Putting

$$\int_0^{\pi} ue^{-u} d\varphi = I + J$$

we have

$$I = \int_0^{\varphi_0} ue^{-u} d\varphi.$$

We are going to prove that

$$\lim_{t\to\infty} t^a I = 0. \tag{19}$$

Since the function $w(\varphi)$ is continuous in $(0, \pi)$ and bounded at $\varphi = 0$, there is a number $N > 0$ such that $w(\varphi) < N$ for $0 < \varphi < \varphi_0$. Thus $I < Nt^{-a/(1-a)}\varphi_0$ and this implies (21).

Since a and b can be chosen arbitrarily close to $(\sin \alpha\pi)^{1/(1-a)}$, it follows from (17), (18), (19) that

$$\lim_{t\to\infty} t^a \int_0^{\pi} ue^{-u} d\varphi = (1-a)\sin \pi a \Gamma(a).$$

This is equivalent to (5).

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Reçu par la Rédaction le 12. 9. 58

STUDIA MATHEMATICA, T. XVIII. (1959)

Consistency theorems for Banach space analogues of Toeplitzian methods of summability

by

A. ALEXIEWICZ and W. ORLICZ (Poznań)

We deal in this paper with the generalized Toeplitz sequence-to--sequence transformations from one Banach space X into another Y; these transformations will be called in conformity with the case of numerical sequences *methods of summability*. One instance of such methods, namely those involving the strong limits, has recently been introduced by Robinson [6] and Melvin-Melvin [4], who derived the Toeplitzian conditions for permanency.

One of the non-trivial results in the theory of summability of numerical sequences is the bounded consistency theorem, stating, roughly speaking, that if two Toeplitzian methods are consistent for convergent sequences and if every bounded sequence summable by the first method is summable by the second, both methods are consistent for bounded sequences [3].

It is the purpose of this paper to prove the bounded consistency theorem in the case of sequence-to-sequence transformations in Banach spaces. Our method consists in considering the spaces of bounded summable sequences as two-norm spaces; in these spaces a notion γ of limit arises in a natural way, leading to the class of continuous distributive functionals called the γ-linear functionals. Essential for the success of our method is the fact that the spaces we are dealing with are such that the limit of any pointwise convergent sequence of γ-linear functionals is γ-linear, which is not the case in all the two-norm spaces.

The authors are indebted to Mr. Bogdanowicz for valuable simplifications and corrections.

1. Preliminaries. We shall deal in this paper with the following methods of summability of sequences of Banach spaces. We are given two Banach spaces X and Y and a system $A = \{A_{i\nu}\}$ of linear operators from X to Y. For a sequence $\boldsymbol{x} = \{x_\nu\}$ of elements of X let us consider the transforms

$$A_i(\boldsymbol{x}) = \sum_{\nu=0}^{\infty} A_{i\nu}(x_\nu).$$

If all the series defining $A_i(\boldsymbol{x})$ are convergent and if $\lim_{i\to\infty} A_i(\boldsymbol{x}) = A_{.}(\boldsymbol{x})$ exists, the sequence $\boldsymbol{x}$ is said to be *strongly A-summable to* $A_{.}(\boldsymbol{x})$ or shortly *s-A-summable to* $A_{.}(\boldsymbol{x})$. Such methods of summability were introduced by Robinson [6] and Melvin-Melvin [4]. We shall also deal with other cases.

The weak case: the series representing $A_i(\boldsymbol{x})$ are weakly convergent and $A_{.}(\boldsymbol{x}) = w\text{-}\lim_{i\to\infty} A_i(\boldsymbol{x})$ (1); then the sequence will be termed *w-A-summable to* $A_{.}(\boldsymbol{x})$.

The mixed case: the series representing $A_i(\boldsymbol{x})$ are weakly convergent and $A_i(\boldsymbol{x})$ tend strongly to $A_{.}(\boldsymbol{x})$; then the sequence will be called *m-A-summable to* $A_{.}(\boldsymbol{x})$.

We shall denote by $\mathfrak{M}$, $\mathfrak{C}$, $\mathfrak{C}_0$ the classes of sequences of elements of X which are bounded, convergent and convergent to zero, respectively. Let us denote by $\mathfrak{A}_0^s$, $\mathfrak{A}_0^m$, $\mathfrak{A}_0^w$, $\mathfrak{A}^s$, $\mathfrak{A}^m$, $\mathfrak{A}^w$ the sets of sequences $\boldsymbol{x}$ which are respectively s-A-, m-A-, w-A-summable to zero, s-A-, m-A-, w-A-summable.

Let k be equal to s, m, or w. The method A will be called *k-null-conservative* if $\mathfrak{C}_0 \subset \mathfrak{A}^k$. Similarly, the method will be called *k-conservative* if $\mathfrak{C} \subset \mathfrak{A}^k$; the method will be termed *k-null-permanent* if $\mathfrak{C}_0 \subset \mathfrak{A}_0^k$.

1.1. Proposition. *The method A is s-null-conservative if and only if the following conditions are satisfied:*

$$(\mathrm{a}_1) \qquad \sup_{n=0,1,\dots}\ \sup_{m=0,1,\dots}\ \sup_{\|x_\nu\|\leqslant 1} \Big\| \sum_{\nu=0}^{m} A_{n\nu}(x_\nu) \Big\| < \infty,$$

(a_2) *for any $x \in X$ there exists*

$$\lim_{n\to\infty} A_{n\nu}(x) = A_{.\nu}(x).$$

If these conditions are satisfied, then

$$A_{.}(\boldsymbol{x}) = \sum_{\nu=0}^{\infty} A_{.\nu}(x_\nu)$$

(1) w-lim denotes the weak limit; a sequence x_n is called *weakly convergent* if there exists an element x such that $\xi(x_n) \to \xi(x)$ for any linear functional ξ; a series is weakly convergent if the sequence of its partial sums is weakly convergent.

for $\boldsymbol{x} = \{x_\nu\} \in \mathfrak{C}_0$, the series involved being strongly convergent. Hence the method A is s-null-permanent if and only if it is s-null-conservative and $A_{.\nu}(x) = 0$ for $\nu = 0, 1, \dots$

The quantity defined by the left-hand side of (a_1) will be denoted by M or $\|A\|$ without further reference.

This proposition and the proposition 1.2 to follow are due to Robinson [6] and Melvin-Melvin [4] (for a simple proof see Zeller [7]).

1.2. Proposition. *The method A is s-conservative if and only if it is s-null-conservative and*

$$(\mathrm{a}_3) \qquad \lim_{n\to\infty} \sum_{\nu=0}^{\infty} A_{n\nu}(x) = S_{.}(x)$$

exists for each $x \in X$, the series involved being strongly convergent.

If these conditions are satisfied, then

$$A_{.}(x) = S_{.}(x_{.}) + \sum_{\nu=0}^{\infty} A_{.\nu}(x_\nu - x_{.})$$

for any $\boldsymbol{x} = \{x_\nu\} \in \mathfrak{C}$ with limit $x_{.}$.

Propositions 1.1 and 1.2 are easily deduced by standard methods of Functional Analysis, $\mathfrak{C}_0$ and $\mathfrak{C}$ being regarded as Banach spaces with the norm $\|\boldsymbol{x}\| = \sup_{\nu=0,1,\dots} \|x_\nu\|$.

It is easily seen that the general form of linear functionals in $\mathfrak{C}_0$ is

$$\xi(\boldsymbol{x}) = \sum_{\nu=0}^{\infty} \xi_\nu(x_\nu)$$

where ξ_ν are linear functionals on X, and

$$\|\xi\| = \sum_{\nu=0}^{\infty} \|\xi_\nu\|.$$

This result directly implies

1.3. Proposition. *Let $\boldsymbol{x}_n = \{x_{n\nu}\} \in \mathfrak{C}_0$, $\boldsymbol{x} = \{x_{.\nu}\} \in \mathfrak{C}_0$. Then $\boldsymbol{x}_n$ converge weakly to $\boldsymbol{x}$ in $\mathfrak{C}_0$ if and only if $\sup_{n=0,1,\dots} \|\boldsymbol{x}_n\| < \infty$ and $w\text{-}\lim_{n\to\infty} x_{n\nu} = x_{.\nu}$ for $\nu = 0, 1, \dots$*

The Toeplitzian conditions for m-summability are deduced similarly. We need the following propositions, whose proof may be obtained by a repeated use of the Banach-Steinhaus theorem:

1.4. Proposition. *Let U_n be linear operations from X to Y and let $w\text{-}\lim_{n\to\infty} U_n(x) = U(x)$ for every $x \in X$. Then $\sup_{n=0,1,\dots} \|U_n\| < \infty$ and U is a linear operation from X to Y.*

1.5. PROPOSITION. *The sequence* $\{U_n\}$ *of linear operations from* X *to* Y *converges weakly for every* $x \in X$ *if and only if* $\sup_{n=0,1,\ldots} \|U_n\| < \infty$ *and* $\{U_n(x)\}$ *converges weakly in a set dense in* X.

From these propositions we easily deduce

1.6. PROPOSITION. *The method* A *is m-null-conservative if and only if it is s-null-conservative. The method* A *is m-null-permanent if and only if it is s-null-permanent. The method* A *is m-conservative if and only if it is s-null-conservative and*

$$\text{(b)} \qquad \lim_{n\to\infty} \sum_{\nu=0}^{\infty} A_{n\nu}(x) = S_{.}(x)$$

exists, the series being weakly convergent.

1.7. PROPOSITION. *The method* A *is w-null-conservative if and only if the condition* (a_1) *and the following conditions are satisfied:*

$$(c_1) \qquad w\text{-}\lim_{n\to\infty} A_{n\nu}(x) = A_{.\nu}(x) \text{ exists for any } x \in X.$$

The method A *is w-null-permanent if and only if it is w-null-conservative and* $A_{.\nu}(x) = 0$ *for each* $x \in X$ *and* $\nu = 0, 1, \ldots$

1.8. PROPOSITION. *The method* A *is w-conservative if and only if it is w-null-conservative and*

$$(c_2) \qquad w\text{-}\lim_{n\to\infty} \sum_{\nu=0}^{\infty} A_{n\nu}(x) = S_{.}(x) \text{ exists for any } x \in X.$$

If these conditions are satisfied, then

$$A_{.}(\boldsymbol{x}) = S_{.}(x_{.}) + \sum_{\nu=0}^{\infty} A_{.\nu}(x_\nu - x_{.})$$

for any $\boldsymbol{x} = \{x_\nu\} \in \mathfrak{C}$ *with limit* $x_{.}$ *(all the above series are weakly convergent).*

For any s-, w-, or m-conservative and null-permanent method A the operation $S.(x)$ will be called the *characteristic of* A and written $\chi_A(x)$.

2. Auxiliary notions. Let Z be a linear space in which there are defined two norms $\| \;\|$ and $\| \;\|^*$; the triplet $\langle Z, \| \;\|, \| \;\|^*\rangle$, called the *two-norm space*, leads to the following notion of convergence: the sequence $\{z_n\}$ is called *γ-convergent to* $z_{.}$ (written $z_n \xrightarrow{\gamma} z_{.}$) if $\sup_{n=0,1,\ldots} \|z_n\| < \infty$ and $\lim_{n\to\infty} \|z_n - z_{.}\|^* = 0$. We suppose the following postulate to be satisfied: the functional $\| \;\|$ is lower semicontinuous with respect to the conver-

gence γ. The space $\langle Z, \| \;\|, \| \;\|^*\rangle$ is called *γ-complete* if each sequence $\{z_n\}$ satisfying $\sup_{n=0,1,\ldots} \|z_n\| < \infty$ and $\lim_{m,n\to\infty} \|z_n - z_m\|^* = 0$ is γ-convergent to an element of Z. The set D is called *γ-dense* (in Z) if each element of Z is equal to the limit of a γ-convergent sequence of elements of D. A functional ζ defined on Z is called *γ-linear* if it is distributive and if $z_n \xrightarrow{\gamma} z_{.}$ implies $\zeta(z_n) \to \zeta(z_{.})$.

In a general two-norm-space case, in contrast to the Banach space case, the limit of a convergent sequence of γ-linear functionals need not be γ-linear. However, there are known some sufficient conditions to be satisfied by the two-norm space, ensuring the γ-linearity of the limit functional. The condition we shall use is the following:

(Σ_1) Given any $z_0 \in S = \{z \colon \|z\| \leqslant 1\}$ and $\varepsilon > 0$ there is a $\delta > 0$ such that any $z \in S$ satisfying $\|z\|^* < \delta$ is of form $z = z_1 - z_2$ where $z_1, z_2 \in S$ and $\|z_1 - z_0\|^* < \varepsilon$, $\|z_2 - z_0\|^* < \varepsilon$.

The following proposition, whose proof may be found in [1], p. 55, or [5], p. 10, is basic for the sequel:

2.1. PROPOSITION. *Let the space* $\langle Z, \| \;\|, \| \;\|^*\rangle$ *be γ-complete and let it satisfy the condition* (Σ_1). *Then the limit of any pointwise convergent sequence of γ-linear functionals is γ-linear.*

We shall deal in the sequel with two two-norm spaces. Let the method A be s-null-permanent. The first space, $\mathfrak{M} \frown \mathfrak{A}_0^s$, consists of bounded sequences $\boldsymbol{x} = \{x_\nu\}$ s-A-summable to zero, the norms being defined as

$$\|\boldsymbol{x}\| = \sup_{\nu=0,1,\ldots} \|x_\nu\|,$$

$$\|\boldsymbol{x}\|^* = \sum_{\nu=0}^{\infty} \frac{1}{2^\nu}\Big(\|x_\nu\| + \sup_{\mu=0,1,\ldots} \Big\|\sum_{\sigma=0}^{\mu} A_{\nu\sigma}(x_\sigma)\Big\|\Big) + \sup_{\nu=0,1,\ldots} \|A_\nu(\boldsymbol{x})\|.$$

The second space, $\mathfrak{M} \frown \mathfrak{A}_0^m$, consists of bounded sequences m-A-summable to zero; in this case

$$\|\boldsymbol{x}\| = \sup_{\nu=0,1,\ldots} \|x_\nu\|, \qquad \|\boldsymbol{x}\|^* = \sum_{\nu=0}^{\infty} \frac{1}{2^\nu} \|x_\nu\| + \sup_{\nu=0,1,\ldots} \|A_\nu(\boldsymbol{x})\|.$$

Obviously $\mathfrak{M} \frown \mathfrak{A}_0^s \subset \mathfrak{M} \frown \mathfrak{A}_0^m$. We shall show that these spaces are γ-complete and satisfy the condition (Σ_1), and that the set $\mathfrak{C}_0^*$ composed of sequences with only a finite number of terms different from zero is γ-dense in both spaces.

2.2. Lemma. *Let A_ν be linear operations from X to Y such that the series $\sum_{\nu=0}^{\infty} A_\nu(x_\nu)$ converges weakly for every sequence $x = \{x_\nu\} \epsilon \mathfrak{C}_0$. Then for every functional η-linear on Y and a fixed k*

$$\sum_{\nu=0}^{\infty} \sup_{\|x_\nu\| \leqslant k} |\eta(A_\nu(x_\nu))| < \infty,$$

whence

$$\lim_{n\to\infty} \sup_{\|x_\nu\|\leqslant 1} \sum_{\nu=n}^{\infty} |\eta(A_\nu(x_\nu))| = 0.$$

Proof. Let us consider the functionals $f_n(x, \eta) = \eta(\sum_{\nu=0}^{n} A_\nu(x_\nu))$ bilinear on the product $\mathfrak{C}_0 \times Y^*$ of $\mathfrak{C}_0$ and the space Y^*, conjugate to Y. By hypothesis the sequence f_n converges on $\mathfrak{C}_0 \times Y^*$, whence the sequence of norms

$$\|f_n\| = \sup_{\|x_\nu\|\leqslant 1} \sup_{\|\eta\|\leqslant 1} \Big|\eta\Big(\sum_{\nu=0}^{n} A_\nu(x_\nu)\Big)\Big|$$

is bounded: $\|f_n\| \leqslant M$. Since $x = \{x_\nu\} \epsilon \mathfrak{C}_0$ implies $\{\varepsilon_\nu x_\nu\} \epsilon \mathfrak{C}_0$ for arbitrary $\varepsilon_n = \pm 1$, we infer that

$$\|f_n\| = \sup_{\|x_\nu\|\leqslant 1} \sum_{\nu=0}^{n} \sup_{\|\eta\|\leqslant 1} |\eta(A_\nu(x_\nu))| \leqslant M$$

and, finally,

$$\sum_{\nu=0}^{\infty} \sup_{\|x_\nu\|\leqslant k} |\eta(A_\nu(x_\nu))| \leqslant Mk\|\eta\|.$$

2.3. Proposition. *The spaces $\langle \mathfrak{M} \frown \mathfrak{A}_0^s, \|\,\|, \|\,\|^* \rangle$ and $\langle \mathfrak{M} \frown \mathfrak{A}_0^m, \|\,\|, \|\,\|^* \rangle$ are γ-complete.*

Proof. We omit the easy proof for the space $\langle \mathfrak{M} \frown \mathfrak{A}_0^s, \|\,\|, \|\,\|^* \rangle$ and consider only the second case. Let $x_n = \{x_{n\nu}\}$, $\sup_{n=0,1,\dots} \|x_n\| < \infty$ and let $\lim_{m,n\to\infty} \|x_m - x_n\|^* = 0$. Hence for a fixed ν, $x_{n\nu}$ is a Cauchy sequence in X, whence it converges to an element $x_{\cdot\nu}$. We shall prove that the series $\sum_{\nu=0}^{\infty} A_{i\nu}(x_{\cdot\nu})$ converges weakly for $i = 0, 1, \dots$ Let i be fixed and let us set

$$y_p = \sum_{\nu=0}^{\infty} A_{i\nu}(x_{p\nu});$$

this sequence converges strongly to an element $y_\cdot$, for

$$\sup_{i=0,1,\dots} \|A_i(x^m) - A_i(x^n)\| \to 0 \quad \text{as } m, n \to \infty.$$

Let us write

$$z_m = \sum_{\nu=0}^{m} A_{i\nu}(x_{\cdot\nu}) - y_\cdot;$$

it is sufficient to prove that z_m converges weakly to zero. For any functional η linear over Y

$$\eta(z_m) = \eta\Big(\sum_{\nu=0}^{m} A_{i\nu}(x_{\cdot\nu}) - y_\cdot\Big) = \eta\Big(\sum_{\nu=0}^{m} A_{i\nu}(x_{\cdot\nu}) - \sum_{\nu=0}^{\infty} A_{i\nu}(x_{p\nu})\Big) + \eta(y_p - y_\cdot)$$

$$= \eta\Big(\sum_{\nu=0}^{m} A_{i\nu}(x_{\cdot\nu} - x_{p\nu})\Big) - \eta\Big(\sum_{\nu=m+1}^{\infty} A_{i\nu}(x_{p\nu})\Big) + \eta(y_p - y_\cdot),$$

$$|\eta(z_m)| \leqslant \overline{\lim_{p\to\infty}} \Big|\eta\Big(\sum_{\nu=1}^{m} A_{i\nu}(x_{\cdot\nu} - x_{p\nu})\Big)\Big| + \overline{\lim_{p\to\infty}} \Big|\eta\Big(\sum_{\nu=m+1}^{\infty} A_{i\nu}(x_{p\nu})\Big)\Big| +$$

$$+ \lim_{p\to\infty} |\eta(y_p - y_\cdot)| \leqslant \sup_{p=0,1,\dots} \Big|\eta\Big(\sum_{\nu=m+1}^{\infty} A_{i\nu}(x_{p\nu})\Big)\Big|$$

$$\leqslant \sup_{p=0,1,\dots} \sum_{\nu=m+1}^{\infty} |\eta(A_{i\nu}(x_{p\nu}))| \leqslant \sup_{\|x_\nu\|\leqslant k} \sum_{\nu=m+1}^{\infty} |\eta(A_{i\nu}(x_\nu))|,$$

where $k = \sup_{p=0,1,\dots} \|x_p\|$, whence, by Lemma 2.2, $\eta(z_m) \to 0$. Thus we have proved the existence of the transforms $A_i(x_\cdot) = \sum_{\nu=0}^{\infty} A_{i\nu}(x_{\cdot\nu})$. The above argument shows also that the sums $\sum_{\nu=0}^{\infty} A_{i\nu}(x_{p\nu})$ converge strongly to $A_i(x_\cdot)$, whence

$$\|A_i(x_p) - A_i(x_\cdot)\| \leqslant \lim_{q\to\infty} \|A_i(x_p) - A_i(x_q)\|,$$

$$\lim_{p\to\infty} \sup_{i=0,1,\dots} \|A_i(x_p) - A_i(x_\cdot)\| \leqslant \lim_{p\to\infty} \sup_{i=0,1,\dots} \lim_{q\to\infty} \|A_i(x_p) - A_i(x_q)\| = 0.$$

This implies strong convergence of $A_i(x_\cdot)$ to zero as $i \to \infty$, whence $x \epsilon \mathfrak{M} \frown \mathfrak{A}_0^m$ and $\|x_p - x_\cdot\|^* \to 0$.

2.4. Proposition. *Let $x = \{x_\nu\} \epsilon \mathfrak{M} \frown \mathfrak{A}_0^m$; then for each $\varepsilon > 0$ and n there exists a p and an element $z = \{z_\nu\} \epsilon \mathfrak{C}_0$ such that*

$$z_\nu = \begin{cases} x_\nu & \text{for } \nu \leqslant n, \\ \vartheta_\nu x_\nu & \text{for } n < \nu \leqslant n+p, \\ 0 & \text{elsewhere}, \end{cases}$$

$$1 \geqslant \vartheta_{n+1} \geqslant \vartheta_{n+2} \geqslant \dots \geqslant \vartheta_{n+p} \geqslant 0,$$

$$\sup_{i=0,1,\dots} \|A_i(x) - A_i(z)\| < \varepsilon.$$

Proof. Let us write

$$y_{ni} = \sum_{\nu=0}^{n} A_{i\nu}(x_\nu), \qquad y_i = A_i(\boldsymbol{x}),$$

$$\boldsymbol{y}_n = \{y_{n\nu}\}, \ \boldsymbol{y} = \{y_\nu\}.$$

From Proposition 1.3 we easily infer that the sequence $\{\boldsymbol{y}_n\}$ converges weakly to $\boldsymbol{y}$ in the space $\mathfrak{C}_0$, whence by the theorem of Mazur ([3], p. 81) there exist non negative constants $\lambda_1, \ldots, \lambda_p$ such that $\lambda_1+\ldots+\lambda_p = 1$ and

$$\|\boldsymbol{y}-(\lambda_1\boldsymbol{y}_{n+1}+\ldots+\lambda_p\boldsymbol{y}_{n+p})\| < \varepsilon.$$

Setting $\boldsymbol{x}^{(n)} = \{x_0, \ldots, x_n, 0, 0, \ldots\}$ we have $y_{n\nu} = A_\nu(\boldsymbol{x}^{(n)})$, whence

$$\lambda_1\boldsymbol{y}_{n+1}+\ldots+\lambda_p\boldsymbol{y}_{n+p} = \{A_\nu(\lambda_1\boldsymbol{x}^{(n+1)}+\ldots+\lambda_p\boldsymbol{x}^{(n+p)})\}$$

and an easy computation gives

$$\lambda_1\boldsymbol{x}^{(n+1)}+\ldots+\lambda_p\boldsymbol{x}^{(n+p)} = \{x_1, \ldots, x_n, \vartheta_{n+1}x_{n+1}, \ldots, \vartheta_{n+p}x_{n+p}, 0, 0, \ldots\},$$

where $\vartheta_{n+\nu} = \lambda_\nu+\ldots+\lambda_p$ for $\nu = 1, \ldots, p$, whence $1 \geqslant \vartheta_{n+1} \geqslant \ldots \geqslant \vartheta_{n+p} \geqslant 0$. It is sufficient to choose $\boldsymbol{z} = \lambda_1\boldsymbol{x}^{(n+1)}+\ldots+\lambda_p\boldsymbol{x}^{(n+p)}$.

2.5. Proposition. *The set $\mathfrak{C}_0^*$ is γ-dense in both the spaces,*

$$\langle\mathfrak{M} \frown \mathfrak{A}_0^m, \|\ \|, \|\ \|^*\rangle \quad \textit{and} \quad \langle\mathfrak{M} \frown \mathfrak{A}_0^s, \|\ \|, \|\ \|^*\rangle.$$

Proof. For the space $\langle\mathfrak{M} \frown \mathfrak{A}_0^m, \|\ \|, \|\ \|^*\rangle$ this follows directly from Proposition 2.4. To prove our proposition for $\langle\mathfrak{M} \frown \mathfrak{A}_0^s, \|\ \|, \|\ \|^*\rangle$ let us notice that in this case Proposition 2.4 may be stated in a stronger form. Indeed, let us maintain the notation of the proof of 2.4: let s be a positive integer, and let us write

$$S_{i\nu}(\boldsymbol{x}) = \sum_{\mu=0}^{\nu} A_{i\mu}(x_\mu);$$

$$\boldsymbol{v}_{in} = \{S_{i0}(\boldsymbol{x}^{(n)}), S_{i1}(\boldsymbol{x}^{(n)}), \ldots\}, \qquad v_i = \{S_{i0}(\boldsymbol{x}), S_{i1}(\boldsymbol{x}), \ldots\}.$$

Then the sequence $(\boldsymbol{y}_n, \boldsymbol{v}_{0n}, \ldots, \boldsymbol{v}_{sn})$ is easily seen to converge weakly to $(\boldsymbol{y}, \boldsymbol{v}_0, \ldots, \boldsymbol{v}_s)$ in the space $\mathfrak{C}_0\times\mathfrak{C}\times\ldots\times\mathfrak{C}$ $(s+2$ factors), whence applying the theorem of Mazur we obtain the conclusion of Proposition 2.4; moreover

$$\sup_{\nu=0,1,\ldots}\left\|\sum_{\mu=0}^{\nu}[A_{i\mu}(x_\mu)-A_{i\mu}(z_\mu)]\right\| < \varepsilon$$

holds in this case for $i = 0, 1, \ldots, s$.

γ-density of $\mathfrak{C}_0^*$ in $\langle\mathfrak{M} \frown \mathfrak{A}_0^s, \|\ \|, \|\ \|^*\rangle$ follows from this version of Proposition 2.4 when s is taken sufficiently large.

2.6. Proposition. *The spaces* $\langle\mathfrak{M} \frown \mathfrak{A}_0^s, \|\ \|, \|\ \|^*\rangle$ *and* $\langle\mathfrak{M} \frown \mathfrak{A}_0^m, \|\ \|, \|\ \|^*\rangle$ *satisfy the condition* (Σ_1).

Proof. We give the proof for the first case only.

By Proposition 2.4 there exists a sequence $\boldsymbol{y} = \{y_\nu\}$ such that $\|y_\nu\| \leqslant 1$, $\|\boldsymbol{y}-\boldsymbol{x}_0\|^* < \varepsilon/2$ and $y_\nu = 0$ for $n > p$, say, and we may suppose p to be so large that $1/2^p < \varepsilon/8M$. Let us choose

$$\varepsilon' = \varepsilon/(232M+32), \qquad \delta < \varepsilon'/2^p.$$

Then $\varepsilon' < 1$, $\delta < \varepsilon' < \varepsilon/16$. Let $\boldsymbol{z} = \{z_\nu\}$, $\|\boldsymbol{z}\|^* < \delta$; then

$$\|z_\nu\| < \varepsilon', \qquad \left\|\sum_{\sigma=0}^{\mu} A_{\nu\sigma}(z_\nu)\right\| < \varepsilon' \quad \text{for} \quad \nu = 0, \ldots, p, \ \sigma = 0, 1, \ldots,$$

$$\|A_i(\boldsymbol{z})\| < \delta \quad \text{for} \quad i = 1, 2, \ldots$$

Let us consider the sets $\Theta = \{\nu: \nu \leqslant p, \|y_\nu+z_\nu\| \leqslant 1\}$, $\Lambda = \{\nu: \nu \leqslant p, \|y_\nu+z_\nu\| > 1\}$ and let us write

$$\sigma_\nu = \frac{\|y_\nu+z_\nu\|+\varepsilon'-1}{\|y_\nu+z_\nu\|} \quad \text{for} \quad \nu \in \Lambda;$$

then $0 \leqslant \sigma_\nu \leqslant 1+\varepsilon'+\varepsilon'-1 = 2\varepsilon'$. Now, let us set

$$z_{1\nu} = \begin{cases} y_\nu+z_\nu & \text{for } \nu\in\Theta, \\ y_\nu+z_\nu-\sigma_\nu(y_\nu+z_\nu) & \text{for } \nu\in\Lambda, \\ z_\nu & \text{elsewhere}, \end{cases}$$

$$z_{2\nu} = \begin{cases} y_\nu & \text{for } \nu\in\Theta, \\ y_\nu-\sigma_\nu(y_\nu+z_\nu) & \text{for } \nu\in\Lambda, \\ 0 & \text{elsewhere}, \end{cases}$$

$$\boldsymbol{z}_1 = \{z_{1\nu}\}, \qquad \boldsymbol{z}_2 = \{z_{2\nu}\}.$$

Obviously $\boldsymbol{z} = \boldsymbol{z}_1-\boldsymbol{z}_2$. For $\nu\in\Lambda$

$$\|z_{1\nu}\| = (1-\sigma_\nu)\|y_\nu+z_\nu\| = 1-\varepsilon' < 1,$$

$$\|z_{2\nu}\| = (1-\sigma_\nu)\|y_\nu\|+\|\sigma_\nu z_\nu\| < 1-\sigma_\nu+\sigma_\nu = 1,$$

whence $\boldsymbol{z}_1, \boldsymbol{z}_2 \in S$. Now

$$A_i(\boldsymbol{y}-\boldsymbol{z}_1) = \sum_{\nu=0}^{\infty} A_{i\nu}(y_\nu-z_{1\nu}) = \sum_{\nu\in\Theta} + \sum_{\nu\in\Lambda} + \sum_{\nu=p+1}^{\infty} = \mathrm{I}+\mathrm{II}+\mathrm{III},$$

$$\|\mathrm{I}\| = \left\|\sum_{\nu\in\Theta} A_{i\nu}(z_\nu)\right\| < M\varepsilon',$$

$$\|\mathrm{II}\| = \Big\|\sum_{\nu\in\Lambda} A_{i\nu}\big(-z_\nu+\sigma_\nu(y_\nu+z_\nu)\big)\Big\| \leqslant M\sup_{\nu\in\Lambda}\|-z_\nu+\sigma_\nu(y_\nu+z_\nu)\|$$

$$\leqslant M[2\varepsilon'+2\varepsilon'(1+\varepsilon')] < 5M\varepsilon',$$

$$\|\mathrm{III}\| = \Big\|\sum_{\nu=p+1}^{\infty} A_{i\nu}(z_\nu)\Big\| = \Big\|A_i(z)-\sum_{\nu=0}^{p} A_{i\nu}(z_\nu)\Big\| \leqslant \delta+M\varepsilon',$$

whence $\|A_i(\boldsymbol{y}-\boldsymbol{z}_1)\| < \delta+7M\varepsilon'$. Let us notice that any subsum of I and II satisfies also the above inequalities. Hence for $i\leqslant p$, setting $\Omega=\{1,\ldots,p\}$, we get:

$$\sum_{\sigma=0}^{\mu} A_{i\sigma}(y_\sigma-z_{1\sigma}) = \sum_{\sigma\in\Theta\frown\Omega} + \sum_{\sigma\in\Lambda\frown\Omega} + \sum_{\sigma=p+1}^{\mu} = \mathrm{I}'+\mathrm{II}'+\mathrm{III}',$$

$$\|\mathrm{I}'\| < M\varepsilon', \quad \|\mathrm{II}'\| < 5M\varepsilon',$$

$$\|\mathrm{III}'\| \leqslant \Big\|\sum_{\sigma=p+1}^{\mu} A_{i\sigma}(y_\sigma-z_{1\sigma})\Big\| = \Big\|\sum_{\sigma=p+1}^{\mu} A_{i\sigma}(z_\sigma)\Big\|$$

$$\leqslant \Big\|\sum_{\sigma=0}^{\mu} A_{i\sigma}(z_\sigma)\Big\| + \Big\|\sum_{\sigma=0}^{p} A_{i\sigma}(z_\sigma)\Big\| < 2\varepsilon',$$

whence

$$\Big\|\sum_{\sigma=0}^{\mu} A_{i\sigma}(y_\sigma-z_{1\sigma})\Big\| < (6M+2)\varepsilon'.$$

Finally

$$\sum_{\nu=0}^{\infty}\frac{1}{2^\nu}\|y_\nu-z_{1\nu}\| = \Big\|\sum_{\nu\in\Theta}+\sum_{\nu\in\Lambda}+\sum_{\nu=p+1}^{\infty}\Big\|$$

$$\leqslant \sum_{\nu\in\Theta}\frac{1}{2^\nu}\|z_\nu\| + \sum_{\nu\in\Lambda}\frac{1}{2^\nu}(\|z_\nu\|+\sigma_\nu\|y_\nu+z_\nu\|) + \sum_{\nu=p+1}^{\infty}\frac{1}{2^\nu}\|z_\nu\|$$

$$< \sum_{\nu=0}^{\infty}\frac{1}{2^\nu}\|z_\nu\| + \sum_{\nu\in\Lambda}\frac{1}{2^\nu}5M\varepsilon' \leqslant \|z\|^*+10M\varepsilon' < \delta+10M\varepsilon',$$

$$\|\boldsymbol{y}-\boldsymbol{z}_1\|^* < \delta+10M\varepsilon'+\delta+7M\varepsilon'+\sum_{\nu=0}^{\infty}\frac{1}{2^\nu}\sup_{\mu=0,1,\ldots}\Big\|\sum_{\sigma=0}^{\mu} A_{\nu\sigma}(y_\sigma-z_{1\sigma})\Big\|$$

$$\leqslant 2\delta+17M\varepsilon'+\sum_{\nu=0}^{p}\frac{1}{2^\nu}(6M+2)\varepsilon'+\frac{1}{2^p}2M < 2\delta+(29M+4)\varepsilon'+\frac{\varepsilon}{4} < \frac{\varepsilon}{2},$$

whence $\|\boldsymbol{x}-\boldsymbol{z}_1\|^* < \varepsilon$; similarly $\|\boldsymbol{x}-\boldsymbol{z}_2\|^* < \varepsilon$.

3. Consistency theorems. Now we are able to prove our principal theorems. A and B will denote two methods of summability with the same spaces X and Y; $B_i(\boldsymbol{x}), B_{\bullet}(\boldsymbol{x})$ will have a meaning analogous to $A_i(\boldsymbol{x}), A_{\bullet}(\boldsymbol{x})$.

3.1. THEOREM. *Let the method A be s-null-permanent, let B be w-null-permanent, and let $k=s$ or $k=m$. If any bounded sequence $\boldsymbol{x}$ k-A-summable to zero is w-B-summable, then $B_{\bullet}(\boldsymbol{x})=0$.*

Proof. The case $k=s$. Let η be any linear functional on Y. The functionals $\eta(B_i(\boldsymbol{x}))$ are γ-linear on $\langle\mathfrak{M}\frown\mathfrak{A}_0^s, \|\,\|, \|\,\|^*\rangle$ in virtue of Proposition 2.1, since $\eta(B_i(\boldsymbol{x})) = \lim_{n\to\infty}\eta\big(\sum_{\nu=0}^{n} B_{i\nu}(x_\nu)\big)$ and the functionals $\eta\big(\sum_{\nu=0}^{n} B_{i\nu}(x_\nu)\big)$ are obviously γ-linear. By the same argument $\eta(B_{\bullet}(\boldsymbol{x}))$ is, again, γ-linear. By hypothesis $\eta(B_{\bullet}(\boldsymbol{x}))=0$ in the set $\mathfrak{C}_0^*$, which is γ-dense in $\langle\mathfrak{M}\frown\mathfrak{A}_0^s, \|\,\|, \|\,\|^*\rangle$, whence $\eta(B_{\bullet}(\boldsymbol{x}))=0$ in $\mathfrak{M}\frown\mathfrak{A}_0^s$, which implies $B_{\bullet}(\boldsymbol{x})=0$ in $\mathfrak{M}\frown\mathfrak{A}_0^s$.

In the case $k=m$ the proof is the same.

3.2. THEOREM. *Let the methods $A^{(p)}$ $(p=0,1,\ldots)$ be s-null-permanent and let B be w-null-permanent. Let $k=s$ or $k=m$; if every bounded sequence $\boldsymbol{x}$ k-$A^{(p)}$-summable to zero for $p=0,1,\ldots$ is w-B-summable, then $B_{\bullet}(\boldsymbol{x})=0$.*

The proof may be carried out by the same method as for Theorem 3.1 if we consider the space $\mathfrak{M}\frown\overline{\mathfrak{A}}_0^s$ or $\mathfrak{M}\frown\overline{\mathfrak{A}}_0^m$ of the sequences s-$A^{(n)}$- or m-$A^{(n)}$-summable to zero respectively for $n=0,1,\ldots$, with the norm $\|\,\|$ as in 2, the norm $\|\,\|^*$ being defined by $\|\boldsymbol{x}\|^* = \sum_{n=0}^{\infty}[2(1+\|A^{(n)}\|)]^{-n}\|\boldsymbol{x}\|_n^*$ where $\|\,\|_n^*$ denotes the starred norm for the space of bounded sequences s-$A^{(n)}$- or m-$A^{(n)}$-summable to zero, constructed in section 2.

The methods A and B are said to be *consistent for constant sequences* if each constant sequence is summable by both methods to the same value.

3.3. THEOREM. *Let the method A be s-conservative and s-null-permanent, let B be w-conservative and w-null-permanent, and let the methods be consistent for constant sequences. Let $k=s$ or $k=m$, and let the characteristic of A be reversible* (²). *If every bounded k-A-summable sequence $\boldsymbol{x}$ is w-B-summable, then $A_{\bullet}(\boldsymbol{x})=B_{\bullet}(\boldsymbol{x})$.*

Proof. It is easily seen that $\chi_A(x)=\chi_B(x)$. Let us set $u=\chi_A^{-1}(A_{\bullet}(\boldsymbol{x}))$, $\boldsymbol{u}=\{u,u,\ldots\}$; the sequence $\boldsymbol{x}^*=\{x_\nu-u\}$ is k-A-summable to zero,

(²) This means that the operation is one-to-one and that the inverse operation is continuous.

whence, by Theorem 3.1, $B_{\cdot}(x^*) = 0$. Thus $0 = B_{\cdot}(x^*) = B_{\cdot}(x) - B_{\cdot}(u) = B_{\cdot}(x) - \chi_B(u) = B_{\cdot}(x) - \chi_A(\chi_A^{-1}(A_{\cdot}(x))) = B_{\cdot}(x) - A_{\cdot}(x)$.

We conclude with the following remarks. The formulation of the generalizations of Theorem 3.2 for the case of sequences not necessarily summable to zero is left to the reader. In Theorem 3.1 the methods A and B need not transform the sequences of X into sequences of the same space Y.

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Reçu par la Rédaction le 7. 10. 1958

STUDIA MATHEMATICA, T. XVIII. (1959)

Spaces of continuous functions (III)

(Spaces $C(\Omega)$ for Ω without perfect subsets)

by

A. PEŁCZYŃSKI (Warszawa) and Z. SEMADENI (Poznań)

A topological space T is said to be *dispersed* [1] if it contains no perfect non-void subset. In this paper we present some investigations on spaces $C(\Omega)$ of real-valued continuous functions defined on dispersed compact Hausdorff spaces Ω.

In the main theorem we give a number of necessary and sufficient conditions for a compact Hausdorff space Ω to be dispersed (in terms of the space $C(\Omega)$, of its conjugate space $V(\Omega)$, or in terms of Borel measures on Ω). In particular, we prove that the *dispersedness of a compact Ω is an invariant of the linear dimension* [2] *of $C(\Omega)$*; conditions (5), (6), (8) and (9) give characterizations of dispersedness by invariants of linear dimension. The most essential condition is the following one:

In order that a compact Hausdorff space Ω be dispersed, it is necessary and sufficient that the condition $\dim_l X \leqslant \dim_l C(\Omega)$ *imply* $\dim_l X \geqslant \dim_l c_0$ *or X is of finite dimension.*

Other characterizations may easily be derived from the above.

By condition (10), the dispersedness of a compact Ω is necessary and sufficient for any Borel measure on Ω to be purely atomic, which enables us to derive some properties of the first and second conjugate to $C(\Omega)$. At the same time, these properties characterize the dispersedness of Ω, e. g. *a compact Ω is dispersed if and only if the space $V(\Omega)$ conjugate to $C(\Omega)$ contains no subspace isomorphic to the space L.*

In a further section of this paper we consider some singular properties of the spaces $C(\Omega)$ for dispersed Ω. The great importance of these spaces consists in the fact that many general questions concerning Banach spaces may be negatively solved by a suitable counter-example of a space $C(\Omega)$ with dispersed Ω.

(1) *clairsemé* in French (see [14], p. 46).

(2) The definition of linear dimension is given by Banach ([1], p. 193; see also [2]). The symbol $\dim_l X \leqslant \dim_l Y$ means that X is isomorphic to a subspace of Y, $\dim_l X = \dim_l Y$ means that $\dim_l X \leqslant \dim_l Y$ and $\dim_l X \geqslant \dim_l Y$, and so on.