Note on general transforms

by

S. KACZMARZ (Lwów).

1. Let L_2 denote the class of functions such that

$$\int_0^\infty f^2(x)\,dx < +\infty$$

and $\chi(x)$ a function of x such that

$$(1.1) \qquad \int_0^\infty \frac{\chi(xy)\,\chi(zy)}{y^2}\,dy = \min(x,z)$$

for all nonnegative values of x and z. Then the following theorem was recently proved by WATSON[1]):

The condition (1.1) is necessary and sufficient that for any $f(x)$, belonging to L_2, there exists a function $g(x)$, belonging to L_2, such that almost everywhere

$$g(x) = \frac{d}{dx}\int_0^\infty \frac{\chi(xy)}{y} f(y)\,dy$$

and

$$f(x) = \frac{d}{dx}\int_0^\infty \frac{\chi(xy)}{y} g(y)\,dy.$$

Simpler proofs are given by TITCHMARSH[2]) and PLANCHEREL[3]). The function $\chi(x)$ in WATSON's theorem does not represent every

1) G. N. Watson, Proc. London Math. Soc. (2) 35 (1933) p. 156–199.
2) E. C. Titchmarsh, Journ. London Math. Soc. 8 (1933) p. 217–220.
3) M. Plancherel, Journ. London Math. Soc. 8 (1933) p. 220–226.

transform in L_2. In this direction the following generalisation is true:

Theorem. *The necessary and sufficient conditions that there exists for every function $f(x) \in L_2$ a pair of functions $g(x)$ and $h(x)$ such that*

$$(1.2) \qquad \begin{cases} \displaystyle\int_0^x g(u)\,du = \int_0^\infty f(t)\,\varphi(x,t)\,dt \\ \displaystyle\int_0^x f(u)\,du = \int_0^\infty g(t)\,\psi(x,t)\,dt \end{cases}$$

$$(1.3) \qquad \begin{cases} \displaystyle\int_0^x h(u)\,du = \int_0^\infty f(t)\,\psi(x,t)\,dt \\ \displaystyle\int_0^x f(u)\,du = \int_0^\infty h(t)\,\varphi(x,t)\,dt \end{cases}$$

and

$$(1.4) \qquad \int_0^\infty f^2(t)\,dt = \int_0^\infty g^2(t)\,dt = \int_0^\infty h^2(t)\,dt)$$

are as follows:

$$(a) \qquad \int_0^\infty \varphi(x,t)\,\varphi(y,t)\,dt = \min(x,y)$$

$$(b) \qquad \int_0^\infty \psi(x,t)\,\psi(y,t)\,dt = \min(x,y) \qquad x \geqslant 0, y \geqslant 0$$

$$(c) \qquad \int_0^y \varphi(z,t)\,dt = \int_0^z \psi(y,t)\,dt.$$

2. Necessity. It follows from (1.4) that

$$(2.1) \qquad \int_0^\infty f(x)\,F(x)\,dx = \int_0^\infty g(x)\,G(x)\,dx,$$

if $g(x)$ and $G(x)$ denote the φ-transforms of $f(x)$ and $F(x)$.

Take $z > 0$ and

$$f(x) = \begin{cases} 1 & \text{for } x \leqslant z \\ 0 & \text{for } x > z. \end{cases} \tag{2.2}$$

Then

$$g(x) = \frac{d}{dx}\int_0^z \varphi(x,t)\,dt$$

almost everywhere.

Let further be $y > 0$ and

$$f_1(x) = \begin{cases} 1 & \text{for } x \leqslant y \\ 0 & \text{for } x > y. \end{cases}$$

Then, almost everywhere,

$$h(x) = \frac{d}{dx}\int_0^y \psi(x,t)\,dt.$$

Put $F(x) = h(x)$, then the equality

$$G(x) = \frac{d}{dx}\int_0^\infty h(t)\,\varphi(x,t)\,dt = f_1(x)$$

by (1.3) and (2.1) gives

$$\int_0^z h(x)\,dx = \int_0^y g(x)\,dx,$$

that is, according to (1.2) and (1.3),

$$\int_0^y \psi(z,t)\,dt = \int_0^z \varphi(y,t)\,dt,$$

which is the condition c).

Take now $f(x)$ as defined by (2.2). We have

$$\int_0^x g(u)\,du = \int_0^z \varphi(x,t)\,dt = \int_0^x \psi(z,t)\,dt.$$

Hence

$$g(x) = \psi(z,x)$$

almost everywhere and the second formula of (1.2) gives

$$\int_0^\infty \psi(z,t)\,\psi(x,t)\,dt = \int_0^x f(u)\,du = \min(x,z),$$

which is the condition b).

Take now $f_1(x)$. Then

$$\int_0^x G(u)\,du = \int_0^\infty \varphi(x,t)\frac{d}{dt}\int_0^y \psi(t,u)\,du\;dt = \int_0^\infty \varphi(x,t)\,\varphi(y,t)\,dt$$

by the condition c). But

$$\int_0^x G(u)\,du = \int_0^x f_1(u)\,du = \min(x,y)$$

and a) follows.

3. S u f f i c i e n c y. Suppose, first, that $f(x)$ is a step-function. Then, by a familiar argument, it is sufficient to prove that (1.2) and (1.3) follows from a), b), c) for

$$f(x) = \begin{cases} 1 & \alpha \leqslant x \leqslant \beta \\ 0 & \text{elsewhere.} \end{cases}$$

If we put $x = y$, the condition a) gives $\int_0^\infty \varphi^2(x,t)\,dt = x$ *i.e.* $\varphi(x,t)$ belongs to L_2. Consider now

$$\int_0^\infty f(t)\,\varphi(x,t)\,dt = \int_\alpha^\beta \varphi(x,t)\,dt.$$

According to c) we have

$$\int_0^\infty f(t)\,\varphi(x,t)\,dt = \int_0^x [\psi(\beta,t) - \psi(\alpha,t)]\,dt,$$

hence $g(x)$, as defined by (1.2), exists and

$$g(x) = \psi(\beta,x) - \psi(\alpha,x).$$

Further

$$\int_0^\infty g^2(x)\,dx = \int_0^\infty \psi^2(\beta,x)\,dx - 2\int_0^\infty \psi(\beta,x)\,\psi(\alpha,x)\,dx + \int_0^\infty \psi^2(\alpha,x)\,dx$$

$$= \beta - 2\alpha + \alpha = \int_0^\infty f^2(x)\,dx,$$

which is (1.4). Again

$$\int_0^\infty g(t)\,\psi(x,t)\,dt = \int_0^\infty [\psi(\beta,t)-\psi(\alpha,t)]\,\psi(x,t)\,dt$$

$$= \min(\beta,x)-\min(\alpha,x) = \int_0^x f(t)\,dt$$

and (1.2) is proved. Similarly we obtain (1.3).

Next let $f(x)$ be any function of L_2. Then there is a sequence of step-functions $f_n(x)$ which converges in mean to $f(x)$ over $(0,\infty)$. Let $g_n(x)$ be the φ-transform of $f_n(x)$; then

$$\int_0^\infty |g_m(x)-g_n(x)|^2\,dx = \int_0^\infty |f_m(x)-f_n(x)|^2\,dx \to 0$$

and the sequence $g_n(x)$ converges in mean to a function $g(x)$, belonging to L_2. Further

$$\int_0^\infty g^2(x)\,dx = \lim_{n\to\infty}\int_0^\infty g_n^2(x)\,dx = \lim_{n\to\infty}\int_0^\infty f_n^2(x)\,dx = \int_0^\infty f^2(x)\,dx$$

and (1.4) is true.

We have

$$\int_0^x g(t)\,dt = \lim_{n\to\infty}\int_0^x g_n(t)\,dt = \lim_{n\to\infty}\int_0^\infty f_n(t)\,\varphi(x,t)\,dt$$

$$= \int_0^\infty f(t)\,\varphi(x,t)\,dt$$

and

$$\int_0^x f(t)\,dt = \lim_{n\to\infty}\int_0^x f_n(t)\,dt = \lim\int_0^\infty g_n(t)\,\psi(x,t)\,dt$$

$$= \int_0^\infty g(t)\,\psi(x,t)\,dt,$$

whence (1.2) follows. The same argument proves (1.3).

Remarks. 1. It follows from (1.4) that for

$$f_n(x) = \begin{cases} f(x) & x \leqslant n \\ 0 & x > n \end{cases}$$

we have

$$\int_0^\infty |f(x)-f_n(x)|^2\,dx = \int_0^\infty |g(x)-g_n(x)|^2\,dx,$$

that is

$$g(x) = \underset{n\to\infty}{\mathrm{l.\,i.\,m.}}\int_0^n f(t)\,\varphi(x,t)\,dt,$$

and similarly

$$f(x) = \mathrm{l.\,i.\,m.}\int_0^n g(t)\,\psi(x,t)\,dt.$$

2. If $\varphi(x,t)=\psi(x,t)=\dfrac{\chi(xt)}{t}$, then the condition c) is always satisfied. But for

$$\varphi(x,t) = \begin{cases} 2 & t \leqslant x \\ 0 & t > x \end{cases} \quad \text{and} \quad \psi(x,t) = \begin{cases} \frac{1}{2} & t \leqslant x \\ 0 & t > x \end{cases}$$

(1.2) and (1.3) are true but not (1.4) and therefore the conditions a), b), c) are false.

(Reçu par la Rédaction le 6. 12. 1933).