, \ldots) = \sum_n |a_n|$ $\big(\psi(a_1, \ldots, a_n, \ldots) = \sup_n |a_n|\big)$. This is not true for $\mathscr{L}_p$-spaces, $1 \leqslant p \neq 2 < +\infty$, and it follows from Pełczyński [10], Theorem 7, and our Proposition 3. However, for the Boolean algebras of "multiplications by characteristic functions" in a space isomorphic to c_0 or L_p, $1 \leqslant p < +\infty$, ψ can be chosen as $\psi(a_1, \ldots, a_n, \ldots) = \sup_n |a_n|$, respectively $\psi(a_1, \ldots, a_n, \ldots) = \big(\sum_n |a_n|^p\big)^{1/p}$.

For a Banach space X with an unconditional basis $\{e_n\}$ and any set of integers $\sigma \subset N$ let us write

$$P(\sigma)\Big(\sum_{n=1}^{\infty} a_n e_n\Big) = \sum_{n\in\sigma} a_n e_n, \quad \sum_{n=1}^{\infty} a_n e_n \in X.$$

Since the basis is unconditional, the projections $\{P(\sigma)\}_{\sigma\subset N}$ form a σ-complete Boolean algebra of projections $\mathfrak{B}$ which will be called *the Boolean algebra of projections generated by the basis* $\{e_n\}$.

The proof of the main results is based on the next proposition which constitutes a characterization of l_p, $1 \leqslant p < \infty$, and c_0 among the Banach spaces with unconditional bases.

Proposition 3. *Let X be a Banach space with a normalized unconditional basis $\{e_n\}$ and $\mathfrak{B}$ the σ-complete Boolean algebra of projections generated by the basis. If $\mathfrak{B}$ has a two-sided estimate, then X is isomorphic either to c_0 or to l_p, $1 \leqslant p < \infty$, and under this isomorphism the basis $\{e_n\}$ is equivalent to the natural unit vectors basis c_0 or l_p.*

Proof. Assume that the two-sided estimate of $\mathfrak{B}$ is given by the constant K and the function ψ as in the Definition 2. Let $\{p_k\}$ be an increasing sequence of non-negative integers and

$$w_k = \sum_{n=p_k+1}^{p_{k+1}} \lambda_n e_n, \quad k = 1, 2, \ldots,$$

with λ_n scalars such that $\|w_k\| = 1$, $k = 1, 2, \ldots$ A sequence having this form is called a *normalized block sequence* with respect to $\{e_n\}$. Consider $x = \sum_{i=q}^{r} a_i e_i$ and write $w = \sum_{i=q}^{r} a_i w_i$. Then

$$\begin{aligned}\|x\| &\leqslant K\psi(|a_q|, |a_{q+1}|, \ldots, |a_r|, 0, 0, \ldots)\\ &\leqslant K\psi(\|P(\sigma_q)w\|, \|P(\sigma_{q+1})w\|, \ldots, \|P(\sigma_r)w\|, 0, 0, \ldots)\\ &\leqslant K^2\|w\|,\end{aligned}$$

where $\sigma_k = \{p_k+1, \ldots, p_{k+1}\}$. By means of symmetry we have also

$$\|x\| \geqslant K^{-2}\|w\|$$

which implies that the unconditional basis $\{e_n\}$ is equivalent to every normalized block sequence. Thus, by Zippin [12], Theorem 3.1, the basis $\{e_n\}$ is equivalent to the natural unit vectors basis of c_0 or l_p, $1 \leqslant p < +\infty$. Hence X is isomorphic to c_0 or l_p, $1 \leqslant p < +\infty$ (this is a well-known consequence of the open mapping theorem and the uniform boundedness principle; cf. [4], Ch. II), q.e.d.

Theorem 4. *A Banach space X is isomorphic to c_0 or L_p, $1 \leqslant p < +\infty$, on some finite measure space (Ω, Σ, μ) if and only if there exists a Boolean algebra of projections $\mathfrak{B}$ in X such that*

(a) *$\mathfrak{B}$ is σ-complete;*

(b) *$X = \operatorname{clm}\{Px_0 \mid P \in \mathfrak{B}\}$ for some $x_0 \in X$;*

(c) *$\mathfrak{B}$ has a two-sided estimate.*

Proof. The necessity is obvious and has been discussed in the introduction. In order to prove the converse, first, let us observe that $\mathfrak{B}$ can be considered as the range of a spectral measure $P(\sigma)$ defined on the

Borel sets $\sigma \in \Sigma$ of a compact Hausdorff topological space Ω and every projection in $\mathfrak{B}$ is countably decomposable (cf. Bade [1], Lemma 2.6). We also can assume that $\mathfrak{B}$ contains an infinite number of disjoint projections otherwise X is finite-dimensional and the assertion is trivial.

Let $0 \neq P(\sigma_n) \in \mathfrak{B}$, $n = 1, 2, \ldots$, be an infinite sequence of disjoint projections whose sum is I and remark that $\{P(\sigma_n)x_0/\|P(\sigma_n)x_0\|\}$ is an unconditional basis for the subspace

$$Y = \operatorname{clm}\{P(\sigma_n)x_0 \mid n = 1, 2, \ldots\}.$$

Since the Boolean algebra of projections generated by this basis is a subalgebra of $\mathfrak{B}$ having still a two-sided estimate, in view of Proposition 3 we conclude that Y is isomorphic to c_0 or l_p, $1 \leqslant p < +\infty$, and under this isomorphism the basis $\{P(\sigma_n)x_0/\|P(\sigma_n)x_0\|\}$ is equivalent to the natural unit vectors basis of c_0 or l_p. Now, consider any other sequence of disjoint projections $0 \neq P(\delta_n) \in \mathfrak{B}$, $n = 1, 2, \ldots$, whose sum is the identity I. Obviously, $\{P(\delta_n)x_0/\|P(\delta_n)x_0\|\}$ will be a normalized unconditional basis for the subspace

$$Z = \operatorname{clm}\{P(\delta_n)|x_0| n = 1, 2, \ldots\}.$$

Moreover, the existence of a two-sided estimate for $\mathfrak{B}$ implies

$$\Big\|\sum_n a_n \frac{P(\delta_n)x_0}{\|P(\delta_n)x_0\|}\Big\| \leqslant K\psi(|a_1|, |a_2|, \ldots) \leqslant K^2 \Big\|\sum_n a_n \frac{P(\sigma_n)x_0}{\|P(\sigma_n)x_0\|}\Big\|$$

and

$$\Big\|\sum_n a_n \frac{P(\delta_n)x_0}{\|P(\delta_n)x_0\|}\Big\| \geqslant K^{-2} \Big\|\sum_n a_n \frac{P(\sigma_n)x_0}{\|P(\sigma_n)x_0\|}\Big\|$$

for every vector

$$\sum_n a_n \frac{P(\delta_n)x_0}{\|P(\delta_n)x_0\|} \in Z.$$

Hence Z is isomorphic to Y and $d(Z, Y) \leqslant K^4$. Thus, the subspaces Z_ι, $\iota \in I$, corresponding to all possible infinite sequences of disjoint projections in $\mathfrak{B}$ whose sum is I (as Z for $\{P(\delta_n)\}$) will form a net of subspaces for which

(*) $X = \overline{\bigcup_{i \in I} Z_i}$;

(**) there exists a Banach space, either c_0 or l_p, $1 \leqslant p < +\infty$, which is isomorphic to all Z_ι, $\iota \in I$ (the same for all Z_ι) and $d(Z_\iota, c_0)$ or $d(Z_\iota, l_p)$ are uniformly bounded for all $\iota \in I$.

It immediately follows from Definition 1 that X is an $\mathscr{L}_p$-space for some $1 \leqslant p \leqslant +\infty$. If X is an $\mathscr{L}_\infty$-space, the proof can be completed by using McCarthy and Tzafriri [9] Theorem 16 and X will be isomorphic

to c_0. Thus we can assume that there exist $M \geqslant 1$ and $1 \leqslant p < \infty$ such that $d(Z_\iota, l_p) \leqslant M^2$, $\iota \in I$, and furthermore

$$M^{-1}\Big(\sum_{n=1}^\infty |a_n|^p\Big)^{1/p} \leqslant \Big\|\sum_{n=1}^\infty a_n \frac{P(\delta_n^{(\iota)})x_0}{\|P(\delta_n^{(\iota)})x_0\|}\Big\| \leqslant M\Big(\sum_{n=1}^\infty |a_n|^p\Big)^{1/p}$$

for every vector

$$\sum_{n=1}^\infty a_n \frac{P(\delta_n^{(\iota)})x_0}{\|P(\delta_n^{(\iota)})x_0\|} \in Z_\iota, \qquad \iota \in I.$$

Now, denote by Ω_0 the set of all atoms of the vector-valued measure $P(\cdot)x_0$. Since $I \in \mathfrak{B}$ is countably decomposable, Ω_0 should be at most countable and therefore $P(\Omega_0)X$ would be either a finite-dimensional space or a space with unconditional basis. In any case, by Proposition 3, $P(\Omega_0)X$ will be isomorphic to l_p for some $1 \leqslant p < +\infty$. Hence, we can assume with no loss of generality that $P(\Omega_0) \neq I$. Let $0 \neq P(\delta_k) \in \mathfrak{B}$, $k = 1, \ldots, m$, be a finite sequence of projections whose sum is $P(\delta)$ and $\sigma \in \Sigma$ such that $P(\sigma) \nleq P(\Omega - \Omega_0)$. Obviously, there exists $\iota \in I$ such that

$$P\big((\sigma \cup \Omega_0) \cap \delta_k\big) = P(\delta_k^{(\iota)}), \qquad k = 1, \ldots, m,$$

which implies

$$\sum_{k=1}^m \|P\big((\sigma \cup \Omega_0) \cap \delta_k\big)x_0\|^p \leqslant M^p \Big\|\sum_{k=1}^m \|P\big((\sigma \cup \Omega_0) \cap \delta_k\big)x_0\| \frac{P(\delta_k^{(\iota)})x_0}{\|P(\delta_k^{(\iota)})x_0\|}\Big\|^p$$

$$\leqslant M^p \|P\big((\sigma \cup \Omega_0) \cap \delta\big)x_0\|^p$$

and in view of the σ-additivity of $P(\cdot)x_0$

$$\sum_{k=1}^m \|P(\delta_k)x_0\|^p \leqslant M^p \|P(\delta)x_0\|^p.$$

Therefore, we are able to define

$$\nu(\delta) = \sup\Big\{\sum_{k=1}^m \|P(\delta_k)x_0\|^p\Big\}, \qquad \delta \in \Sigma,$$

where the supremum is taken over all finite partitions of δ. Evidently,

$$\|P(\delta)x_0\|^p \leqslant \nu(\delta) \leqslant M^p \|P(\delta)x_0\|^p, \qquad \delta \in \Sigma,$$

and if $\delta, \sigma \in \Sigma$, $\delta \cap \sigma = \emptyset$, we have

$$\nu(\delta \cup \sigma) \geqslant \sup\Big\{\sum_{i=1}^r \|P(\delta_i)x_0\|^p + \sum_{j=1}^s \|P(\sigma_j)x_0\|^p\Big\},$$

where this supremum is taken over all the partitions δ_i of δ and σ_j of σ. It follows that

$$\nu(\delta\cup\sigma)\geqslant\nu(\delta)+\nu(\sigma).$$

Now, let us set

$$\mu(\delta)=\inf\sum_{i=1}^{m}\nu(\eta_i),\qquad \delta\in\Sigma,$$

where infimum is taken over all partitions η_i of δ. If $\delta,\sigma\in\Sigma$, $\delta\cap\sigma=\emptyset$

$$\mu(\delta\cup\sigma)\leqslant\inf\Big\{\sum_{i=1}^{r}\nu(\delta_i)+\sum_{j=1}^{s}\nu(\sigma_j)\Big\}=\mu(\delta)+\mu(\sigma),$$

where δ_i and σ_j are partitions of δ, respectively σ. Conversely, for every $\varepsilon>0$ there is a partition η_i^{ε} of $\delta\cup\sigma$ such that

$$\mu(\delta\cup\sigma)\geqslant\sum_{i=1}^{m}\nu(\eta_i^{\varepsilon})-\varepsilon$$

and if $\eta_i^{\varepsilon}=\delta_i^{\varepsilon}\cup\sigma_i^{\varepsilon}$, $\delta_i^{\varepsilon}\subset\delta$, $\sigma_i^{\varepsilon}\subset\sigma$, then

$$\mu(\delta\cup\sigma)\geqslant\sum_{i=1}^{m}\nu(\delta_i^{\varepsilon})+\sum_{i=1}^{m}\nu(\sigma_i^{\varepsilon})-\varepsilon\geqslant\mu(\delta)+\mu(\sigma)-\varepsilon.$$

Consequently, μ is an additive measure on (Ω,Σ) which satisfies

$$M^{-p}\|P(\delta)x_0\|^p\leqslant\mu(\delta)\leqslant\nu(\delta)\leqslant M^p\|P(\delta)x_0\|^p,\qquad \delta\in\Sigma.$$

Thus, μ is σ-additive.

The next and the final step will be to construct an isomorphism τ from X onto $L_p(\Omega,\Sigma,\mu)$. τ will be defined on the set (dense in X) of all vectors $\sum_{k=1}^{m}\beta_kP(\delta_k)x_0$ for which $P(\delta_k)\in\mathfrak{B}$, $k=1,2,\ldots,m$, are disjoint projections, as follows:

$$\tau\Big(\sum_{k=1}^{m}\beta_kP(\delta_k)x_0\Big)=\sum_{k=1}^{m}\beta_k\chi_{\delta_k}\in L_p(\Omega,\Sigma,\mu).$$

Then

$$\Big\|\tau\Big(\sum_{k=1}^{m}\beta_kP(\delta_k)x_0\Big)\Big\|^p=\sum_{k=1}^{m}|\beta_k|^p\mu(\delta_k)\leqslant M^p\sum_{k=1}^{m}|\beta_k|^p\|P(\delta_k)x_0\|^p$$

$$\leqslant M^{2p}\Big\|\sum_{k=1}^{m}\beta_k\Big\|P(\delta_k)x_0\Big\|\frac{P(\delta_k)x_0}{\|P(\delta_k)x_0\|}\Big\|^p\leqslant M^{2p}\Big\|\sum_{k=1}^{m}\beta_kP(\delta_k)x_0\Big\|^p$$

and by similar arguments

$$\Big\|\tau\Big(\sum_{k=1}^{m}\beta_kP(\delta_k)x_0\Big)\Big\|^p\geqslant M^{-2p}\Big\|\sum_{k=1}^{m}\beta_kP(\delta_k)x_0\Big\|^p.$$

Hence, τ can be extended to an isomorphism from X onto $L_p(\Omega,\Sigma,\mu)$ and this completes the proof, q.e.d.

The previous theorem is mostly a characterization of c_0 and *separable* L_p-spaces. The separability can be dropped as follows:

THEOREM 5. *A Banach space X is isomorphic to $c_0(\Gamma)$ for some abstract set Γ or to L_p, $1\leqslant p<+\infty$, on some measure space (Ω,Σ,μ) if and only if there exists a Boolean algebra of projections $\mathfrak{B}$ in X such that:*

(a) *$\mathfrak{B}$ is complete;*

(b) *$I\in\mathfrak{B}$ has multiplicity one;*

(c) *$\mathfrak{B}$ has a two-sided estimate.*

Proof. First, remark that the definition of multiplicity insures the existence of a set of disjoint projections $P(\sigma_\gamma)\in\mathfrak{B}$, $\gamma\in\Gamma$, such that

$$I=\bigvee_{\gamma\in\Gamma}P(\sigma_\gamma)$$

and

$$P(\sigma_\gamma)X=\operatorname{clm}\{P(\sigma)x_\gamma\,|\,P(\sigma)\in\mathfrak{B}\}$$

for some $x_\gamma\in\Gamma$, $\|x_\gamma\|=1$. If Γ is countable, by taking $x_0=\sum_{\gamma=1}^{\infty}\frac{x_\gamma}{2^\gamma}$ we get

$$X=\operatorname{clm}\{P(\sigma)x_0\,|\,P(\sigma)\in\mathfrak{B}\}$$

and we are again in the case covered by the previous theorem. Thus, we can assume with no loss of generality that Γ is uncountable and every subspace $P(\sigma_\gamma)X$, $\gamma\in\Gamma$, is infinite-dimensional (otherwise we can construct another decomposition of the identity which satisfies this condition). Let $\{\gamma_n\}$ be an infinite sequence in Γ and observe that $\{x_{\gamma_n}\}$ is an unconditional basis for its closed span which generates a Boolean algebra of projections included in $\mathfrak{B}$ and having a two-sided estimate. Hence, $\{x_{\gamma_n}\}$ is equivalent to the natural basis of c_0 or l_p, $1\leqslant p<\infty$ (cf. Proposition 3). Since $\mathfrak{B}$ has a two-sided estimate, every subspace of X having an unconditional basis which generates a Boolean algebra of projections included in $\mathfrak{B}$ will be isomorphic to $\operatorname{clm}\{x_{\gamma_n}\}$ and therefore to c_0 or l_p (c_0 or the same p for all these subspaces) and all this family of isomorphisms will be uniformly bounded. Thus, X is isomorphic either to $\sum_{\gamma\in\Gamma}{}_0\oplus P(\sigma_\gamma)X$ (direct sum in c_0-sense) or to $\sum_{\gamma\in\Gamma}{}_p\oplus P(\sigma_\gamma)X$ (direct sum in l_p-sense), where in the first case the subspaces $P(\sigma_\gamma)X$ are uniformly isomorphic to c_0 (i.e. $d(P(\sigma_\gamma)X,c_0)$, $\gamma\in\Gamma$, are uniformly bounded) and in the second case to L_p, $1\leqslant p<\infty$, for some measure space $(\sigma_\gamma,\Sigma_\gamma,\mu_\gamma)$. Consequently, X is isomorphic to $c_0(\Gamma)$ or L_p, $1\leqslant p<+\infty$, on some measure space, q.e.d.

References

[1] W. G. Bade, *On Boolean algebras of projections and algebras of operators*, Trans. Amer. Math. Soc. 80 (1955), p. 345-359.

[2] — *A multiplicity theory for Boolean algebras of projections in Banach spaces*, ibidem 92 (1958), p. 508-530.

[3] M. M. Day, *Normed linear spaces*, Berlin 1962.

[4] N. Dunford and J. Schwartz, *Linear operators I*, New York 1958.

[5] J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in $\mathscr{L}_p$-spaces and their applications*, Studia Math. 29 (1968), p. 275-326.

[6] J. Lindenstrauss and M. Zippin, *Banach spaces with a unique unconditional basis* (to appear).

[7] — *Banach spaces with sufficiently many Boolean algebras of projections* (to appear).

[8] G. Mackey, *Commutative Banach algebras*, Notes de Matematica 4, Rio de Janeiro.

[9] C. A. McCarthy and L. Tzafriri, *Projections in $\mathscr{L}_1$ and $\mathscr{L}_\infty$ spaces*, Pacific J. Math. (to appear).

[10] A. Pełczyński, *Projections in certain Banach spaces*, Studia Math. 19 (1960), p. 209-228.

[11] J. Werner, *Commuting spectral measures on Hilbert space*, Pacific J. Math. 4 (1954), p. 255-261.

[12] M. Zippin, *On perfectly homogeneous bases in Banach spaces*, Israel J. Math. 4 (1966), p. 265-272.

UNIVERSITY OF MINNESOTA

Reçu par la Rédaction le 6. 4. 1968

STUDIA MATHEMATICA, T. XXXII. (1969)

Note on the class $L\log L$

by

E. M. STEIN (Princeton)

1. The purpose of this note is to prove two theorems. Each of these incidentally characterizes the class $L\log L$ in terms of the converse of some well-known inequality.

In Theorem 1 the setting is $\boldsymbol{R}^n$, and for a given integrable function $f(x)$, we define the maximal function $Mf(x)$ by

$$(1)\qquad (Mf)(x) = \sup_{r>0} \frac{1}{m\big(B(x,r)\big)} \int_{B(x,r)} |f(y)|\,dy,$$

where $B(x,r)$ denotes the ball of radius r centered at x and $m\big(B(x,r)\big)$ is its Lebesgue measure.

THEOREM 1. *Suppose that f is integrable and is supported on some finite ball B. Then $\int_B Mf\,dx < \infty$ if and only if*

$$\int_B |f|\log^+|f|\,dx < \infty.$$

One direction, that $f\in L\log L$ implies $Mf\in L$, is very well known; but the converse although not really deeper, seems to have been overlooked all these years.

We shall also obtain a consequence of this result dealing with the Hilbert transform and its n-dimensional generalization, the Riesz transforms. The most appropriate setting for this will be periodic functions, i.e. those that satisfy $f(x+m) = f(x)$, where $m = (m_1, m_2, \ldots, m_n)$ is any vector with integral coordinates. We denote by Q the "fundamental cube" $-\frac{1}{2} < x_j \leqslant \frac{1}{2}$, $j = 1, \ldots, n$. Let

$$(2)\qquad f \sim \sum_m a_m e^{2\pi i m\cdot x}$$

be the Fourier series of a periodic function integrable over Q, and let its Riesz transforms be given by

$$(3)\qquad R_k(f) \sim i \sum_m{}' \frac{m_k}{|m|} a_m e^{2\pi i m\cdot x}, \qquad k = 1, \ldots, n.$$