A problem of Hewitt on restrictions of multiplicative linear functionals

by

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In his paper [1] on measure algebras, Hewitt posed the following problem: Find those closed linear subspaces of commutative Banach algebras on which every linear functional of norm at most 1 is the restriction (to the subspace) of a multiplicative linear functional on the algebra. He remarked that the question was open even for the algebra $C(X)$ of all continuous complex valued functions on the compact Hausdorff space X. The purpose of this note is to show that some well known ideas may be applied to obtain a solution to this special case of the problem.

Suppose that X is a compact Hausdorff space and that A is a closed linear subspace of $C(X)$. We will say that the pair (X, A) *has property* (H) if every linear functional L in A^* with $\|L\| \leqslant 1$ is the restriction to A of a multiplicative linear functional on $C(X)$. What we will show is that, roughly speaking, the pair (X, A) has property (H) if and only if X is (homeomorphic to) a symmetric compact convex subset of a locally convex space, and A is (canonically isometric to) the space of continuous linear functions on X. As a first step, we reformulate property (H) in the following manner.

As is well known, every non-zero multiplicative linear functional on $C(X)$ is obtained by evaluation at some point x of X. Consider the map $\varphi: X \to U^*$ (where U^* is the unit ball of A^*) defined by $(\varphi x)(f) = f(x)$, f in A, x in X. If $\varphi(X) = U^*$, then clearly (X, A) has property (H). Conversely, if (X, A) has property (H), then the non-zero functionals in U^* are restrictions of non-zero multiplicative functionals on $C(X)$, hence $U^* \sim \{0\} \subset \varphi(X) \subset U^*$. But if we use the weak* topology in A^*, then φ is continuous and hence $\varphi(X)$ is compact; it follows that $U^* = \varphi(X)$. (Note that φ is one-to-one, and hence a homeomorphism, if and only if A separates points of X.) Next, we exhibit a large class of pairs (X, A) having property (H); we will call them *canonical* pairs.

Suppose that E is a locally convex topological linear space and that X is a compact convex subset of E which is *symmetric*, i. e. if $x \in X$ and $|a| \leqslant 1$, then $ax \in X$. Let A be the subspace of $C(X)$ consisting of all continuous linear functions on X, that is, all linear functions on E whose restrictions to A are continuous. (Equivalently, A consists of all continuous affine functions f on X for which $f(0) = 0$.) Then A is a closed linear subspace of $C(X)$; furthermore, (X, A) *has property* (H). Indeed, suppose that $L_0 \epsilon A^*$ but $L_0 \notin \varphi(X)$. Since the functions in A are affine on X, the map φ is itself affine and hence $\varphi(X)$ is a weak* compact *convex* subset of A^*. By the separation theorem (applied to A^* in its weak* topology) there exists a function f in A such that

$$\operatorname{Re} L_0(f) > \sup\{\operatorname{Re} L(f) : L \epsilon \varphi(X)\} = \sup\{\operatorname{Re} f(x) : x \epsilon X\}.$$

Since f is the restriction to X of a linear functional and since $\alpha x \epsilon X$ whenever $|\alpha| = 1$ and $x \epsilon X$, we see that

$$\sup\{\operatorname{Re} f(x) : x \epsilon X\} = \sup\{|f(x)| : x \epsilon X\} = \|f\|,$$

so that $\|f\| < \operatorname{Re} L_0(f) \leqslant |L_0(f)|$ and hence $\|L_0\| > 1$; it follows that $\varphi(X)$ is the entire unit ball of A^*.

Suppose now that A and B are closed linear subspaces of the spaces $C(X)$ and $C(Y)$, respectively. We will say that the pair (X, A) is *isomorphic* to the pair (Y, B) if there is a homeomorphism τ from X onto Y such that the induced isometry T from $C(Y)$ onto $C(X)$ carries B *onto* A. (As usual, T is defined by $Tf(x) = f(\tau x)$, f in $C(Y)$, x in X.) The precise formulation of what we have to say is the following:

Suppose that A is a closed linear subspace of $C(X)$ which separates points of X. Then (X, A) has property (H) *if and only if (X, A) is isomorphic to a canonical pair (Y, B).*

Proof. If (X, A) has property (H), let $E = A^*$ in its weak* topology and let Y be the unit ball of A^*. The evaluation map φ defined above is a homeomorphism from X onto Y; it remains to show that the induced isometry T carries the space B of continuous linear functions on Y onto the subspace A. Now, B is closed in $C(Y)$, hence is complete, and therefore TB is a complete subspace of $C(X)$. From the fact that A may be canonically identified with the dual E^* of E it follows simply that $Tf \epsilon A$ if f is the restriction to Y of an element of E^*. It is well known (see, e. g., [2], 16.8) that B is the closure in $C(Y)$ of all such restrictions, and hence we conclude that $TB \subset A$. On the other hand, the canonical identification of A with E^* shows that every function in A may be regarded as a continuous linear function on Y, so $TB = A$. Conversely, suppose that (X, A) is isomorphic to a pair (Y, B) and that the latter has property (H). By utilizing the adjoint map T^* from A^* onto B^* it is an easy task to verify that (X, A) also has property (H). The proof then follows from the fact that canonical pairs have property (H).

In conclusion, note that if the notion of isomorphism is weakened by merely requiring that τ be a continuous map of X onto Y (instead of

a homeomorphism), then the above proof is still valid, without assuming that A separates points of X. The latter property is, of course, possessed by *all* canonical pairs. Also, all the above holds true for real valued functions and real linear spaces.

Bibliography

[1] E. Hewitt, *Measure algebras on locally compact groups: a case history in functional analysis*, Studia Math., ser. spec. 1 (1963), p. 41-52.

[2] J. Kelley, I. Namioka, et al, *Linear topological spaces*, Princeton 1963.

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