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A note on parabolic fractional and singular integrals*

by

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Introduction. A. P. Calderón and A. Zygmund have considered in [3] singular integral operators given by convolution with kernels that are homogeneous of degree $-n$ and have mean value zero on the n-dimensional unit sphere, and they proved that, under suitable conditions, these are continuous operators from L^p to L^p, $1 < p < \infty$, where the norms are taken with respect to the Lebesgue measure dx. Assuming this result and the boundedness of the kernel on the unit sphere, Stein [10] completed it by proving that the operators are continuous from L^p to L^p with weighted measures $|x|^\beta dx$. (This was known for $n = 1$, see [1] and [5]).

The purpose of this paper is to obtain an analogue of this last theorem for parabolic singular transforms that we introduce in § 3. (See theorem 3 (1) below). We use a different method of proof. For this we define in § 2 a "parabolic" fractional integral operator and prove some of its properties of continuity, which may be of independent interest.

1. Preliminaries. We begin with some notation and basic definitions.

In the following we shall denote by $(x, t) = (x_1, \ldots, x_n, t)$, $(y, s) = (y_1, \ldots, y_n, s)$ points in the Euclidean $(n+1)$-dimensional half space $E_{n+1}^+ = E_n \times (0, \infty)$. L^p will be understood as $L^p(E_{n+1}^+)$, the class of complex valued measurable functions $f(x, t)$ defined on E_{n+1}^+ such that $\|f\|_p = \left(\int f(x, t)^p\, dx dt\right)^{1/p}$ is finite. An integral without specification of the domain of integration will be understood as being taken over the entire E_{n+1}^+. C with various subscripts will stand for a constant, not necessarily the same at each occurrence, depending only on the variables displayed. Dependence on the dimension, though, will not be indicated.

We introduce the metric $[x, t] = (|x|^{2m} + t^2)^{1/2m}$, where m is a fixed positive integer.

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(1) This result was announced in *Notices*, Amer. Math. Soc. 12, no. 1, 65T-45.

A linear operator T, defined on functions is L^p, is said to be of *type* (p,q), $1 \leqslant p, q \leqslant \infty$, if there exists a constant C_{pq} such that $\|Tf\|_q \leqslant C_{pq}\|f\|_p$, independently of f.

For an $(n+1)$-tuple $P = (p_1, p_2, \ldots, p_{n+1})$, a mixed or vectorial norm P is defined as

$$\|f\|_P = \left(\int \cdots \left(\int \left(\int f(x_1, x_2, \ldots, t)^{p_1} dx_1\right)^{p_2/p_1} dx_2 \ldots dt\right)^{1/p_{n+1}}\right.$$

(see [2]). We say that an operator T is of *vectorial type* (P, Q) for $P = (p_1, p_2, \ldots, p_{n+1})$, $Q = (q_1, q_2, \ldots, q_{n+1})$, $1 \leqslant p_i, q_i \leqslant \infty$, $i = 1, 2, \ldots, n+1$, when $\|Tf\|_Q \leqslant C_{PQ}\|f\|_P$, independently of f.

The distribution function of f is defined by $D(f;\alpha) = D_f(\alpha) =$ (Lebesgue) measure $\{(x,t): |f(x,t)| > \alpha\}$, $\alpha > 0$. $D_f(\alpha)$ is non-increasing and continuous from the right. The non-increasing rearrangement of f onto $(0,\infty)$ is then defined by $f^*(\tau) = \inf\{\alpha > 0: D_f(\alpha) \leqslant \tau\}$, $\tau > 0$. $f^*(\tau)$ is also continuous from the right and has the same distribution function as f.

A sublinear operator T, defined on functions in L^p, is said to be of *weak type* (p,q), $1 \leqslant p \leqslant \infty$, $1 \leqslant q < \infty$, if there exists a constant C_{pq} such that $D_f(\alpha) \leqslant C_{pq}(\|f\|_p/\alpha)^q$ holds independently of f.

As

$$\sup_{\alpha>0} \alpha\left(D_f(\alpha)\right)^{1/r} = \sup_{\tau>0} \tau^{1/r} f^*(\tau),$$

the weak (p,q) condition of an operator T can be stated as

$$\sup_{\tau>0} \tau^{1/q}(Tf)^*(\tau) \leqslant C_{pq}\|f\|_p.$$

In the following paragraph we shall use a property of f^* that we are going to prove next.

LEMMA 1. $(fg)^*(\tau_1+\tau_2) \leqslant f^*(\tau_1)g^*(\tau_2)$.

Proof. By definition, $f^*(\tau_1) = \inf\{\alpha: D_f(\alpha) \leqslant \tau_1\}$, $g^*(\tau_2) = \inf\{\beta: Dg(\beta) \leqslant \tau_2\}$ and $(fg)^*(\tau_1+\tau_2) = \inf\{\eta: D_{fg}(\eta) \leqslant \tau_1+\tau_2\}$. As the distribution functions are non-increasing, if $\alpha \geqslant f^*(\tau_1)$, then $D_f(\alpha) \leqslant \tau_1$, and if $\beta \geqslant g^*(\tau_2)$, then $Dg(\beta) \leqslant \tau_2$. But $\{|fg| > \alpha\beta\} \subset \{|f| > \alpha\} \cup \{|g| > \beta\}$. So $D_{fg}(\alpha\beta) \leqslant D_f(\alpha) + D_g(\beta)$.

Then, it is $D_{fg}(\alpha\beta) \leqslant \tau_1+\tau_2$ for all α, β such that $\alpha \geqslant f^*(\tau_1)$ and $\beta \geqslant g^*(\tau_2)$, and this implies $(fg)^*(\tau_1+\tau_2) \leqslant \alpha\beta$ for all such α, β. So $(fg)^*(\tau_1+\tau_2) \leqslant f^*(\tau_1)g^*(\tau_2)$.

2. Parabolic fractional integrals. In one dimension we know that the fractional integral operator that we denote by $H_{\gamma,1}$, $0 < \gamma < 1$,

$$(H_{\gamma,1}f)(x) = \int_{-\infty}^{+\infty} f(y)\,|x-y|^{\gamma-1}dy,$$

is of type (p,r), where $1/p - 1/r = \gamma < 1/p < 1$ (see, for instance, [12], vol. II, p. 142) and of weak type $\left(1, 1/(1-\gamma)\right)$ ([11], Th. 6, p. 242).

Let us now consider the one-dimensional operator

$$(T_{\gamma,1}f)(x) = |x|^{-\gamma}(H_{\gamma,1}f)(x) = |x|^{-\gamma}\int_{-\infty}^{+\infty} f(y)\,|x-y|^{\gamma-1}dy.$$

LEMMA 2. *$T_{\gamma,1}$, $0 < \gamma < 1$, is an operator of type (p,p) for $1 < p < 1/\gamma$ and of weak type* $(1,1)$.

Proof. It is sufficient to show that $T_{\gamma,1}$ is of weak type (p,p) for $1 \leqslant p < 1/\gamma$ and then apply Marcinkiewicz theorem [11].

Using the fact proved in Lemma 1 with $\tau_1 = \tau_2 = \tau/2$ we have $(T_{\gamma,1}f)^*(\tau) \leqslant (\tau/2)^{-\gamma}(H_{\gamma,1}f)^*(\tau/2)$ and

$$\sup_{\tau>0} \tau^{1/p}(T_{\gamma,1}f)^*(\tau) \leqslant 2^{\gamma}\sup_{\tau>0} \tau^{1/p}(H_{\gamma,1}f)^*\left(\frac{\tau}{2}\right). \tag{1}$$

Taking into account that $1/p - \gamma = 1/r$ and that $H_{\gamma,1}$ is of weak type (p,r) for $1 \leqslant p < 1/\gamma$, inequality (1) becomes

$$\sup_{\tau>0} \tau^{1/p}(T_{\gamma,1})^*(\tau) \leqslant C_{pr}\|f\|_q$$

and the conclusion of the theorem follows.

In the $(n+1)$-dimensional case we define, for $0 < \gamma < n+1$, the following "parabolic" fractional integral operators:

$$(H_\gamma f)(x,t) = \int f(y,s)[x-y, t-s]^{-n-m+\gamma}dyds,$$

$$(T_\gamma f)(x,t) = [x,t]^{-\gamma}(H_\gamma f)(x,t)$$

$$= [x,t]^{-\gamma}\int f(y,s)[x-y, t-s]^{-n-m+\gamma}dyds.$$

THEOREM 1. (a) *The operator H_γ is of vectorial type (P,R), $P = (p, \ldots, p, p)$, $R = (r, \ldots, r, r^{\#})$, where $1/p - 1/r = \gamma/(n+1)$, $1/p - 1/r^{\#} = \gamma/m(n+1)$ for $1 < p < (n+1)/\gamma$.*

(b) *The operator T_γ is of (ordinary) type (p,p) for $1 < p < 1/\gamma$.*

Proof. We have

$$[x-y, t-s]^{n+m-\gamma} \geqslant C_m\left(|x_1-y_1|^{2m} + \ldots + |x_n-y_n|^{2m} + |t-s|^2\right)^{(n+m-\gamma)/2m}$$

$$\geqslant C_m|x_1-y_1|^{1-\gamma/(n+1)} \ldots |x_n-y_n|^{1-\gamma/(n+1)}|t-s|^{1-\gamma/(n+1)m}.$$

Without loss of generality, we may assume that $f \geqslant 0$. Then $H_\gamma f$ is dominated by the iteration of $n+1$ one-dimensional operators:

$$(H_\gamma f)(x,t) \leqslant C_m H_{\gamma/(n+1),1} \ldots (n \text{ times}) \ldots H_{\gamma/(n+1),1}H_{\gamma/m(n+1),1}f(x_1, \ldots, x_n, t).$$

But under the conditions of the hypothesis the conclusion follows.

(b) As before, we have

$$[x,t]^\gamma \geqslant C_m(|x_1|)^{2m}+\ldots+|x_n|^{2m}+t^2)^{\gamma/2m}$$
$$\geqslant C_m|x_1|^{\gamma/(n+1)}\ldots|x_n|^{\gamma/(n+1)}t^{\gamma/m(n+1)},$$

so that

$$(T_\gamma f)(x,t) \leqslant C_m(|x_1|^{-\gamma/(n+1)}H_{\gamma/(n+1),1})\ldots(|x_n|^{-\gamma/(n+1)}H_{\gamma/(n+1),1})\times$$
$$\times(t^{-\gamma/(mn+1)}H_{\gamma/m(n+1),1})f(x_1,\ldots,x_n,t)$$

for $f \geqslant 0$.

Combining what we know about the types of these one-dimensional operators, we infer that T_γ is of type (p,p) for $1 < p < (n+1)/\gamma$.

The properties of type of the parabolic fractional integrals given in Theorem 1 are all that will be used in §3.

However, it should be noticed that the following proposition, that is the natural analogue of Sobolev's lemma, holds:

THEOREM 2. (a) *The operator H_γ is of type (p,q) where $1/p-1/q = \gamma/(n+m)$ for $1 < p < (n+m)/\gamma$.*

(b) *The operator H_γ is of weak type $\big(1,(n+m)/(n+m-\gamma)\big)$.*

Proof. To prove (b) it is sufficient to consider $f\in L^1$ such that $\|f\|_1 = 1$. We now write $[x,t]^{-n-m+\gamma} = F(x,t)+G(x,t)$ where $F(x,t) = [x,t]^{-n-m+\gamma}$ if $[x,t] > \varrho$ and zero elsewhere and $G(x,t) = [x,t]^{-n-m+\gamma}-F(x,t)$. Then $F\in L^\infty$ and $G\in L^1$. So $\|F*f\|_\infty \leqslant \|F\|_\infty$ and $\|G*f\|_1 \leqslant \|G\|_1$.

We choose ϱ in such a way that $\|F\|_\infty = \varrho^{-n-m+\gamma} = \gamma/2$. But then the set $\{|F*f| > \alpha/2\}$ is empty and as it is always $\{|H_\gamma f| > \alpha\} \subset \{|F*f| > \alpha/2\} \cup \{|G*f| > \alpha/2\}$, we obtain

$$D(H_\gamma f;\alpha) \leqslant D(G*f;\alpha/2). \tag{2}$$

Also it is

$$D(G*f;\alpha/2) \leqslant (2/\alpha)\|G*f\|_1 \leqslant (2/\alpha)\|G\|_1. \tag{3}$$

By the definition of G its L^1-norm is equal to a constant times $\varrho^\gamma = \alpha^{\gamma/(\gamma-n-m)}$.

Replacing in (2) and (3) we obtain

$$D(H_\gamma f;\alpha) \leqslant C_{nm}\alpha^{\gamma/(\gamma-n-m)-1} = C_{nm}\alpha^{(n+m)/(\gamma-n-m)}$$

and this inequality is the weak type $\big(1,(n+m)/(n+m-\gamma)\big)$ condition of the operator H_γ.

It may be proved in a similar way that H_γ is of weak type (p,q) where $1/q = 1/p-\gamma/(n+m)$ and $1 < p < (n+m)/\gamma$.

Then (a) follows as an application of Marcinkiewicz theorem.

The preceding theorem follows inmediately from the convolution theorem for Lorentz spaces (see [8]). Essentially, the given proof contains the same ideas that would be used to show that the result on Lorentz spaces is applicable to this case (²).

3. The analogue of Stein-Babenko theorem for parabolic singular integrals. As an analogue of the singular integral kernels of Calderón and Zygmund it is possible to introduce the concept of a parabolic kernel $k(x,t)$, defined in E^+_{n+1}, with the "homogeneity" condition

$$k(\lambda x,\lambda^m t) = \lambda^{-n-m}k(x,t), \qquad \lambda > 0, \tag{4}$$

where $m > 1$ is a fixed integer [6].

B. F. Jones, Jr., considered the operators given as convolutions with the family of truncated kernels, that we are going to denote as $k^\#_\varepsilon$, coinciding with $k(x,t)$ for $t > \varepsilon$, and vanishing for $0 < t \leqslant \varepsilon$, and their limits for $\varepsilon\to 0$. He proved [7] that, under certain conditions of $k(x,1)$, these operators are of type (p,p), $1 < p < \infty$.

We shall consider the parabolic singular integral operators

$$(Kf)(x,t) = \lim_{\varepsilon\to 0} k_\varepsilon(x,t)*f(x,t) = \lim_{\varepsilon\to 0}\int_{[x-y,t-s]>\varepsilon} k(x-y,t-s)f(y,s)\,dy\,ds \tag{5}$$

that is, the limit of convolutions with kernels truncated in every variable, as to coincide with $k(x,t)$ for $[x,t] > \varepsilon$.

In spite of the different ways of truncating the kernels, Jones's theorem holds for (5), since the difference between k_ε and $k^\#_{\varepsilon^m}$, that is a kernel equal to $k(x,t)$ if $[x,t] > \varepsilon$ and $0 < t < \varepsilon^m$ and zero elsewhere, is easily seen to belong to L^1, with L^1-norm bounded independently of ε (see [9]).

Moreover, the hypothesis of the integrability of the kernel and its mean value zero on the hyperplane $t = 1$, used in [7], are equivalent to similar assumptions given on the set $\{(x,t)\colon [x,t] = 1\}$. This is so because

$$\int_{1<[x,t]<2} |k(x,t)|\,dx\,dt$$
$$= \int_{\substack{x\in E_n\\ 0<t<2^m}} |k(x,t)|\,dx\,dt + \int_{\substack{[x,t]>1\\ 0<t<1}} |k(x,t)|\,dx\,dt - \int_{\substack{[x,t]>2\\ 0<t<2^m}} |k(x,t)|\,dx\,dt$$

(²) Using theorem 2, (b) of theorem 1 can be proved by the method used in lemma 2, with the additional result of the weak type $(1,1)$ of T_γ, $0 < \gamma < n+m$.

and the last two integrals are easily seen to be equal, taking into account the "homogeneity" property (4) of $k(x, t)$.

So, it is natural to impose conditions on the kernel to be satisfied on the set $\{(x, t): [x, t] = 1\}$.

Remark. The hypotheses of [7] are not the more general under which the theorem of type (p, p), $1 < p < \infty$, of the parabolic singular integrals holds (see [4]). Nevertheless we shall not specify less restrictive hypothesis here as in our main theorem we already assume the type (p, p) of the singular integrals.

THEOREM 3. *If*

$$(Kf)(x, t) = \lim_{\varepsilon\to 0} \int_{[x-y, t-s]>\varepsilon} k(x-y, t-s) f(y, s)\, dy ds$$

is such that $\|Kf\|_p \leqslant C_p \|f\|_p$, $1 < p < \infty$, *and* $|k(x', t')| \leqslant B$ *whenever* $[x', t'] = 1$, *then* K *is of weighted type* (p, p), *i.e.*

$$\|(Kf)(x, t)[x, t]^\beta\|_p \leqslant C_{p\beta} \|f(x, t)[x, t]^\beta\|_p, \qquad 1 < p < \infty,$$

for $-(n+1)/p < \beta < (n+1)/p'$ $(1/p + 1/p' = 1)$ *where* $C_{p\beta}$ *is a constant depending only on* k, p, β, m *and* n. *If* $-(n+1) < \beta \leqslant 0$, K *is of weighted weak type* $(1, 1)$,

Proof. By the first hypothesis we infer that

$$\int \big|K\big(f(y, s)[y, s]^\beta\big)\big|^p dx dt \leqslant C_p^p \int |f(x, t)[x, t]^\beta|^p dx dt.$$

Then, it will be enough to prove that

$$\int \big|K\big(f(y, s)[y, s]^\beta\big) - [x, t]^\beta K\big(f(y, s)\big)\big|^p dx dt$$

is similarly bounded. Being $x' = x/[x, t]$ and $t' = t/[x, t]^m$, we consider the difference

$$\begin{aligned}
&\big|K\big(f(y, s)[y, s]^\beta\big) - [x, t]^\beta K\big(f(y, s)\big)\big| \\
&\quad = \Big|\int [x-y, t-s]^{-n-m} k\big((x-y)', (t-s)'\big) f(y, s)[y, s]^\beta dy ds - \\
&\qquad - \int [x, t]^\beta [x-y, t-s]^{-n-m} k\big((x-y)', (t-s)'\big) f(y, s) dy ds\Big| \\
&\quad = \Big|\int k\big((x-y)', (t-s)'\big) f(y, s)[x-y, t-s]^{-n-m}\big([y, s]^\beta - [x, t]^\beta\big) dy ds\Big| \\
&\quad \leqslant \int B|f(y, s)|\,[y, s]^\beta |\big([y, s]^\beta - [x, t]^\beta\big)/[y, s]^\beta|\,[x-y, t-s]^{-n-m} dy ds.
\end{aligned}$$

So, the theorem will be proved if we show that, under the stated conditions, the operator

$$UF(x, t) = \int F(y, s)\,|1 - ([x, t]/[y, s])^\beta|\,[x-y, t-s]^{-n-m} dy ds$$

is of type (p, p) and of weak type $(1, 1)$.

LEMMA 3. *Let*

$$K(x, t; y, s) = |1 - ([x, t]/[y, s])^\beta|\,[x-y, t-s]^{-n-m}$$

be the kernel of the operator U, *given by*

$$UF(x, t) = \int K(x, t; y, s) F(y, s)\, dy ds.$$

Then U *is of type* (p, p) *for* $1 < p < \infty$ *and* $-(n+1)/p < \beta < (n+1)/p'$ *and of weak type* $(1, 1)$ *for* $-(n+1) < \beta < 0$.

Proof. We shall show that the operator U is dominated by a sum of operators T_γ's, that is, that its kernel K is dominated by sums of the kernels of these operators.

First case. Let $-\beta = \gamma > 0$. Then

$$K(x, t; y, s) = \frac{[x, t]^\gamma - [y, s]^\gamma}{[x, t]^\gamma [x-y, t-s]^{n+m}}.$$

(I) Let $[y, s] < 2[x-y, t-s]$.

Let $(z, w) = (x-y, t-s)$ or $(x, t) = (z+y, w+s)$. Then

$$[y+z, s+w]^\gamma \leqslant ([y, s] + [z, w])^\gamma < 2^\gamma [y, s]^\gamma + 2^\gamma [z, w]^\gamma,$$

$$\begin{aligned}
[y+z, s+w]^\gamma - [y, s]^\gamma &\leqslant (2^\gamma - 1)[y, s]^\gamma + 2^\gamma [x-y, t-s]^\gamma \\
&\leqslant (2^\gamma - 1) 2^\gamma [x-y, t-s]^\gamma + 2^\gamma [x-y, t-s]^\gamma,
\end{aligned}$$

so that $[x, t]^\gamma - [y, s]^\gamma \leqslant C_\gamma [x-y, t-s]^\gamma$. Similarly, calling $(z, w) = (y-x, s-t)$ we obtain that $|[x, t]^\gamma - [y, s]^\gamma| \leqslant C_\gamma [x-y, t-s]^\gamma$, which implies $K(x, t; y, s) \leqslant C_\gamma [x, t]^{-\gamma} [x-y, t-s]^{-(n+m-\gamma)}$. In this case UF is dominated by $T_\gamma F$ and so will be of type (p, p) by the condition $\gamma/(n+1) < 1/p$ that is satisfied by hypothesis, being equivalent to $\beta > -(n+1)/p$.

(II) Let $[y, s] > 2[x-y, t-s]$.

It is true that $[y, s] > 2[x, t] - 2[y, s]$ and that $[y, s] > 2[y, s] - 2[x, t]$, so

(6) $$3[y, s] > 2[x, t] > [y, s] \quad \text{or} \quad \tfrac{1}{2} < [x, t]/[y, s] < \tfrac{3}{2}.$$

By the mean-value theorem we have

$$[x, t]^\gamma - [y, s]^\gamma = ([x, t] - [y, s])\gamma [\xi, \zeta]^{\gamma-1},$$

where (ξ, ζ) is intermediate between (x, t) and (y, s). From inequality (6) it follows that

$$|[x, t]^\gamma - [y, s]^\gamma| \leqslant C_\gamma [x-y, t-s][x, t]^{\gamma-1}$$

which implies that

$$K(x,t;y,s) \leqslant C_\gamma \frac{[x,t]^{\gamma-1}[x-y,t-s]}{[x,t]^\gamma [x-y,t-s]^{n+m}} = C_\gamma [x,t]^{-1}[x-y,t-s]^{-(n+m-1)}.$$

In this case UF is dominated by T_1F. This will be of type (p,p) if $\gamma \geqslant 1$ because this implies $1/(n+1) \leqslant \gamma/(n+1) < 1/p$. If $\gamma < 1$ we bound UF by $T_\gamma F$:

$$[x-y,t-s] \leqslant [x,t]+2[x,t]$$

so

$$K(x,t;y,s) \leqslant C_\gamma \frac{[x,t]^{\gamma-1}[x-y,t-s]}{[x,t]^\gamma [x-y,t-s]^{n+m}}$$

$$\leqslant C_\gamma \frac{[x-y,t-s]^{1-\gamma}}{[x,t]^\gamma [x-y,t-s]^{n+m-\gamma}[x,t]^{1-\gamma}} \leqslant C_\gamma [x,t]^{-\gamma}[x-y,t-s]^{-(n+m-\gamma)}.$$

In this case UF is dominated by $T_\gamma F$. So, UF is dominated by $T_\gamma F$ or by $T_1F+T_\gamma F$.

Second case. Let now $\beta > 0$. We have

$$\|Uf\|_p = \sup_{\|g\|_{p'}=1} \int Uf(x,t)\,g(x,t)\,dxdt$$

$$= \sup_g \int \Big(\int |1-([x,t]/[y,s])^\beta| [x-y,t-s]^{-(n+m)} f(y,s)\,dyds \Big) g(x,t)\,dxdt$$

$$= \sup_g \int f(y,s) \Big(\int |1-([x,t]/[y,s])^\beta| [x-y,t-s]^{-(n+m)} g(x,t)\,dxdt \Big)\,dyds.$$

The last bracket is in the conditions of the preceding case with $-(n+1)/p' < -\beta \leqslant 0$, that is, $0 \leqslant \beta < (n+1)/p'$, which is the hypothesis. So the conclusion follows.

References

[1] K. I. Babenko, *On conjugate functions*, Doklady Akademii Nauk SSSR (N. S.) 62 (1948), p. 157-160.

[2] A. Benedek and R. Panzone, *The spaces L^p with mixed norms*, Duke Math. J. 28. 3 (1961), p. 301-324.

[3] A. P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Math. 88 (1952), p. 85-139.

[4] E. B. Fabes and N. Rivière, *Parabolic Singular Integrals*, to be published.

[5] G. H. Hardy and J. E. Littlewood, *Some more theorems concerning Fourier series and Fourier power series*, Duke Math. J. 2 (1936), p. 351-382.

[6] B. F. Jones, Jr., *Singular integrals and parabolic equations*, Bull. Amer. Math. Soc. 64. 4 (1963), p. 501-503.

[7] — *A class of singular integrals*, Amer. J. Math. 86.2 (1964), p. 441-462.

[8] R. O'Neil, *Convolution operators and $L(p,q)$ spaces*, Duke Math. J. 30 (1963), p. 129-142.

[9] C. Sadosky, *On class preservation and pointwise convergence for parabolic singular operators*, Thesis, University of Chicago, March 1965.

[10] E. M. Stein, *Note on singular integrals*, Proc. Amer. Math. Soc. 8 (1957), p. 250-254.

[11] A. Zygmund, *On a theorem of Marcinkiewicz concerning interpolation of operators*, J. Math. Pures App. 35 (1956), p. 223-248.

[12] — *Trigonometric series*, second edition, Cambridge 1959, 2 volumes.

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