(b) A^T *maps* S'_x *one to one onto* $(S_y^0)'$;

(c) $A(S_y^0) = S_x$ *and* $A^T(S'_x) = (S_y^0)'$;

(d) (i) $\sum_{i=1}^{\infty} t_i y_i = 0$ *implies* $t_i = 0$ *for each* i, (ii) A^T *is one to one on* S'_x, (iii) $A^T(S'_x)$ *is closed in* $(S_y^0)'$.

Proof. (a) $\Leftrightarrow$ (b). This follows from (a) $\Leftrightarrow$ (e) of Theorem 1 and Lemma 2. This statement is Theorem 3.2 of [3].

(a) $\Leftrightarrow$ (c). By (e) of Theorem 1, $A^T(S'_x) = (S_y^0)'$ if and only if $\mathcal{Y}$ is basic. In the proof of Theorem 2.1 of [3] it is shown that $A(S_y^0) = S_x$ if and only if $\mathcal{Y}$ is fundamental in X.

(a) $\Leftrightarrow$ (d). This follows from (f) $\Leftrightarrow$ (a) in Theorem 1 and from Lemma 2.

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On the characterization of sequence spaces associated with Schauder bases

by

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1. Introduction. An F-space which has a Schauder basis is essentially a space of sequences ([9], p. 207). This paper discusses the question: What kinds of sequence spaces are associated with a Schauder basis of a locally convex F-space? The chief results are contained in Theorems 3.1, 3.2 and 3.3. They are correspondences between (a) Schauder bases and γ-perfect FK-spaces (b) unconditional bases and α-perfect FK-spaces (c) symmetric bases and σ-perfect FK-spaces. (See 2.1 and 2.2 for definitions of γ, α and σ-perfect.)

1.1. Definition. An *F-space* is a complete linear metric space.

A sequence $\chi = \{x_1, x_2, \ldots\}$ is a *basis* for the F-space X if each point x of X has a unique representation

$$x = \sum_{i=1}^{\infty} t_i x_i \tag{1.1}$$

where (t_i) is a sequence of scalars.

The sequence χ is an *unconditional basis* if the convergence in (1.1) is unconditional.

In the sequel we shall limit our consideration to χ a basis for a locally convex F-space.

It is known ([9], p. 207) that the linear functionals defined by

$$f_n\Big(\sum_{i=1}^{\infty} t_i x_i\Big) = t_n$$

are continuous. Thus if the linear space of sequences

$$S = \Big\{(t_i)\colon \sum_{i=1}^{\infty} t_i x_i \text{ converges in } X\Big\}$$

is given the identity topology with respect to the isomorphism

$$\sum_{i=1}^{\infty} t_i x_i \leftrightarrow (t_i). \tag{1.2}$$

S is an FK-space i.e. an F-space of sequences on which the coordinate functionals $s \to s_i$ are continuous ([9], p. 202). Under the equivalence (1.2) x_i corresponds to e_i, the sequence which has a 1 in the i-th coordinate and 0's elsewhere. Consequently, whenever we are to consider a locally convex F-space with a basis, we may assume it is a locally convex FK-space with basis $\mathscr{E} = \{e_1, e_2, \ldots\}$.

1.2. Definition. A sequence χ is a *symmetric basis* for a locally convex F-space X if the topology for X is determined by a sequence $\{p_1, p_2, \ldots\}$ of seminorms, each of which have the property:

$$\sup_{\pi} \sup_{\substack{|a_i| \leqslant 1 \\ 1 \leqslant n < \infty}} p_k\Big(\sum_{i=1}^{n} a_i f_i(x) x_{\pi(i)}\Big) < \infty \quad \text{for all } x \in X. \tag{1.3}$$

Here π ranges over the set of all permutations on the natural numbers and $\{f_n\}$ denotes the sequence of continuous functionals biorthogonal to χ.

Condition (1.3) corresponds to the condition (SB_1) of Singer [8], given for Banach spaces. The following lemma is a generalization of his result (SB_1) $\Rightarrow$ (SB_3) for a Banach space. We shall prove the converse of this Lemma at the end of Section 3.

1.3. Lemma. *If $\{x_n\}$ is a symmetric basis for the locally convex F-space X, then every permutation $\{x_{\pi(n)}\}$ of $\{x_n\}$ is a basis for X equivalent to $\{x_n\}$.*

Proof. A basis $\{x_n\}$ of an F-space X is equivalent to a basis $\{y_n\}$ of a space Y if

$$\sum_{i=1}^{\infty} t_i x_i \text{ converges in } X \Leftrightarrow \sum_{i=1}^{\infty} t_i y_i \text{ converges in } Y.$$

This condition holds if and only if there is a topological isomorphism T from X onto Y such that $Tx_n = y_n$ for each n [1].

In this case if the topology on X is given by a sequence $\{p_k\}$ of seminorms which satisfy (1.3) the mapping

$$T_\pi\Big(\sum_{i=1}^{\infty} t_i x_i\Big) = \sum_{i=1}^{\infty} t_i x_{\pi(i)}$$

is an isometry for each permutation π on the natural numbers.

2. Preliminary results. A *coordinate space* or *sequence space* is a set S of scalar (real or complex) sequences which is closed under coordinatewise addition and scalar multiplication. We denote the sequence whose i-th term is s_i by (s_i) or simply by s.

2.1. Definition. For a subset T of a coordinate space S,

T^α, the *α-dual* of T is the set of all s such that $\sum_{i=1}^{\infty} |s_i t_i|$ converges for each t in T.

T^γ, the *γ-dual* of T is the set of all s such that $\sup_n |\sum_{i=1}^{n} s_i t_i| < \infty$ for each t in T.

T^σ, the *σ-dual* of T is the set of all s such that $\sum_{i=1}^{\infty} |s_i t_{\pi(i)}| < \infty$ for each $t \in T$ and each permutation π on the positive integers.

In particular, we write s^χ for $\{s\}^\chi$ where s is a sequence $\chi = \alpha, \gamma, \sigma$. The χ-dual of T is a coordinate space for $\chi = \alpha, \gamma, \sigma$ and $T^\sigma \subseteq T^\alpha \subseteq T^\gamma$.

2.2. Definition. The sequence space S is *χ-perfect* if $S^{\chi\chi} = S$ ($\chi = \alpha, \gamma, \sigma$).

It is always true that $S^{\chi\chi} \supseteq S$; for the case $\chi = \alpha$, see [4], p. 197.

2.3. Lemma. *If S is χ-perfect, then*

$$S = \cap \{s^\chi : s \in S^\chi\}.$$

Proof. By definition $T^\chi = \cap \{s^\chi : x \in T\}$ so that the first equality follows when T is replaced by S^χ.

2.4. Lemma. *If $\{S_\mu\}$ is a family of χ-perfect sequence spaces, then $\bigcap_\mu S_\mu$ is χ-perfect ($\chi = \alpha, \gamma, \sigma$).*

Proof. It suffices to show that $\bigcap_\mu S_\mu \supseteq (\bigcap_\mu S_\mu)^{\chi\chi}$. Since $\bigcap_\mu S_\mu \subseteq S_\mu$ for each μ, $(\bigcap_\mu S_\mu)^\chi \supseteq \bigcup_\mu S_\mu$ so that $(\bigcap_\mu S_\mu)^{\chi\chi} \subseteq (\bigcup_\mu S_\mu^\chi)^\chi$. Also $\bigcup_\mu S_\mu^\chi \supseteq S_\mu^\chi$ for each μ so that $(\bigcup_\mu S_\mu^\chi)^\chi \subseteq S_\mu^{\chi\chi}$ for each. Therefore,

$$(\bigcap_\mu S_\mu)^{\chi\chi} \subseteq \bigcap_\mu S_\mu^{\chi\chi} = \bigcap_\mu S_\mu$$

because S_μ is χ-perfect for each μ.

2.5. Definition. The sequence space S is *normal* if for each $s \in S$ and each bounded sequence (a_i), $(a_i s_i) \in S$. The sequence space S is *symmetric* if for each $s \in S$ and each permutation π on the positive integers $(s_{\pi(i)}) \in S$.

2.6. Lemma. *If S is normal, $S^\gamma = S^\alpha$; if S is symmetric, $S^\alpha = S^\sigma$.*

Proof. If S is normal, and $t \in S^\gamma$,

$$\sup_n \Big|\sum_{i=1}^{n} s_i t_i\Big| < \infty$$

for each s in S. By definition $([s_i t_i / |s_i t_i|] s_i)$ is in S so that $\sum_{i=1}^{\infty} |s_i t_i| < \infty$ for each s in S. Thus $S^\gamma \subseteq S^\alpha$.

Suppose S is symmetric, $t \in S^a$ and π is any permutation on the positive integers. For each s in S, $(s_{\pi^{-1}(i)}) \in S$ so that

$$\sum_{i=1}^{\infty} |s_{\pi^{-1}(i)} t_i| = \sum_{i=1}^{\infty} |s_i t_{\pi(i)}| < \infty.$$

The opposite inclusions have already been noted.

2.7. Lemma. *If S is a locally convex FK-space which contains $\mathscr{E}$ and t is in S^γ, there is a continuous linear functional f on S such that $f(e_i) = t_i$ for each i.*

Proof. For $n = 1, 2, \ldots$ define

$$F_n(s) = \sum_{i=1}^{n} t_i s_i.$$

Then for each i, $\lim_n F_n(e_i) = t_i$ and for each s in S,

$$\{F_n(s): n = 1, 2, \ldots\}$$

is a bounded set. Therefore, by [3], II.1.18,

$$F(s) = \lim_n F_n(s) = \sum_{i=1}^{\infty} t_i s_i$$

is continuous on the closed linear span of $\mathscr{E}$ in S. By the Hahn-Banach theorem ([3], II.3.10), F can be extended to a continuous linear functional f defined on all of S.

Let $\Phi = \{Q_i: i = 1, 2, \ldots\}$ be an increasing sequence of seminorms which determines the topology of an FK-space S with basis $\mathscr{E}$ ([9], pp. 216, 217). A new sequence of seminorms $\pi = \{P_i: i = 1, 2, \ldots\}$ may be constructed by means of the relations

$$P_i\Big(\sum_{j=1}^{\infty} s_j e_j\Big) = \sup_n Q_i\Big(\sum_{j=1}^{n} s_j e_j\Big). \tag{2.1}$$

Each seminorm is *monotonic* with respect to $\mathscr{E}$, i.e.

$$P_i\Big(\sum_{j=1}^{n} s_j e_j\Big) \leqslant P_i\Big(\sum_{j=1}^{n+1} s_j e_j\Big)$$

for each n.

2.8. Lemma. *The topology defined by π on S coincides with that defined by Φ.*

Proof. The topology induced by π on S is evidently stronger than that induced by Φ. It is routine to verify that S is complete with this stronger topology. Thus the topologies coincide ([9], p. 199, Cor. 1).

2.9. Definition. A *sequential seminorm* (s. s.) is a function, P, from s, the space of all sequences into R^* which satisfies the following conditions:

(a) P is an extended seminorm, i.e. P can assume the value ∞ and

(i) $P(as) = |a| P(s)$ for each scalar a,

(ii) $P(s+t) \leqslant P(s) + P(t)$.

(b) $P(e_i) < \infty$ for each i.

(c) $P(s) = \sup_n P(\sum_{i=1}^{n} s_i e_i)$ for each sequence s.

An s. s. P is a *sequential norm* (s. n.) if in addition

(d) $P(s) = 0$ implies $s = 0$.

An s. s. P is *balanced* if

(e) $P(s) = \sup\{P(a_i x_i): |a_i| \leqslant 1 \text{ for each } i\}$.

An s. n. P is *symmetric* if

(f) $P(s) = P\big((s_{\pi(i)})\big)$ for each permutation π on the natural numbers.

2.10. Lemma. *If $\{P_\lambda: \lambda \in \Lambda\}$ is a family of s. s., then P defined by*

$$P(s) = \sup_\lambda P_\lambda(s)$$

is an s. s. providing that $P(e_i) < \infty$ for each i. If each P_λ is balanced (symmetric) P is balanced (symmetric).

Proof. All of the properties (a)-(f) of 2.9 are preserved by supremums with the exception of (b) which holds by hypothesis.

For an s. s. P let S_P denote the collection of all s for which $P(s) < \infty$. Define a topology on S_P by means of the seminorm P and the coordinate seminorms

$$E_k(s) = |s_k|, \quad k = 1, 2, \ldots$$

2.11. Lemma. *With the topology defined above S_P is an FK-space and is a basic sequence in S_P.*

Proof. It is evident that S_P is a linear space; it is a metric space since the determining family is countable ([9], p. 217).

We show that S_P is complete. Let $\{s^{(n)}\}$ be a Cauchy sequence in S_P. This means that $\{s^{(n)}\}$ is a Cauchy sequence with respect to the seminorms $P, E_1, E_2, \ldots$ Thus for each i, $\{s_i^{(n)}\}$ is a Cauchy sequence of scalars and so

$$s_i = \lim_n s_i^{(n)}$$

exists for each i. Then

$$\lim_n E_k(s - s^{(n)}) = 0 \quad \text{for each } k$$

is evident. If $n, m \geqslant N$ implies $P(s^{(n)} - s^{(m)}) < \varepsilon$ we have

$$P\Big(\sum_{i=1}^{k} [s_i^{(n)} - s_i^{(m)}] e_i\Big) < \varepsilon$$

for each k so that

$$P\Big(\sum_{i=1}^{k} [s_i^{(n)} - s_i] e_i\Big) \leqslant \varepsilon$$

for each k. Thus from (c) of 2.9

$$P(s^{(n)} - s) \leqslant \varepsilon$$

for $n \geqslant N$. This implies that $s \in S_P$ and that

$$\lim_n P(s - s^{(n)}) = 0.$$

The continuity of the coordinate functionals on S_P was established above.

Finally note that the operators

$$F_n(s) = \sum_{i=1}^{n} s_i e_i$$

are uniformly bounded on S_P and that

$$\lim_n F_n(e_i) = e_i$$

for each i. Thus by [3], II.1.18,

$$s = \lim_n F_n(s) = \sum_{i=1}^{\infty} s_i e_i$$

exists for each s in the closed linear span of $\mathscr{E}$. Therefore, $\mathscr{E}$ is a basic sequence in S_P.

2.12. Lemma. *If P is an s. s., S_P is γ-perfect.*

Proof. This was proved in [5] for a special case of P and s. n. The same proof holds for P an arbitrary s. n.

Assume $P(e_{k_i}) \neq 0$ and $P(e_{h_i}) = 0$ for $i = 1, 2, \ldots$ In the case when the sequence (k_i) terminates then $S_P = s$, which we know is γ-perfect. If the sequence does not terminate define

$$N(s) = \sup_n P\Big(\sum_{i=1}^{n} s_i e_{k_i}\Big).$$

Then it can be directly verified that N is an s. n. and that S_P consists of all s for which $(s_{k_1}, s_{k_2}, \ldots) \in S_N$. Therefore, $S_P^{\gamma\gamma}$ consists of all s for which $(s_{k_1}, s_{k_2}, \ldots) \in S_N^{\gamma\gamma}$ so that S_P is γ-perfect because S_N is γ-perfect.

2.13. Lemma. *Let $\Phi = \{P_i : i = 1, 2, \ldots\}$ be a sequence of s. s. such that $P_i(s) = 0$ for each i implies $s = 0$ and let*

$$S_\Phi = \bigcap_{i=1}^{\infty} S_{P_i}.$$

Then S_Φ is a γ perfect FK-space with the topology determined by Φ. The sequence $\mathscr{E}$ is basic in S_Φ. If each P_i is balanced, $\mathscr{E}$ is an unconditional basis. If each P_i is symmetric, $\mathscr{E}$ is a symmetric basis.

Proof. By [9], p. 205, Theorem 3, S_Φ is an FK-space with the seminorms $\Phi \cup \{E_k : k = 1, 2, \ldots\}$. But the seminorms E_k, $k = 1, 2, \ldots$, are not necessary since if $P_j(e_k) > 0$

$$E_k(s) P_j(e_k) = P_j(s_k e_k) \leqslant P_j\Big(\sum_{i=1}^{k} s_i e_i\Big) + P_j\Big(\sum_{i=1}^{k-1} s_i e_i\Big) \leqslant 2P_j(s).$$

That $\mathscr{E}$ is basic in S_Φ can be shown either directly or as in 2.11.

The space S_Φ is γ-perfect by 2.12 and 2.4.

Suppose each P_i is balanced. Let s be in the closed linear span of $\mathscr{E}$ and let (a_i) be a sequence with $|a_i| \leqslant 1$ for each i. Then for $m = 1, 2, \ldots, n > m$ and each k,

$$P_k\Big(\sum_{i=m}^{n} a_i s_i e_i\Big) \leqslant P_k\Big(\sum_{i=m}^{\infty} s_i e_i\Big)$$

so that $\sum_{i=1} s_i e_i$ converges unconditionally to s by [2], p. 59 (1)(d).

If each P_k is symmetric, $\mathscr{E}$ is a symmetric basis by Definition 1.2 since we have already seen that $\mathscr{E}$ is a basic sequence.

For a given sequence t we can define three sequential seminorms which we shall employ in the next section:

$$P_t^\gamma(s) = \sup_n \Big|\sum_{i=1}^{n} s_i t_i\Big|,$$

$$P_t^\alpha(s) = \sum_{i=1}^{\infty} |s_i t_i|,$$

$$P_t^\sigma(s) = \sup\Big\{\sum_{i=1}^{\infty} |s_i t_{\pi(i)}| : \pi \text{ is a permutation on the natural numbers}\Big\}.$$

Note that P_t^α is obviously balanced and that P_t^σ is obviously symmetric.

2.14. Lemma. *The space of all s for which $P_t^\gamma(s) < \infty$ is t^γ. The space of all s for which $P_t^\alpha(s) < \infty$ is t^α. The space for all s for which $P_t^\sigma(s) < \infty$ is m (bounded sequences) if t is finitely non-zero and t^σ if t has an infinite number of non-zero coordinates.*

Proof. The first two statements are obvious. The third is not difficult to obtain and discussed in [6] and [7].

3. The main theorems. In the following three theorems we shall say that a space of sequences S is *associated with a basis* if the correspondence (1.2) exists between a space with basis χ and S.

3.1. Theorem. *A coordinate space S is associated with a basis of a locally convex F-space if and only if S is the closed linear span of $\mathscr{E}$ in a γ-perfect locally convex FK-space T. The space T is equal to $S^{\gamma\gamma}$.*

3.2. Theorem. *A coordinate space S is associated with an unconditional basis of a locally convex F-space if and only if S is the closed linear span of $\mathscr{E}$ in an a-perfect locally convex FK-space T. The space T is equal to S^{aa}.*

3.3. Theorem. *A coordinate space S can be associated with a symmetric basis of a locally convex F-space if and only if S is the closed linear span of $\mathscr{E}$ in a σ-perfect locally convex FK-space T. The space T is equal to $S^{\sigma\sigma}$.*

Proof of Theorem 3.1. Suppose S is associated with a basis of a locally convex F-space. Then the correspondence (1.2) shows that S can be regarded as a locally convex FK-space with basis $\mathscr{E}$. According to 2.8 the FK-topology of S can be given by an increasing sequence of seminorms $\{p_1, p_2, \ldots\}$ which are monotonic with respect to the basis $\mathscr{E}$. For each k, we can extend p_k to the s. s. P_k by

$$P_k(s) = \sup_n p_k\Big(\sum_{i=1}^n s_i e_i\Big).$$

By 2.13, $T = \bigcap_{i=1}^{\infty} S_{P_i}$ is a γ-perfect FK-space with the topology determined by $\{P_k: k = 1, 2, \ldots\}$. Since P_k restricted to S is p_k, S is a closed subspace of T. Since $\mathscr{E}$ is fundamental in S, S must be the closed linear span of $\mathscr{E}$ in T.

In order to prove that $T = S^{\gamma\gamma}$ it suffices to show that $S^\gamma = T^\gamma$. That $S^\gamma \supseteq T^\gamma$ follows since $S \subseteq T$. Assume $u \in S^\gamma$; then by 2.7 there is a continuous linear functional F on T such that $F(e_i) = u_i$ for each i. For each $s \in T$ the set $\{\sum_{i=1}^n s_i e_i: n = 1, 2, \ldots\}$ is bounded because

$$p_k\Big(\sum_{i=1}^n s_i e_i\Big) \leqslant p_k(s) \quad \text{for each } n.$$

Therefore,

$$\Big\{F\Big(\sum_{i=1}^n s_i e_i\Big): n = 1, 2, \ldots\Big\} = \Big\{\sum_{i=1}^n s_i u_i: n = 1, 2, \ldots\Big\}$$

is a bounded set so that $u \in T^\gamma$.

It remains to show that if T is a γ-perfect FK-space, $\mathscr{E}$ is a basic sequence in T. Let $\{Q_i: i = 1, 2, \ldots\}$ be an increasing sequence of seminorms which determine the topology of T. For each $u \in T^\gamma$ define the s. s.

$$P_u^\gamma(s) = \sup_n \Big|\sum_{i=1}^n s_i u_i\Big|,$$

and let u^γ be given the FK topology discussed in 2.11. By 2.3, $T = \cap \{u^\gamma: u \in T^\gamma\}$ so that $T \subseteq u^\gamma$ for each $u \in T^\gamma$. Thus the topology of T is stronger than that of u^γ ([9], p. 204) which implies that there is a number $k_u > 0$ and a seminorm Q_i such that

$$k_u P_u^\gamma(t) \leqslant Q_i(t), \quad t \in T. \tag{3.1}$$

For $i = 1, 2, \ldots$ form the collection

$$\mathscr{Q}_i = \{k_u P_u^\gamma: u \in T^\gamma;\ k_u P_u^\gamma(t) \leqslant Q_i(t) \text{ for each } t \text{ in } T\}.$$

If $Q_i \neq 0$, then $Q_i(e_j) \neq 0$ for some j so that $\mathscr{Q}_i$ will contain the non-zero seminorm $Q_i(e_j) P_{e_j}^\gamma$.

Let $P_i = \sup \mathscr{Q}_i$. Then $\Phi = \{P_i: i = 1, 2, \ldots\}$ is a countable collection of s. s. by 2.10. If $Q_i(x) \neq 0$, $P_i(x) \neq 0$ so that S_Φ is an FK-space by 2.13. Since $P_i(t) \leqslant Q_i(t)$ for $t \in T$, $S_\Phi \supseteq T$. On the other hand, by (3.1) for each u in T^γ there is P_i and k_u such that

$$k_u P_u(t) \leqslant P_i(t), \quad t \in T,$$

which means that $u^\gamma \supseteq S_{P_i}$. Hence $S_\Phi \subseteq T$. Consequently, Φ determines the FK-topology for T by [9], p. 204, so that $\mathscr{E}$ is a basic sequence in T by 2.13.

Proof of Theorem 3.2. Suppose S is associated with an unconditional basis of a locally convex F-space. By Theorem 3.1, $S^{\gamma\gamma}$ has an FK-topology in which S is the closed linear span of $\mathscr{E}$. It is necessary to show that $S^{\gamma\gamma} = S^{aa}$. However, if $s \in S$ and $|a_i| \leqslant 1$ for each i, then $(a_i s_i) \in S$ by [2], p. 59 (1) (d), so that S is normal. According to 2.6, $S^\gamma = S^a$; and since S^a is also normal $S^{\gamma\gamma} = S^{aa}$.

If T is an a-perfect locally convex FK-space we show that $\mathscr{E}$ is an unconditional basic sequence by an argument analogous to that in the γ-perfect case. In this instance, instead of P_u^γ we use the s. s.

$$P_u^a(s) = \sup_n \sum_{i=1}^n |s_i u_i|$$

defined for each u in T^a. Each seminorm P_i defined as a supremum of seminorms of the form P_u^a will be balanced so that $\mathscr{E}$ will be an unconditional basic sequence in $\cap S_{P_i} = T$ by 2.13.

Proof of Theorem 3.3. In the case of S associated with a symmetric basis of an F-space it is necessary to prove that $S^{\sigma\sigma} = S^{\gamma\gamma}$, and this follows from 1.3 and 2.6.

If S is a σ-perfect FK-space, then either $S = s$ in which case we are finished or $S \subseteq m$ [4]. In the second case for each u in S^{α} define

$$P_u^{\sigma}(s) = \sup_{\pi} \Big\{ \sum_{i=1}^{\infty} |u_{\pi(i)} s_i| : \pi \text{ is a permutation on the natural numbers} \Big\}$$

and proceed as in the γ-perfect case. The seminorms P_i defined in the course of the argument will all be symmetric by 2.10 so that $\mathscr{E}$ will be a symmetric basis for its closed linear span by Definition 1.2.

The following is a generalization of $(\mathrm{SB}_1) \Leftrightarrow (\mathrm{SB}_3)$ of Singer [8], 6, Theorem 5.3.

3.4. Corollary. *A basis $\{x_n\}$ of a locally convex F-space X is a symmetric basis if and only if every permutation $\{x_{\pi(n)}\}$ of $\{x_n\}$ is a basis of X equivalent to $\{x_n\}$.*

Proof. Without loss of generality we may restrict our attention to $\mathscr{E}$ a basis for a locally convex space S.

The necessity of the condition was given in Lemma 1.3.

If every permutation of $\mathscr{E}$ is a basis for S equivalent to $\mathscr{E}$, then $s \in S$ implies $(s_{\pi(i)}) \in S$ so that S is symmetric. Therefore, $S^{\alpha\alpha} = S^{\sigma\sigma}$ so that, by 3.2 and 3.3, $\mathscr{E}$ is a symmetric basis for S.

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An example concerning reflexivity

by

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The spaces c_0 and l not only fail to be reflexive but contain no infinite-dimensional reflexive subspace [7, 12]. It is natural to conjecture that each non-reflexive space contains an infinite-dimensional closed subspace with this property; this conjecture is false. Here we give an example of a Banach space which is not reflexive (or even quasi-reflexive [4]) yet has the property that each of its closed infinite-dimensional subspaces contains a subspace isomorphic to the Hilbert space l^2. We also discuss this type of non-reflexive space and show that it has some properties in common with reflexive and quasi-reflexive spaces.

Lemma 1. *Let X be the quasi-reflexive space constructed by R. C. James* ([8], *also see* [9], *p.* 198). *Every infinite-dimensional closed subspace of X contains a subspace isomorphic to l^2.*

Proof. We recall that the space X consists of vectors $x = (a_1, a_2, \ldots)$, x a sequence of scalars, where x is in X if and only if $\lim a_n = 0$ and

$$\|x\| = \sup \Big[\sum_{i=1}^{n} |a_{p_{2i-1}} - a_{p_{2i}}|^2 + |a_{p_{2n+1}}|^2 \Big]^{1/2}$$

is finite, where the supremum is taken over all finite increasing (or one term) sequences. The vectors x_i, with a one in the i-th place and zeros elsewhere, constitute a Schauder basis for X.

Using a theorem due to Bessaga and Pełczyński ([2], C.2, p. 157), each infinite-dimensional closed subspace M of X contains a sequence $\{y_n\}$ which is basic ($\{y_n\}$ is a Schauder basis for its closed linear span $[y_n]$) and is equivalent to a block basis $[z_n]$, with respect to x_n, i.e. each z_n is given by

$$z_n = \sum_{i=q_n+1}^{q_{n+1}} a_i^n x_i$$

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