Theorems of Saks' type for abstract polynomials

by

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Using the method of my previous Note [1] I wish to give the generalization for abstract polynomials of some results obtained by Alexiewicz [2][1]). Because of the close connection of this Note with the paper [2] I use throughout the definitions adopted there.

1. In particular $(T,\mathfrak{E},\mu)$ denotes a measure space on which μ is σ-additive and $\mu(T)<\infty$. L denotes an arbitrary linear set and X an arbitrary separable F-space. Y denotes an F-space, elements of which are functions from T to L. Denoting by y_e, for every $y\in Y$ and $e\in\mathfrak{E}$, the function defined by equations

$$y_e=\begin{cases} y(t) & \text{for } t\in e,\\ 0 & \text{elsewhere,}\end{cases}$$

we suppose the following postulate to be satisfied: if $y\in Y$ and $e\in\mathfrak{E}$, then $y_e\in Y$; if $y\in Y$ and $e\in\mathfrak{E}$, then $\|y_e\|\leqslant\|y\|$;

(α) if $e_n\in\mathfrak{E}$ $(n=1,2,\ldots)$ and $\mu(e_n)\to 0$, then $\|y_{e_n}\|\to 0$.

B denotes an analytic set (Kuratowski [3], p. 360) in Y which satisfies the following condition:

(β) if $y\in B$ and $e\in\mathfrak{E}$, then $y_e\in B$.

When the set B is also linear, we suppose that the additional condition to be satisfied: if $y_e\in B$ and $y_h\in B$, then $y_{e\cup h}\in B$[2]). $U(x)$ will stand for a polynomial operation of degree m from X to Y.

It is known [4] that we can represent the polynomial operation $U(x)$ in the canonical form

$$U(x)=\sum_{k=0}^{m}U_k(x), \tag{1}$$

[1]) I express my indebtedness to Professor A. Alexiewicz for kindly placing the manuscript of [2] at my disposal.

[2]) The symbols $\cup,\cap,\setminus$ denote set-theoretical union, product or difference of sets respectively.

where $U_k(x)$ is a homogeneous polynomial of degree m, moreover

$$U_k(x)=\sum_{\nu=0}^{m}a_{k\nu}U(\nu x) \tag{2}$$

with $a_{k\nu}$ independent of U and x.

2. The Mazur-Orlicz theorem for a sequence of polynomial operations ([5], Theorem VII, p. 185) and postulate (α) implies the following

LEMMA 1. *The operation $U(x)_e$, $e\in\mathfrak{E}$, is continuous in the space $X\times(T,\mathfrak{E},\mu)$.*

We shall denote by $P(B,h,\varepsilon)$ a set of elements x for which the operation $U(x)$ has the following property: there exists a set $e\in\mathfrak{E}$ such that $\mu(h\setminus e)<\varepsilon$ and $U(x)_e\in B$.

It is easily seen that (in analogy to [2]) the following lemma always holds:

LEMMA 2. *For every $h\in\mathfrak{E}$ and $\varepsilon>0$ the set $P(B,h,\varepsilon)$ is analytic.*

LEMMA 3. *Let B be a linear and analytic set in Y. If the set $P(B,h,\varepsilon)$ is of the second category, then the set $P\big(B,h,(m+1)\varepsilon\big)$ is residual.*

Proof. Since the set $P(B,h,\varepsilon)$ is analytic and is of the second category, there exists a sphere $K(0,r)$ with centre 0[3]) and radius r in which this set is residual. Therefore $K(0,r)\setminus M\subset P(B,h,\varepsilon)$, where M denotes a set of the first category. Setting

$$N=\bigcup_{k=0}^{m}kM^{4)},\qquad K=K(0,r/m)\setminus N,$$

we see (as in [1]) that if $x\in K$, then $kx\in K(0,r)\setminus N$ for $k=0,1,\ldots,m$, and therefore $K\subset K(0,r)\setminus M\subset P(B,h,\varepsilon)$. This shows that if $x\in K$, then there exists a set $e_\nu\in\mathfrak{E}$ such that $\mu(h\setminus e_\nu)<\varepsilon$ and $U(\nu x)_{e_\nu}\in B$ for $\nu=0,1,\ldots,m$. Therefore if $x\in K$ and $e=\bigcap_{\nu=0}^{m}e_\nu$, we have

$$\mu(h\setminus e)<(m+1)\varepsilon\quad\text{ and }\quad U_k(x)_e\in B\quad\text{ for }\quad k=0,1,\ldots,m$$

(this is the immediate consequence of (2) and condition (β)).

Since the set B is linear and (1) holds, we see that if $x\in K$, then $nx\in P\big(B,h,(m+1)\varepsilon\big)$ for $n=0,1,2,\ldots$ Hence we finally obtain

$$X-\bigcup_{n=0}^{m}nN=\bigcup_{n=0}^{m}nK\subset P\big(B,h,(m+1)\varepsilon\big).$$

[3]) In the contrary case it is sufficient to consider the polynomial $\overline{U}(x)=U(x+x_0)$.

[4]) kM denotes the set of elements kx where $x\in M$.

3. Using the above Lemmata we can prove by the same method as in the paper of A. Alexiewicz the following

THEOREM 1. *If the set B is linear and analytic, then there exists a decomposition $T=e\cup h$ and a residual set R in X such that*

(a) *for every x and every $\varepsilon>0$ there exists a set e' such that $\mu(e\setminus e')<\varepsilon$ and $U(x)_{e'}\in B$;*

(b) *for every $x\in R$ and every set $h'\subset h$ of positive measure $U(x)_{h'}\bar{\in} B$.*

THEOREM 2. *When the set B is analytic and satisfies the following condition:*

$$\textit{from } y_{e_n}\in B\ (n=1,2,\ldots) \textit{ and } e=\bigcup_{n=1}^{\infty} e_n \textit{ results } y_e\in B,$$

then there exists a decomposition $T=e\cup h$ and a residual set $R\subset X$ such that

(a_1) *$U(x)_e\in B$ for every x,*

(a_2) *$U(x)_{h'}\bar{\in} B$ for every $x\in R$ and every set $h'\subset h$ of positive measure.*

References

[1] J. Albrycht, *On Saks' theorem for abstract polynomials*, Studia Mathematica 14 (1953), p. 79-81.

[2] A. Alexiewicz, *A theorem on the structure of linear operations*, Studia Mathematica 14 (1953), p. 1-12.

[3] C. Kuratowski, *Topologie I*, 2 éd., Warszawa 1948.

[4] S. Mazur und W. Orlicz, *Grundlegende Eigenschaften der polynomischen Operationen* (Erste Mitteilung), Studia Mathematica 5 (1934), p. 50-68.

[5] — *Grundlegende Eigenschaften der polynomischen Operationen* (Zweite Mitteilung), Studia Mathematica 5 (1934), p. 179-189.

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On the estimation of the norm of the n-linear symmetric operation

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Let X and Y be two Banach spaces. An operation $U(x_1,\ldots,x_n)$ from $\underbrace{X\times\ldots\times X}_{n}$ to Y is called *n-linear* if it is linear in each variable x_i separately. It is called *symmetric* if $U(x_1,\ldots,x_n)=U(x_{\pi_1},\ldots,x_{\pi_n})$ for every permutation $\pi_1,\ldots,\pi_n$ of the numbers $1,\ldots,n$. The operation $U(x_1,\ldots,x_n)$ being n-linear and symmetric, we call the operation $U(x)=U(x,x,\ldots,x)$ the *power of degree n*; $U(x_1,\ldots,x_n)$ is then called the *primitive* (or *polar*) operation of $U(x)$. Between the norms of these operations an inequality

$$\sup_{\|x_1\|\leqslant 1,\ldots,\|x_n\|\leqslant 1}\|U(x_1,\ldots,x_n)\|\leqslant B_n\sup_{\|x\|\leqslant 1}\|U(x)\|$$

holds, with B_n depending only on n. A. E. Taylor[1]) has shown that $B_n\leqslant n^n/n!$. We shall show that this estimation is the best possible.

Let $X=L$, $Y=R^1$ (the space of reals), $\Delta_k=\langle(k-1)/n,\ k/n\rangle$ $(k=1,2,\ldots,n)$. Let us consider the operation

$$U(x_1,\ldots,x_n)=\sum_{(\pi_1,\ldots,\pi_n)}\int_{\Delta_1}x_{\pi_1}(t)\,dt\ldots\int_{\Delta_n}x_{\pi_n}(t)\,dt,$$

the summation being extended over all permutations $\pi_1,\ldots,\pi_n$ of the numbers $1,\ldots,n$. This operation is obviously n-linear and symmetric.

We shall prove that

$$(*)\qquad \sup_{\|x_1\|\leqslant 1,\ldots,\|x_n\|\leqslant 1}\|U(x_1,\ldots,x_n)\|=\frac{n^n}{n!}\sup_{\|x\|\leqslant 1}\|U(x)\|.$$

Let

$$\|x_i\|=\int_0^1|x_i(t)|\,dt\leqslant 1$$

1) A. E. Taylor, *Additions to the Theory of Polynomials in Normed Linear Spaces*, The Tôhoku Math. Journal 44 (1938), p. 302-318, theorems 2.5 and 2.6.