Consequently we have

$$D_{n+1}\begin{pmatrix}\xi_0 A, \xi_1, \ldots, \xi_n \\ x_0, x_1, \ldots, x_n\end{pmatrix}$$

$$= \sum_{i=0}^{n} (-1)^i \cdot \xi_0' A' x_i' \cdot D_n \begin{pmatrix}\xi_1, \ldots\ldots\ldots\ldots\ldots, \xi_n \\ x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\end{pmatrix} +$$

$$+ \sum_{i=0}^{n} (-1)^i \cdot \xi_0'' A'' x_i'' \cdot D_n \begin{pmatrix}\xi_1, \ldots\ldots\ldots\ldots\ldots, \xi_n \\ x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\end{pmatrix}$$

$$= \sum_{i=0}^{n} (-1)^i \cdot \xi_0 A x_i \cdot D_n \begin{pmatrix}\xi_1, \ldots\ldots\ldots\ldots\ldots, \xi_n \\ x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\end{pmatrix}$$

since $\xi_0' A' x_i' + \xi_0'' A'' x_i'' = \xi_0 A x_i$ by the definition (19) of $A$.

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# On Leżański endomorphisms


by

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This paper is a supplement to my paper [5][1].

An endomorphism $T$ of a complex Banach space $X$ is said to be a *Leżański endomorphism* provided the functional

(1) $$F_0(K) = \operatorname{trace} KT \qquad (K \in \mathfrak{K}_0)$$

is continuous on the space $\mathfrak{K}_0$ of all finitely dimensional endomorphisms $K$ in $X$ (with respect to the usual norm of $K$), i. e. if it satisfies the hypotheses of Leżański's [1], [2] determinant theory of the linear equation

(2) $$x + \lambda T x = x_0.$$

In [5] I quoted an example of a Leżański endomorphism $T$ (in the space $L$) which was not compact (= completely continuous). However, the endomorphism $T^2$ was compact. The subject of this paper is to prove that this is true for every Leżański endomorphism. More precisely:

THEOREM. *If $T$ is a Leżański endomorphism in $X$, then $T^2$ (and, consequently, $T^n$ for $n = 2, 3, \ldots$) is the limit (in the norm) of a sequence of finitely dimensional endomorphisms.*

Let $F$ be any continuous linear extension of $F_0$ (see (1)) over the space $\mathfrak{K}$ of all linear continuous endomorphisms (with the usual norm) in $X$. Let $D_0(\lambda)$ be the Leżański determinant of (2), determined by $F$. $D_0(\lambda)$ is an entire function of $\lambda$ and, for small $\lambda$,

(3) $$D_0(\lambda) = \exp\left(\frac{\sigma_1\lambda}{1} - \frac{\sigma_2\lambda^2}{2} + \frac{\sigma_3\lambda^3}{3} - \frac{\sigma_4\lambda^4}{4} + \ldots\right)$$

where

(4) $$\sigma_n = F(T^{n-1}) \quad \text{for} \quad n = 1, 2, \ldots$$

[1] Errata to [5]. In footnote [4] on p. 106 instead of "$\mathfrak{K}^*$ is identical with $\mathfrak{K}_*$" we should have "$K_*$ is identical with the class of all $T$ satisfying (*)".

Errata to [4]. The lines 18-30 on p. 46 should be omitted.

(see e. g. Sikorski [4], (47)). We recall that $D_0(\lambda)$ is not uniquely determined by $T$. It is uniquely determined by $F$, which can be a completely arbitrary extension of $F_0$.

The Leżański subdeterminant $D_r(\lambda)$ of order $r$ $(r = 1, 2, \ldots)$ is a $2r$-linear functional $D_r\binom{\xi_1, \ldots, \xi_r}{x_1, \ldots, x_r}$ of variables $x_1, \ldots, x_r \in X$ and $\xi_1, \ldots, \xi_r \in \Xi$ = the space of all linear continuous functionals on $X$. The endomorphism $A(\lambda) = I + \lambda T$ (where $I$ is the identity mapping in $X$) has an inverse $A(\lambda)^{-1}$ if and only if $D_0(\lambda) \neq 0$. Then (Ruston [3]; see also Sikorski [6], (28)), $D_r(\lambda) = D_0(\lambda)\cdot\mathcal{D}_r(\lambda)$ for $r = 1, 2, \ldots$ where $\mathcal{D}_r(\lambda)$ is a $2r$-linear functional defined by the following formula:

$$\mathcal{D}_r\left(\begin{matrix}\xi_1, \ldots, \xi_r \\ x_1, \ldots, x_r\end{matrix}\middle|\,\lambda\right) = \begin{vmatrix} \xi_1 A(\lambda)^{-1}x_1, \ldots, \xi_1 A(\lambda)^{-1}x_r \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots \\ \xi_r A(\lambda)^{-1}x_1, \ldots, \xi_r A(\lambda)^{-1}x_r \end{vmatrix}$$

for $\xi_1, \ldots, \xi_r \in \Xi$, $x_1, \ldots, x_r \in X$. The functions $\mathcal{D}_r(\lambda)$ (with values in the Banach space of all $2r$-linear continuous functionals on the Cartesian product $\Xi^r \times X^r$) are holomorphic in the open subset $R = [\lambda: D_0(\lambda) \neq 0]$ of the plane, and are determined by $T$ only. The set $S = [\lambda: D_0(\lambda) = 0]$ is isolated. All functions $D_r(\lambda)$ (with values in the Banach space of all $2r$-linear continuous functionals on $\Xi^r \times X^r$) are holomorphic on the whole plane and, for every complex number $\lambda$, at least one of $D_0(\lambda)$, $D_1(\lambda), D_2(\lambda), \ldots$ is not zero. This implies that if $\lambda_0 \in S$, then all $\mathcal{D}_r(\lambda)$ have at most poles at $\lambda_0$, and the orders of those poles at $\lambda_0$ are bounded. More exactly, the order of zero of $D_0(\lambda)$ at $\lambda_0$ is the maximal order of poles of $\mathcal{D}_r(\lambda)$ $(r = 1, 2, \ldots)$ at $\lambda_0$. Hence we infer that the order of every zero point $\lambda_0$ of $D_0(\lambda)$ (i. e. $\lambda_0 \in S$) is uniquely determined by $T$ only.

Let $\lambda_1, \lambda_2, \ldots$ $(0 < |\lambda_1| \leqslant |\lambda_2| \leqslant \ldots)$ be the sequence of all zero points of $D_0(\lambda)$. For every $\lambda \in S$, if $\lambda$ is a zero point of order $k$, then $\lambda$ appears exactly $k$ times in the sequence $\{\lambda_n\}$.

Since (Leżański [1], p. 253)

$$D_0(\lambda) = 1 + \sum_{n=1}^{n} a_n\lambda^n \quad \text{where} \quad |a_n| \leqslant \|F\|^n \cdot \frac{n^{n/2}}{n!},$$

$D_0(\lambda)$ is an entire function of order $\leqslant 2$. By an old theorem on entire functions,

$$D_0(\lambda) = \exp(a\lambda + \beta\lambda^2)\prod_n \left(1 - \frac{\lambda}{\lambda_n}\right)\exp\left(\frac{\lambda}{\lambda_n} + \frac{1}{2}\left(\frac{\lambda}{\lambda_n}\right)^2\right)$$



and the series

$$s_m = \sum_n \frac{1}{\lambda_n^m}$$

converges for $m = 3, 4, \ldots$ Hence, for small $\lambda$,

$$D_0(\lambda) = \exp\left(a\lambda + \beta\lambda^2 - \frac{s_3\lambda^3}{3} - \frac{s_4\lambda^4}{4} - \ldots\right). \tag{5}$$

It follows from (3), (4), (5) that $F(T^2) = -s_3$ for every extension $F$ of $F_0$. Since $s_3$ is uniquely determined by $T$, all extensions $F$ of the functional $F_0$ defined by (1) on $\mathfrak{K}_0$ assume the same value. This implies that $T^2$ belongs to the closure of $\mathfrak{K}_0$, q. e. d.

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