The theorem of Hildebrandt [1])

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In his monograph on linear operators S. BANACH obtains a sufficient condition for the uniform boundedness of a sequence of continuous and non-negative functionals [2]). It can be shown that this condition of BANACH is intimately related to the well known theorem of T. H. HILDEBRANDT on systems of functional operators [3]). In order to exploit this relationship to its fullest extent the author generalizes BANACH's condition by a considerable weakening of hypotheses. This new result is easily established and yields the HILDEBRANDT theorem as an immediate corollary. In fact this corollary is obtained under hypotheses which are actually weaker than those used by HILDEBRANDT; moreover the entire proof is shorter than any other proof of this theorem known to the author.

Throughout this note $\mathfrak{L}$ will signify an arbitrary class of elements l and R any binary relation between the elements of $\mathfrak{L}$. Then the generalization of BANACH's result is

Theorem I. *Let $\mathfrak{Y}$ be a set of the second category, which is contained in a metric space $\mathfrak{X}$; further let F be a real-valued function defined on the composite class* [4]) $(\mathfrak{L}, \mathfrak{X})$ *and possessing the following properties: for each l, F is a lower semi-continuous*

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[1]) Presented to the American Mathematical Society, April 10, 1937.

[2]) S. Banach, Théorie des opérations linéaires (Warsaw), 1932, p. 19.

[3]) See T. H. Hildebrandt, On uniform limitedness of sets of functional operations, Bulletin of the Amer. Math. Soc. 29 (1923) p. 309—315, and L. M. Graves, Topics in the functional calculus, Bull. Amer. Math. Soc. 41 (1935) p. 647 f.

[4]) I. e. the cartesian product of $\mathfrak{L}$ and $\mathfrak{X}$.

function on $\mathfrak{X}$*;* F *is semi-uniformly bounded above on* $\mathfrak{Y}$*, i. e., there exists a denumerable sequence* $\{l_n\}$ *of elements of* $\mathfrak{L}$ *such that to each* y *of* $\mathfrak{Y}$ *there correspond integers* m_y *and* n_y *with the property that* $F(l, y)$ *does not exceed* m_y*, whenever* l *is in the* R*-relation to* l_{n_y}*. Then there are integers* m *and* n *and a sphere* K *in* $\mathfrak{X}$ *such that if* l *is in the* R*-relation to* l_n *and* x *is in* K*,* $F(l, x)$ *does not exceed* m*.*

For each pair (i, j) of integers let the set $\mathfrak{X}_{ij}$ be defined as the class of all x in $\mathfrak{X}$ such that $F(l, x) \leqslant i$, whenever l is in the R-relation to l_j. Hence each set $\mathfrak{X}_{ij}$ is closed and the double sum of these sets contains the set $\mathfrak{Y}$. Then, since $\mathfrak{Y}$ is of the second category, at least one of the sets $\mathfrak{X}_{ij}$ — say $\mathfrak{X}_{mn}$ — must contain the desired sphere.

From this result we are able to demonstrate the theorem of Hildebrandt on systems of functional operations:

Theorem II. *Let* F *be a real-valued function defined on the composite of the classes* $\mathfrak{L}$ *and* $\mathfrak{X}$*, a Banach space, with the following properties: the absolute value of* $F(l, x)$ *is not greater than some integer* m_l *times the norm of* x*, for each* l *and* x*; there is a denumerable sequence* $\{l_n\}$ *such that to each* x *correspond integers* m_x *and* n_x *for which* $|F(l, x)| \leqslant m_x$*, whenever* l *is in the* R*-relation to* l_{n_x}*; the absolute value of* F *satisfies the triangle inequality, i. e.,*

$$|F(l, x_1 a_1 + x_2 a_2)| \leqslant |F(l, x_1)| \cdot |a_1| + |F(l, x_2)| \cdot |a_2|.$$

Then there are integers m *and* n *such that the absolute value of* $F(l, x)$ *does not exceed* $m \cdot \|x\|$*, whenever* x *is in* $\mathfrak{X}$ *and* l *is in the* R*-relation to* l_n*.*

The proof follows at once from the fact that every metric complete space is of the second category [5]) and Theorem I above.

[5]) See S. Banach, op. cit., p. 14.

(Reçu par la Rédaction le 2. 8. 1937).