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Sequential theory of the convolution of distributions

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1. In [3], a concept of *regular operations* has been introduced. Such operations extend automatically from functions to distributions and retain their properties. Operations which are not regular, are called *irregular*. Irregular operations can not be extended on arbitrary distributions. However, there exists a general method which permits, in cases where it is possible, to perform such an extension (see [2]).

Convolution is one of the most important irregular operations. Its extension on distributions was largely investigated by Laurent Schwartz [4] and other authors. The sequential approach which is the subject of the present paper makes use of the general method of defining irregular operations so that the definition of the convolution is nothing else but a particular case of it. It turns out that this definition embraces all cases in which the convolution was defined previously by other methods. This uniform approach can also be considered as more elementary, because it does not need any concepts of functional analysis or topology.

Beside the new approach to known facts, there is also a number of theorems which are stated, in this paper, for the first time.

In what follows we shall use the notation and the terminology of [3].

2. If φ is a smooth function of bounded carrier, then the convolutions

$$(1)\qquad f*\varphi = \int_{-\infty}^{\infty} f(x-t)\varphi(t)\,dt \quad\text{and}\quad \varphi*f = \int_{-\infty}^{\infty} \varphi(x-t)f(t)\,dt$$

are defined for every distribution f, as regular operations, performed on f. (It should be emphasized that the convolution is an irregular operation only if it is considered as an operation on two functions or distributions. Otherwise it is regular.) Such convolutions preserve, for any distribution f, their ordinary properties:

$$(2)\qquad f*\varphi = \varphi*f,$$

$$(3)\quad \begin{cases} (f_1+f_2)*\varphi = f_1*\varphi+f_2*\varphi, \\ f*(\varphi_1+\varphi_2) = f*\varphi_1+f*\varphi_2, \\ (\lambda f)*\varphi = f*(\lambda\varphi) = \lambda(f*\varphi) \quad (\lambda \text{ real or complex number}), \\ f^{(k)}*\varphi = f*\varphi^{(k)} = (f*\varphi)^{(k)}, \end{cases}$$

$$(4)\qquad (f*\varphi_1)*\varphi_2 = f*(\varphi_1*\varphi_2).$$

All the above facts follow from the general theory of regular operations, presented in [3]. The distributions and functions are considered as defined in a Euclidean space $\boldsymbol{R}^q$ of any fixed number of dimensions q. There is no reason to restrict oneself to functions and distributions whose values are real or complex numbers. Although such a restriction was made, for didactic purposes in [3], the whole theory presented in that paper applies also, when admitting the values in a Banach space. In what follows, all results can be interpreted for real or complex valued functions and distributions as well as for functions and distributions with values in an arbitrary given Banach space. However, in the last case, some explanations are needed. Namely, in order to make sensible the products under the integral sign in (1), we have to say, generally, what a product ab of elements means, when none of factors a and b is a number. Let B_1, B_2 and B_3 be three given Banach spaces. We shall assume that, if $a\in B_1$ and $b\in B_2$, then $ab\in B_3$ and that the following properties are fulfilled (see [1]):

1° $(a_1+a_2)(b_1+b_2) = a_1b_1+a_1b_2+a_2b_1+a_2b_2$;

2° $\lambda a\cdot \varkappa b = \lambda\varkappa\cdot ab$ (λ, $\varkappa$ real numbers);

3° $|ab| \leqslant |a|\cdot|b|$.

The modulus sign in equality 3° is used for norms in the corresponding Banach spaces. Properties 1°-3° ensure that equalities (3) are true. If we assume, moreover, that $ab = ba$, then also (2) holds. In order to ensure (4), where the values of f, φ_1, φ_2 belong to Banach spaces B_1, B_2, B_3 respectively, we have to assume that the commutativity of multiplication, $a(bc) = (ab)c$, holds generally for $a\in B_1$, $b\in B_2$, $c\in B_3$ (the product may then be an element of another Banach space B_4).

3. In the contrary to the preceding case, if we consider the convolution $f*g$ as an operation on two functions or distributions f and g, then this operation is irregular; it is not defined for every pair of distributions (and even not for every pair of functions). We say that the convolution $f*g$ *is defined* for distributions f and g, iff, for any delta sequence δ_n, the sequence $(f*\delta_n)*(g*\delta_n)$ is defined and converges distributionally to a distribution h. Then we put, by definition, $h = f*g$. This definition needs some explanations. First of all, we must say what a delta sequence is. This will be, in the sequel, any sequence of real-valued smooth functions δ_n with the following properties (see [2]):

1° There is a sequence of positive numbers α_n, convergent to 0, such that $\delta_n(x) = 0$ for $|x| \geqslant \alpha_n$.

2° $\int\limits_{-\infty}^{\infty} \delta_n = 1$.

3° There are numbers $M_0, M_1, \ldots$ such that

$$\alpha_n^k \int\limits_{-\infty}^{\infty} |\delta_n^{(k)}| < M_k$$

holds for $n = 1, 2, \ldots$ and every order k.

In 3°, if the order of differentiation is $k = (\varkappa_1, \ldots, \varkappa^q)$, then by α_n^k we understand the number $\alpha_n^{\varkappa_1+\ldots+\varkappa_q}$.

Now, the convolutions $f*\delta_n$ and $g*\delta_n$ are fully defined in the sense of section 1. Moreover, it is known that those convolutions are smooth functions. Still, the convolutions $(f*\delta_n)*(g*\delta_n)$ need not be defined, in general. Denoting $f*\delta_n$ and $g*\delta_n$ by f_n and g_n respectively we shall say that the convolution f_n*g_n is defined, iff, given any orders k and l, the integral $\int\limits_{-\infty}^{\infty} f_n^{(k)}(x-t)\,g_n^{(l)}(t)\,dt$ exists in the Bochner sense, i.e. if the integral $\int\limits_{-\infty}^{\infty} |f_n^{(k)}(x-t)\,g_n^{(l)}(t)|\,dt$ converges. Then we put

$$(f*\delta_n)*(g*\delta_n) = \int\limits_{-\infty}^{\infty} f_n(x-t)\,g_n(t)\,dt.$$

Directly from the definition it follows that, if convolutions f_1*g and f_2*g exist, then also the convolution $(f_1+f_2)*g$ exists and

$$(f_1+f_2)*g = f_1*g+f_2*g.$$

If the convolution $f*g$ exists, then also the convolutions $(\lambda f)*g$ and $f*(\lambda g)$ exist for every real number λ and we have

$$(\lambda f)*g = f*(\lambda g) = \lambda(f*g).$$

If $f*g$ exists, then also $f^{(k)}*g$ and $f*g^{(k)}$ exist for every order k and we have

$$f^{(k)}*g = f*g^{(k)} = (f*g)^{(k)}.$$

The associativity $(f*g)*h = f*(g*h)$ does not hold in general, even if both sides are meaningful. But this is not astonishing, because the associativity of the convolution does not hold, even if both factors are real-valued smooth functions and the convolution is defined in the ordinary way. E.g., let x be a real variable and let

$$f(x) = 1, \qquad g(x) = xe^{-x^2}, \qquad h = \int\limits_{-\infty}^{x} e^{-t^2}\,dt.$$

Then it is easy to verify that $(f*g)*h = 0$ and $f*(g*h) = \pi/2$.

Note that, if one of distributions f or g reduces to a smooth function of bounded carrier, then the convolution $f*g$, defined in the sense of this section, does exist and is equal to the convolution in the sense of section 1. Therefore the same symbol $f*g$ may be used in both cases.

We are going to investigate, now, a few particular cases in which the existence of the convolution is ensured.

4. In this section we shall be concerned with the case which is related to the theory of p-integrable ($1 \leqslant p < \infty$) functions in $\boldsymbol{R}^q$. The space of those functions will be denoted by L_p. The following classical theorem, due to Young, plays here the striking role:

If $f \in L_p$, $g \in L_q$ and $1/r = 1/p+1/q-1 \geqslant 0$ $(0 \leqslant p, q < \infty)$, then $f*g \in L_r$. Moreover, $\|f*g\|_r \leqslant \|f\|_p \cdot \|g\|_q$.

This theorem is known mainly for real-valued functions, but it is true also for functions whose values are in Banach spaces. The proof remains the same, needs only a more general interpretation.

In the above theorem, the convolution is understood in the sense of the Bochner or of the Lebesgue integral. Our first task is to show that, under the hypotheses of the theorem, the convolution $f*g$ exists also in the sense of section 2, and represents the same function.

If $f \in L_p$ and δ_n is a delta sequence, then

$$|f*\delta_n - f| = \left|\int [f(x-t)-f(x)]\,\delta_n(t)\,dt\right| \leqslant \int |f(x-t)-f(x)|\,|\delta_n(t)|^{1/p}\cdot|\delta_n(t)|^{1-1/p}\,dt.$$

Hence, by the Hölder inequality,

$$|f*\delta_n - f| \leqslant \left(\int |f(x-t)-f(x)|^p\,|\delta_n(t)|\,dt\right)^{1/p}\cdot M_0,$$

in view of property 3° of delta sequences. This implies

$$\|f*\delta_n - f\|_p \leqslant M_0 \int \left(\int |f(x-t)-f(x)|^p\,|\delta_n(t)|\,dt\right) dx = M_0 \int |\delta_n(t)|\,dt \int |f(x-t)-f(x)|^p\,dx.$$

Since $\int |f(x-t)-f(x)|^p\,dx \to 0$, as $t \to 0$, there is a sequence of numbers $\varepsilon_n > 0$, tending to 0, such that $\int |f(x-t)-f(x)|^p\,dx < \varepsilon_n$ for $|t| \leqslant a_n$. Hence

$$\|f*\delta_n - f\|_p \leqslant M_0 \int |\delta_n(t)|\,dt\cdot\varepsilon_n \leqslant M_0^2\,\varepsilon_n \to 0.$$

We can similarly prove that $f \in L_p$ implies $f*\delta_n^{(k)} \in L_p$ for every k and n.

If f and g satisfy the hypotheses of the Young theorem, we have

$$(f*\delta_n)*(g*\delta_n) = (f*g)*(\delta_n*\delta_n). \tag{5}$$

Since $\bar{\delta}_n = \delta_n*\delta_n$ is a delta sequence as well as δ_n, the above equality implies that $(f*\delta_n)*(g*\delta_n)$ converges in the r-th mean, thus also distributionally, to $f*g$. Thus the convolution $f*g$ exists in the sense of section 2 and represents the same function as the convolution in the ordinary sense.

In the preceding argument, one might ask about the validity of equality (5). This easily follows from the following known theorem (being a simple consequence of the Fubini theorem on multiple integrals, see e.g. [1]):

If f, g, h are measurable functions and the convolutions $(|f|*|g|)*|h|$ and $g*h$ exist, then also all the convolutions in the equality $(f*g)*h = f*(g*h)$ exist and the equality holds.

The assertion remains true if we replace the hypothesis that $g*h$ exists by the hypothesis that $f \neq 0$ on a set of positive measure.

If f and g satisfy the hypotheses of the Young theorem, then the convolution of distributions $f^{(k)}$ and $g^{(l)}$ exists, in the sense of section 2, for any orders k and l. If $\mathscr{D}'_{L_p}$ (see [4]) denotes the linear space generated by the set of distributions of the form $f^{(k)}$ with $f \in L_p$ and arbitrary k, then it follows from the preceding argument that the convolution $f*g$ exists for distributions $f \in \mathscr{D}'_{L^p}$ and $g \in \mathscr{D}'_{L^q}$, where $1/r = 1/p+1/q-1$. Evidently $f*g \in \mathscr{D}'_{L^r}$.

Combining the Young theorem with the associativity theorem, quoted above, it follows that, if $f \in L^p$, $g \in L^q$, $h \in L^r$ with $1/p+1/q+1/r \geqslant 2$, then $(f*g)*h = f*(g*h)$. This implies the following associativity theorem for the convolution of distributions:

THEOREM 1. *If $f \in \mathscr{D}'_{L^p}$, $g \in \mathscr{D}'_{L^q}$, $h \in \mathscr{D}'_{L^r}$ with $1/p+1/q+1/r \geqslant 2$, then all the convolutions in the equality $(f*g)*h = f*(g*h)$ exist and the equality holds.*

5. If $f \in L_p$, then for every $\varepsilon > 0$ the set of points at which $|f| > \varepsilon$ is of finite measure. In this sense we may say that the functions of class L_p are small at infinity. Consequently, the theory presented in the preceding section applies to a restricted domain and fails for polynomials, for instance, and other functions which increase at infinity.

However, the convolution $f*g$ makes sense even when f increases at infinity, provided the second factor g decreases then rapidly enough. Such cases are important in the theory of Fourier transform of distributions, for instance. We are going to gather them, in this section, in a more general scheme.

We shall say that a real-valued function u is *subexponential*, if it is positive, locally integrable, and satisfies the inequality

$$u(x+y) \leqslant u(x)\,u(y).$$

The functions e^x, $2^n+|x|^n$ and $e^{\sqrt{|x|}}$ are examples of subexponential functions. In these examples, x may be a real variable or a point $x = (\xi_1, \ldots, \xi_q)$ of the q-dimensional space $\boldsymbol{R}^q$; in the last case e^x is to be read as $e^{\xi_1+\ldots+\xi_q}$.

THEOREM 2. *If f and g are locally integrable functions such that the quotient $f(x)/u(x)$ is bounded and the product $u(-x)g(x)$ is integrable for a subexponential function $u(x)$, then the convolution $h = f*g$ exists in the ordinary sense, as well as an irregular operation, and represents a locally integrable function such that the quotient $h(x)/u(x)$ is bounded.*

Proof. Let $|f(x)| \leqslant Ku(x)$. Then

$$|f*g| \leqslant |f|*|g| \leqslant K\cdot u*|g| = K\int u(x-t)|g(t)|dt$$
$$\leqslant Ku(x)\int u(-t)|g(t)|dt \leqslant K_1 u(x).$$

We can also verify that the quotient $(f*\delta_n^{(k)})/u$ is bounded and the product $\bar{u}(g*\delta_n^{(k)})$, where $\bar{u}(x) = u(-x)$, is integrable. Evidently, we have

$$(f*\delta_n)*(g*\delta_n) = (f*g)*\bar{\delta}_n,$$

where $\bar{\delta}_n = \delta_n*\delta_n$ is a delta sequence. This completes the proof.

Retaining the assumption on u, if there are two functions g_1 and g_2 such that the products $u(-x)g_1(x)$ and $u(-x)g_2(x)$ are integrable, then the convolutions $(u*g_1)*g_2$ exist, which is asserted by a repeated application of Theorem 2. Similarly, the convolutions $(|u|*|g_1|)*|g_2|$ exist. Since $u > 0$, this implies that also the convolutions $u*(|g_1|*|g_2|)$ exist. Let $G = |g_1|*|g_2|$. Then,

$$\int u(-t)G(t)\,dt \leqslant u(-x)\int u(x-t)G(t)\,dt = u*(|g_1|*|g_2|),$$

which implies that the product $u(-x)G(x)$ is integrable. Consequently also the product $u(-x)\big(g_1(x)*g_2(x)\big)$ is integrable.

Thus, if E_u denotes the set of all functions g such that the product $u(-x)g(x)$ is integrable, then the convolution g_1*g_2 exists and belongs to E_u, provided g_1 and g_2 belong to E_u. Similarly, if g_1, g_2, g_3 belong to E_u, then the convolutions $(|g_1|*|g_2|)*|g_3|$ and g_2*g_3 exist. This implies that $(g_1*g_2)*g_3 = g_1*(g_2*g_3)$. This proves that E_u is a ring under convolution as multiplication.

Every tempered distribution is a derivative of some order k of a continuous function f bounded by a polynomial $u(x) = 2^p+|x|^p$. If g is a rapidly decreasing function, then the product $u(-x)g(x)$ is integrable and the convolution $f*g$ exists. This is a particular case of Theorem 2. Subsequently, there exists also the convolution $f^{(k)}*(g_1^{(l_1)}\ldots g_p^{(l_p)})$, where $g_1, \ldots, g_p$ are rapidly decreasing functions. In other words, if f is a tempered distribution and g a rapidly decreasing distribution, the convolution $f*g$ exists. It follows also easily that, if f, g, h are tempered distributions, two at least of them being rapidly decreasing, we have $(f*g)*h = f*(g*h)$.

6. We are coming now to next important case in which the existence of the convolution $f*g$ of two distributions is ensured: we shall make no restriction on the growth of distributions, but some restriction will be imposed on their carriers.

Given any subset Y of $\boldsymbol{R}^q$ and a positive number α, we shall denote by Y_α the α-neighbourhood of Y, i.e., the set of all points $x\in\boldsymbol{R}^q$ whose distances from Y are less than α. Evidently, the neighbourhood Y_α is always an open set.

We shall say that two subsets Y and Z of $\boldsymbol{R}^q$ are *compatible*, iff the intersection $(-Y)_\alpha \cap Z_\alpha$ is bounded for any fixed number $\alpha > 0$; here, by $-Y$ we understand the set of all $x\in\boldsymbol{R}^q$ such that $-x\in Y$. Evidently, we may also say that the sets Y and Z are compatible, iff the intersection $Y_\alpha \cap (-Z)_\alpha$ is bounded for any fixed number α; thus, the relation of being compatible is symmetric. The definition of compatible sets may also be formulated as follows: the sets Y and Z are compatible iff, given any number $\alpha > 0$, the set of points whose distances from $-Y$ and Z (or from Y and $-Z$) are less than α is bounded.

Evidently, if one of sets Y or Z is bounded, then Y and Z are compatible. Another example: if $x \geqslant a$ for $x\in Y$ and $x \geqslant b$ for $x\in Z$, then Y and Z are compatible.

If Y and Z are compatible, then so are the sets Y_α and Z_β, no matter what are the positive numbers α and β.

THEOREM 3. *If Y is the carrier of a distribution of finite order $f = F^{(k)}$, where F is a continuous function in $\boldsymbol{R}^q$, then for any given $\alpha > 0$, f can be represented as a finite sum of derivatives of continuous functions whose carriers are in Y_α.*

Proof. Let φ be a smooth function which admits the value 1 on $Y_{\alpha/2}$ and vanishes outside Y_α. Then the assertion follows from the identity

$$f = \varphi F^{(k)} = \sum_{0\leqslant m\leqslant k}(-1)^m\binom{k}{m}(\varphi^{(m)}F)^{(k-m)}.$$

THEOREM 4. *If f and g are distributions whose carriers Y and Z are compatible, then the convolution exists as an irregular operation.*

Proof. We shall first assume that f and g are continuous functions. Then the sequences $f_n = f*\delta_n$ and $g_n = g*\delta_n$ converge almost uniformly to f and g respectively. Since Y and Z are carriers of f and g, we have $f_n(-x) = 0$ outside $(-Y)_\alpha$ and $g_n(x) = 0$ outside Z_α, where $\alpha = \sup\alpha_n$. Thus, if ϱ is a positive number, the product $f_n(x-t)g_n(t)$ vanishes for $t\notin(-Y)_{\varrho+\alpha}\cap Z_\alpha$. This implies that the products $f_n(x-t)g_n(t)$ converge to $f(x-t)g(t)$ uniformly for $|x| < \varrho$ and every $t\in\boldsymbol{R}^q$, as $n\to\infty$. This implies that the convolutions

$$f_n^{(k)}*g_n^{(l)} = \int f_n^{(k)}(x-t)\,g_n^{(l)}(t)\,dt \tag{6}$$

exist for every k, l, and that, for $k = l = 0$, this sequence converges uniformly in $|x| < \varrho$ as $n \to \infty$. Since ϱ is arbitrary, the assertion is proved for continuous f and g. If f and g are distributions of finite order, then the assertion follows by Theorem 3.

It remains to consider the case, when we do not assume that f and g are of finite order. If $|x| < \varrho$, then as before the products $f_n(x-t)g(t)$ vanish outside $(-Y)_{\varrho+a} \cap Z_a$, but we cannot assert that the sequence converges uniformly. Let Ω be a smooth function, equal to 1 on the intersection $(-Y)_{2\varrho+2a} \cap Z_{2a}$ and vanishing outside $(-Y)_{2\varrho+3a} \cap Z_{3a}$. Let $\bar{f} = f\Omega$, $\bar{g} = g\Omega$, $\bar{f}_n = \bar{f} * \delta_n$, $\bar{g}_n = \bar{g} * \delta_n$. Then we have

$$\bar{f}_n(x-t)\,\bar{g}_n(t) = f_n(x-t)\,g_n(t) \tag{7}$$

for $|x| < \varrho$ and $t \in (-Y)_{\varrho+a} \cap Z_a$. On the other hand, we have $\bar{f}_n(x-t)\,\bar{g}_n(t) = 0$ for $|x| < \varrho$ and $t \notin (-Y)_{\varrho+a} \cap Z_a$, which can be proved by the same argument as for f_n and g_n, for the carriers of $\bar{f}$ and $\bar{g}$ are included in the carriers of f and g respectively. Thus (7) holds for every t, provided $|x| < \varrho$. Hence

$$f_n * g_n = \bar{f}_n * \bar{g}_n \quad \text{for} \quad |x| < \varrho. \tag{8}$$

Since $\bar{f}$ and $\bar{g}$ are of finite order, the sequence $\bar{f}_n * \bar{g}_n$ converges distributionally in $\boldsymbol{R}^q$. By (8), the sequence $f_n * g_n$ converges distributionally for $|x| < \varrho$. Since ϱ is arbitrary, it converges distributionally in $\boldsymbol{R}^q$ and the proof is complete.

The preceding theorem can be completed by the almost obvious statement that the carrier of the convolution $f*g$ is included in the set $Y+Z$ which consists of elements $x = y+z$ such that $y \in Y$ and $z \in Z$.

Note that, letting $n \to \infty$ in (8), we obtain incidently

$$f*g = \bar{f}*\bar{g} \quad \text{for} \quad |x| < \varrho.$$

This remark will be useful in the proof of

THEOREM 5. *Let f_n and g_n be sequences of distributions such that $f_n \to f$ and $g_n \to g$. Let the carriers of f_n be included in a set Y and the carriers of g_n in another set Z which is compatible with Y. Then $f_n * g_n \to f*g$.*

Proof. We first assume that f_n and g_n are continuous and that the sequences converge almost uniformly. If ϱ is a positive number and $|x| < \varrho$, the products $f_n(x-t)g_n(t)$ vanish for t not belonging to $(-Y)_\varrho \cap Z$ and, as $n \to \infty$, converge to $f(x-t)g(t)$ uniformly for $|x| < \varrho$ and $t \in \boldsymbol{R}^q$. Hence, as in the preceding proof, we conclude that the convolutions $f_n * g_n$ exist and converge to $f*g$ almost uniformly in $\boldsymbol{R}^q$.

If there are orders k and l such that $F_n^{(k)} = f_n$, $G_n^{(l)} = g_n$, F_n and G_n being continuous and almost uniformly convergent for $n \to \infty$, then the assertion follows by Theorem 3.

Consider the general case. Let $\varrho > 0$ and let Ω be a smooth function, equal to 1 on $(-Y)_{2\varrho+2a} \cap Z_{2a}$ and vanishing outside $(-Y)_{2\varrho+3a} \cap Z_{3a}$. Let $\bar{f}_n = f_n\Omega$ and $\bar{g}_n = g_n\Omega$. Then

$$f_n * g_n = \bar{f}_n * \bar{g}_n \quad \text{for} \quad |x| < \varrho, \tag{9}$$

by the remark just before Theorem 5. Since the sequences $\bar{f}_n$ and $\bar{g}_n$ are convergent and their terms have their carriers included in a common bounded set (carrier of Ω), there are orders k and l such that $\bar{F}_n^{(k)} = \bar{f}_n$ and $\bar{G}_n^{(l)} = \bar{g}_n$, $\bar{F}_n$ and $\bar{G}_n$ being continuous functions, convergent almost uniformly for $n \to \infty$. Thus, by the preceding result, $\bar{f}_n * \bar{g}_n$ converges in $\boldsymbol{R}^q$. By (9), the sequence $f_n * g_n$ converges in $|x| < \varrho$. Since ϱ is arbitrary, $f_n * g_n$ converges in $\boldsymbol{R}^q$.

We shall prove that *the sets X and Y are compatible, iff*

$$x \in X,\ y \in Y,\ |x|+|y| \to \infty \ \text{implies}\ |x+y| \to \infty. \tag{*}$$

In fact, assume that X and Y are compatible and $x_n \in X$, $y_n \in Y$, $|x_n|+|y_n| \to \infty$. If $|x_n+y_n| \to \infty$ does not hold, there is an increasing sequence of positive integers p_n such that $|x_{p_n}+y_{p_n}| < M$. The distances of $-x_{p_n}$ from $-X$ are 0, and the distances of $-x_{p_n}$ from Y are less than M. Consequently, all x_{p_n} are contained in a bounded set, for X and Y are compatible. Similarly, all y_{p_n} are contained in a bounded set. This contradicts $|x_n|+|y_n| \to \infty$.

Now, assume that X and Y are not compatible. Then there is a sequence z_n such that the distances of z_n from $-X$ and Y are bounded and $|z_n| \to \infty$. Thus there are sequences $-x_n \in -X$ and $y_n \in Y$ such that $|z_n+x_n| < M$ and $|z_n-y_n| < M$. This implies that $|x_n| \to \infty$, $|y_n| \to \infty$ and $|x_n+y_n| < 2M$, which proves that condition (*) is not satisfied.

Condition (*) is more adequate for generalizations, when considering three (or more) sets. We shall say that three subsets X, Y, Z of $\boldsymbol{R}^q$ are *compatible*, iff $x \in X$, $y \in Y$, $z \in Z$, $|x|+|y|+|z| \to \infty$ implies $|x+y+z| \to \infty$. Evidently, if three sets are compatible, then each two of them so are. Moreover, if X, Y, Z are compatible, then the sets $X+Y$ and Z are compatible; similarly X and $Y+Z$ are compatible.

THEOREM 6. *If the carriers of distributions f, g, h are compatible, then $(f*g)*h = f*(g*h)$.*

Proof. Let $f_n = f*\delta_n$, $g_n = g*\delta_n$ and $h_n = h*\delta_n$. If the carriers X, Y, Z of f, g, h are compatible, then also the carriers of f_n, g_n, h_n are included in compatible sets X_a, Y_a, Z_a and therefore are themselves compatible. This implies that the convolutions $|f_n| * |g_n|$, $(|f_n| * |g_n|) * |h_n|$ and $g_n * h_n$ exist. Consequently, we have $(f_n * g_n) * h_n = f_n * (g_n * h_n)$. Hence the assertion follows by Theorem 5.

From theorem 6 it follows, in particular, that $(f*g)*h = f*(g*h)$ holds always, if two of distributions f, g, h are of bounded carrier.

7. We shall still prove that the commutativity $(f*g)*h = f*(g*h)$ holds also, if one of distributions f, g, h is of bounded carrier, provided the convolution of two remaining ones exists. We have namely:

THEOREM 7. *If the convolution $f*g$ exists and h is a distribution of bounded carrier, then, in the equality $(f*g)*h = f*(g*h)$, all the convolutions exist and the equality holds.*

Proof. The existence of convolutions $(f*g)*h$ and $g*h$ follows, as a particular case, from Theorem 5. It remains to prove the existence of $f*(g*h)$ and the equality. Let $k_n = (f*\delta_n)*(g*\delta_n)$. Then $k_n \to f*g$ by the hypothesis that $f*g$ exists. Now, by Theorem 5 we have $k_n*h \to (f*g)*h$. Assume that h is a continuous function. Then

$$k_n*h = [(f*\delta_n)*(g*\delta_n)]*h = (f*\delta_n)*[(g*\delta_n)*h] = (f*\delta_n)*[(g*h)*\delta_n];$$

here, the second equality follows from the fact that $f*\delta_n$, $g*\delta_n$ and h are functions for which the convolutions $(|f*\delta_n|*|g*\delta_n|)*|h|$ and $(g*\delta_n)*h$ exist, and the last equality follows from the remark at the end of section 6. Since k_n*h converges, the convolution $f*(g*h)$ exists, by definition, and the equality $(f*g)*h = f*(g*h)$ holds. If h is not a continuous function, then the assertion follows by Theorem 3.

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On the uniqueness of the ideals of compact and strictly singular operators *

by

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The purpose of this note is to extend a result of [3]. In particular, it is shown there that the ideal of compact operators is unique in $[X]$, the bounded linear operators from X to X, where $X = lp$, $1 \leqslant p < \infty$ and c_0. An obvious question is do there exist other spaces for which this is true? We obtain a partial result in this direction by requiring our space to have two properties which lp and c_0 enjoy. In addition, using one of these properties, we show that the ideal of compact and strictly singular operators agree. The phrase "partial result" is used since we cannot at this time exhibit a space with the above-mentioned properties other than lp or c_0. However, the proofs given here have the advantage of treating all cases simultaneously, as opposed to what is done in [3].

We will assume that the reader is somewhat familiar with the theory of Schauder bases in Banach spaces. Results used from this theory may be found in [2].

1. Definition. $\{e_i\}$ is a *Schauder basis* for X if for each $x \in X$, $x = \sum_{1}^{\infty} a_i x_i$ uniquely. In this case $a_i = g_i(x)$, $g_i \in X^*$.

2. Definition. $\{z_k\}$ is said to be a *block basis* if

$$z_k = \sum_{a_k+1}^{a_{k+1}} a_i^{(k)} e_i, \qquad a_1 < a_2 < \ldots$$

If $\{e_i\}$ is a Schauder basis for X, then $\{z_k\}$ is a Schauder basis for $\overline{\mathrm{sp}}\{z_k\}$, [1].

3. Definition. We will say that a Banach space X with a basis has $(+)$ if given a block basis $\{z_k\}$, there exists $P: X \to \overline{\mathrm{sp}}\{z_k\}$, P a projection.

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