On a representation of the resolvent

by

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Let A be a linear (= additive and continuous) transformation of a (real or complex) Banach space X. The purpose of this paper is to give a sufficient condition for the solvability of the equation

$$(1) \qquad Ax = y_0$$

with y_0 fixed. This yields as a corollary a sufficient condition for the existence of the inverse transformation A^{-1}, and gives a sufficient condition for the existence of the dissolvent[1]) of an element A in a Banach algebra with no unit.

In this paper vector-valued functions are used. A vector-valued function x_t of a real variable t with values in X is said to *converge weakly to* x_0 *as* $t\to\infty$ (in symbols $\operatorname{w-lim}_{t\to\infty} x_t = x_0$) if $\varphi(x_0) = \lim_{t\to\infty}\varphi(x_t)$ for every functional φ linear on X. The function x_t is said to be *weakly differentiable to* x_t' if $\lim_{h\to 0} h^{-1}\varphi(x_{t+h}-x_t) = \varphi(x_t')$ for every t and every linear functional φ.

LEMMA 1. *Let the function x_t be weakly differentiable to x_t' and let x_t' converge weakly as $t\to\infty$. If for every linear functional φ the function $\varphi(x_t)$ has a limit as $t\to\infty$, then*

$$a = \operatorname{w-lim}_{t\to\infty} x_t' = 0.$$

Proof. Suppose that $a \neq 0$. There exists a linear functional φ_0 such that

$$\lim_{t\to\infty}\varphi_0(x_t') = \varphi_0(a) = \alpha > 0;$$

then

$$\varphi_0(x_t') > \alpha/2 \quad \text{for} \quad t > t_1,$$

[1]) The *dissolvent* of A is an element V satisfying the equation $A+V+AV = A+V+VA = 0$.

whence

$$|\varphi_0(x_t) - \varphi_0(x_{t_1})| = \Big|\int_{t_1}^{t}\frac{d}{ds}\varphi_0(x_s)\,ds\Big| = \Big|\int_{t_1}^{t}\varphi_0(x_s')\,ds\Big| \geqslant |t-t_1|\frac{\alpha}{2},$$

which contradicts the hypothesis that $\lim_{t\to\infty}\varphi_0(x_t)$ exists. Thus $\operatorname{w-lim}_{t\to\infty} x_t' = 0$.

Let $\mathfrak{R}$ be a (real or complex) Banach algebra. Set for real t and $A \in \mathfrak{R}$

$$(2) \qquad F(A) = \sum_{n=1}^{\infty}\frac{1}{n!}A^n,$$

$$(3) \qquad V(t,A) = \int_0^t e^{-s}F(-sA)\,ds.$$

The series (2) converges strongly and the function $F(-sA)$ is continuous in $(-\infty,\infty)$, moreover the function $V(t,A)$ is strongly differentiable (with respect to t). We shall prove

LEMMA 2. *The function $V(t,A)$ satisfies the differential equation*

$$V_t'(t,A) = -[V(t,A) + (1-e^{-t})A + AV(t,A)].$$

Proof. The function

$$-e^{-s}F(-sA) - \int_0^s e^{-r}[A + AF(-rA)]\,dr$$

has a (strong) derivative equal to $e^{-s}F(-sA)$, and $F(0) = 0$. Hence

$$V(t,A) = \int_0^t e^{-s}F(-sA)\,ds = e^{-t}F(-tA) - \int_0^t e^{-s}[A + AF(-sA)]\,ds$$

$$= e^{-t}F(-tA) - \Big[\int_0^t e^{-s}\,ds\Big]A - A\int_0^t e^{-s}F(-sA)\,ds$$

$$= -\frac{d}{dt}V(t,A) - (1-e^{-t})A - AV(t,A).$$

LEMMA 3. *Let A, B, B_n $(n=1,2,\ldots)$ be in $\mathfrak{R}$. If $B = \operatorname{w-lim}_{n\to\infty} B_n$ and $AB_n = B_nA$ $(n=1,2,\ldots)$, then $AB = BA$.*

Proof. Let φ be an arbitrary linear functional on $\mathfrak{R}$. Set

$$\psi(X) = \varphi(AX), \qquad \chi(X) = \varphi(XA) \qquad \text{for} \qquad X \in \mathfrak{R};$$

then ψ and χ are linear functionals on $\mathfrak{R}$, and

$$|\varphi(AB - BA)| \leqslant |\varphi(A(B_n - B))| + |\varphi((B_n - B)A)|$$

$$= |\psi(B_n - B)| + |\chi(B_n - B)|.$$

Since B_n converges weakly to B, the right-hand side of the above inequality converges to 0 as $n\to\infty$, whence $\varphi(AB-BA)=0$. The linear functional φ being arbitrary, we get $AB=BA$.

THEOREM 1. *If $V(t,A)$ converges weakly to $V\in\mathfrak{R}$ as $t\to\infty$, then*

$$A+V+AV=A+V+VA=0.$$

Proof. By lemma 2

$$V_t'(t,A)=-[V(t,A)+(1-e^{-t})A+AV(t,A)],$$

whence, by hypothesis, $V_t'(t,A)$ converges weakly to $-(V+A+AV)$ as $t\to\infty$. By lemmata 1 and 3 it follows that $V+A+AV=0$. By lemma 3, V is permutable with A for it may be represented as the weak limit of a (triple) sequence of elements permutable with A, whence $A+V+VA=0$.

Suppose now that R has a unit element I. Set

$$\exp A=\sum_{n=0}^{\infty}\frac{1}{n!}A^n,$$

$$U(t,A)=\int_0^t\exp(-sA)\,ds$$

(the integral being taken in the strong sense). It is easily seen that $U(t,A)$ is strongly differentiable with respect to t; the function $\exp(-tA)$ has the strong derivative equal to $-A\exp(-tA)$.

LEMMA 4. $U_t'(t,A)=-A[U(t,A)-I]$.

Proof. Indeed,

$$U_t'(t,A)-I=\exp(-tA)-I=\int_0^t\frac{d}{ds}\exp(-sA)\,ds$$

$$=\int_0^t A\exp(-sA)\,ds=-AU(-t,A).$$

THEOREM 2. *Let $U(t,A)$ converge weakly to V as $t\to\infty$; then $AU=UA=I$.*

Proof. By lemma 4 $U_t'(t,A)=-[AU(t,A)-I]$, whence $U_t'(t,A)$ converges weakly to $-(AU-I)$ as $t\to\infty$, which implies $AU-I=0$ by lemma 2. Moreover, U is permutable with A as the weak limit of elements permutable with A.

Let X be a (real or complex) Banach space; $\mathfrak{R}$ will stand for the Banach algebra of the linear transformations of the space X. Let V_t

be a function of the real variable t taking on values from the space $\mathfrak{R}$. The function V_t is called *weakly convergent to V as $t\to\infty$* if

$$\lim_{t\to\infty}\varphi(V_t(x))=\varphi(V(x))$$

for every $x\in X$ and every functional φ linear on X.

THEOREM 3. *Let $U\in\mathfrak{R}$, $y_0\in X$. If $U(t,A)(y_0)$ converges weakly to x_0 as $t\to\infty$, then $Ax_0=y_0$.*

Proof. Set $x_t=U(t,A)(y_0)$; the function $U(t,A)$ has the strong derivative equal to $-A[U(t,A)-I]$, whence x_t is strongly differentiable to $-(Ax_t-y_0)$. By hypothesis x_t converges weakly to x_0 as $t\to\infty$, therefore x_t' converges weakly to $-(Ax_0-y_0)$, whence, by lemma 1, $Ax_0-y_0=0$.

COROLLARY. *If $U(t,A)=\int_0^t\exp(-sA)\,ds$ converges weakly to $U\in\mathfrak{R}$ as $t\to\infty$, then $AU=UA=I$.*

It remains to prove that U is permutable with A. This follows from the fact that U is the weak limit of operations which are permutable with A.

THEOREM 4. *Set*

$$F(A)=\sum_{n=1}^{\infty}\frac{1}{n!}A^n,\qquad V(t,A)=\int_0^t e^{-s}F(-sA)\,ds.$$

If, given $y_0\in X$, $V(t,A)(y_0)$ converges weakly to $x_0\in X$ as $t\to\infty$, then $Ay_0+x_0+Ax_0=0$.

Proof. Set $x_t=V(t,A)(y_0)$. By lemma 2 the function $V(t,A)$ is strongly differentiable to $-[V(t,A)+(1-e^{-t})A+AV(t,A)]$. This implies that x_t is strongly differentiable to $x_t'=-[x_0+(1-e^{-t})Ay_0+Ax_0]$.

Since x_t converges weakly to x_0 as $t\to\infty$, x_t' converges weakly to $-(x_0+Ay_0+Ax_0)$, whence, by lemma 1, $x_0+Ay_0+Ax_0=0$.

COROLLARY. *If $V(t,A)=\int_0^t e^{-s}F(-sA)\,ds$ converges weakly to $V\in\mathfrak{R}$ as $t\to\infty$, then $A+V+AV=A+V+VA=0$, i. e. V is the dissolvent of the operation A.*

The proof is similar to that of the first corollary.

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