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Heat kernels for class 2 nilpotent groups

by

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Abstract. The formulas for the heat kernels for simply connected nilpotent Lie groups of class 2 are obtained by the use of the non-Euclidean Fourier transform and the Plancherel formula.

Introduction. Let G be a Lie group with its Lie algebra generated by the left-invariant vector fields $X_1, \ldots, X_n$ on G. We consider the *heat equation*

$$\frac{\partial}{\partial s} h = (X_1^2 + \ldots + X_n^2) h \quad \text{on } \boldsymbol{R}^+ \times G. \tag{1}$$

In the case of G being the Heisenberg group, the fundamental solution of (1) has been explicitly calculated by Gaveau [4] by use of the stochastic integral and by Hulanicki [8] by calculations with the representations of G. For the $N_{n,2}$ group, i.e. the simply connected nilpotent Lie group whose Lie algebra is the free nilpotent Lie algebra of class 2 with n generators, such a formula was given by Gaveau [6], [13], see Added in proof.

The aim of this note is to describe the procedure giving the formulas for the heat kernels for all simply connected nilpotent Lie groups of class 2. This is done as follows. Using the non-Euclidean Fourier transform on the $N_{n,2}$ group, we transform equation (1) into the operator equations on the spaces of representations of the group. Then instead of $X_1^2 + \ldots + X_n^2$ there appear harmonic oscillators. We solve the operator equations and retransform the solutions via the Plancherel formula. The calculations of the trace for the Plancherel formula are based on the formula of Mehler. The general case is deduced from the resulting formulas for the $N_{n,2}$ groups by the method of descent. The formulas thus obtained are given by (5.5).

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1. Detailed procedure. Let us recall (cf. e.g. [1], [9]) that for a locally compact unimodular group G and its irreducible unitary representation T acting on the Hilbert space H the *non-Euclidean Fourier transform* of the function $\varphi \in L^1(G)$ at the "point" T is an operator $(\mathscr{F}\varphi)(T)$ on H

given by the weak integral

(F) $$(\mathcal{F}\varphi)(T) = \int_G \varphi(g)T(g)\,dg,$$

and for $\varphi \in A(G)$ (the Fourier algebra of G) for a wide class of groups, in particular for nilpotent Lie groups, there is a *reciprocal formula* (or the *Plancherel formula*)

(R) $$\varphi(g) = \int_{\hat{G}} \operatorname{tr}[T(g^{-1})(\mathcal{F}\varphi)(T)]\,d\mu([T])$$

where $\hat{G}$ is the set of the equivalence classes of the irreducible unitary representations of G, $[T]$ denotes the equivalence class of the representation T, and μ is called the *Plancherel measure* for G.

If G is a Lie group, we denote by dT the corresponding representation of its Lie algebra $\mathfrak{g}$ on the Gårding space $V^\infty \subset H$ of C^∞-vectors for the representation T. Representation dT prolongs to the representation of the enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$. We observe that the operator $X_1^2 + \ldots + X_n^2$ is an element of $\mathfrak{U}(\mathfrak{g})$.

Following Folland [3], we call a Lie group G *stratified* if it is nilpotent and simply connected and its Lie algebra $\mathfrak{g}$ admits a vector space decomposition $\mathfrak{g} = V_1 \oplus \ldots \oplus V_m$ such that $[V_1, V_k] = V_{k+1}$ for $1 \leqslant k < m$ and $[V_1, V_m] = \{0\}$, and we call the operator $\mathcal{L} = X_1^2 + \ldots + X_n^2$ a *sub-Laplacian* for G if $X_1, \ldots, X_n$ is a basis for V_1.

From now on we shall be interested in the heat equation (1) for the stratified group G and its sub-Laplacian $\mathcal{L}$. We proceed as follows. Let $p(s, g) = p_s(g)$ be the $C^\infty\big((0, \infty) \times G\big)$ solution of (1) with the property

(2) $$\varphi * p_s \to \varphi \quad \text{in } L^1(G) \text{ as } s \to 0^+$$

for every $\varphi \in L^1(G)$; cf. [3] (3.1), (3.3); [7].

Putting the non-Euclidean Fourier transform with respect to the *space variable* $g \in G$, we transform (1) into the operator equation

(1_T) $$\langle [(\partial/\partial s)P_s - P_s dT(\mathcal{L})]\xi, \eta\rangle = 0,$$

$P_s = (\mathcal{F}p_s)(T)$ being the Fourier transform of p_s; $\xi, \eta \in V^\infty$, the interchanging of the orders of integration and differentiation in this procedure being justified by Corollary (3.6) of [3]. Property (2) gives rise to the boundary condition

(2_T) $$\langle \mathcal{F}\varphi P_s \xi, \eta\rangle \to \langle \mathcal{F}\varphi\xi, \eta\rangle \quad \text{as} \quad s \to 0^+$$

or every fixed $\xi, \eta \in H$ and $\varphi \in L^1(G)$; here $\mathcal{F}\varphi = (\mathcal{F}\varphi)(T)$. Moreover, $P: \boldsymbol{R}^+ \to \mathfrak{B}(H)$ is a C^∞-function.

Now, for every fixed T we find a $C^\infty\big(\boldsymbol{R}^+, \mathfrak{B}(H)\big)$ solution P of (1_T) and (2_T) (see Section 3). Since the solution is unique, it coincides with our transform $(\mathcal{F}p_s)(T)$ of p_s. In Section 4 we reconstruct from $\mathcal{F}p_s$ the

solution p of (1) and (2) by the reciprocal formula (R), since the fundamental solution p of (1) is in $A(G)$ for every $s > 0$. In fact, $p_s = p_{s/2} * p_{s/2}$ and $p_{s/2} \in L^2(G)$ as a rapidly decreasing function, cf. [3] (3.1), (3.5); [7]. Note that we need to work only with the representations T supporting the Plancherel measure for G. All the above-mentioned calculations in Sections 3 and 4 are done for the case of $N_{n,2}$ groups, since this is sufficient for passing to the case of an arbitrary simple connected nilpotent Lie group of class 2, which is described in Section 5. The necessary formulas for the $N_{n,2}$ groups have been collected in Section 2.

2. Harmonic analysis on $N_{n,2}$. Following Gaveau [6], we recall some facts about the $N_{n,2}$ groups (all the formulas were obtained by Kirillov's method). Let $\mathcal{N}_{n,2}$ be the *free nilpotent Lie algebra* of order 2 with n generators $X_1, \ldots, X_n$ and put $X_{kl} = [X_k, X_l]$, $k < l$. $N_{n,2}$ is the corresponding Lie group with the exponential map

$$\exp\Big(\sum_{j=1}^{n} u_j X_j + \sum_{1 \leqslant k < l \leqslant n} u_{kl} X_{kl}\Big) = (u_j, u_{kl})$$

(as a rule we will write (u_j, u_{kl}) instead of $u = \big(u_1, \ldots, u_n, (u_{kl})_{k<l}\big)$ for the points of $N_{n,2}$) and the group law

$$u \cdot v = \big(u_j + v_j,\ u_{kl} + v_{kl} + \tfrac{1}{2}(u_k v_l - v_k u_l)\big)$$

with $u = (u_j, u_{kl})$, $v = (v_j, v_{kl})$. The bi-invariant measure on $N_{n,2}$ is the Lebesgue measure on $\boldsymbol{R}^{n+\binom{n}{2}} \approx N_{n,2}$. We normalize it in such a way that the volume of the unit cube is 1. We denote by $\Delta_{n,2}$ the sub-Laplacian $X_1^2 + \ldots + X_n^2$ on $N_{n,2}$.

Let $\mathcal{N}_{n,2}^*$ be the dual vector space to $\mathcal{N}_{n,2}$ with the coordinates (a_j, a_{kl}) in the basis dual to $X_1, \ldots, X_n, X_{kl}$, $k < l$. The coadjoint representation of $N_{n,2}$ is then given by

$$\operatorname{Ad}^*(u)(a_j, a_{kl}) = \big(a_j + 2^{-1}(AU)_j,\ a_{kl}\big)$$

where $U = (u_1, \ldots, u_n)$, $(AU)_j$ denotes the jth component of the vector AU and

(2.0) $A = (a_{kl})$ is the skew-symmetric $n \times n$ matrix with $a_{kl} = a_{kl}$ for $k < l$, $a_{kl} = -a_{lk}$ for $l < k$, $a_{kl} = 0$ for $k = l$.

There is an orthogonal matrix $\Omega = (\omega_{kl})$ such that ${}^t\Omega A \Omega$ is a skew-symmetric matrix formed from $[n/2]$ diagonal blocks

$$\begin{bmatrix} 0 & P_{2h-1} \\ -P_{2h-1} & 0 \end{bmatrix}, \quad 1 \leqslant h \leqslant [n/2]$$

and supplemented by the row and column composed of zeros if n is an odd number.

Now we shall still use the above notations but treat separately the two cases depending on the parity of n. We put ν for $[n/2]$.

(a) $n = 2\nu$. The set (of the equivalence classes) of the representations supporting the Plancherel measure for $N_{n,2}$ may be identified with the set of orbits of maximal dimension in $\mathcal{N}^*_{n,2}$ under the action of Ad^*, and this set in turn may be parametrized by the functionals $f \in \mathcal{N}^*_{n,2}$ of the form

$$(2.1a)\qquad f = (0, a_{kl}) \quad \text{with} \quad \det A \neq 0, \ (\text{cf. } (2.0))$$

so this is $\boldsymbol{R}^d$ with $d = \binom{n}{2}$ up to a set of Lebesgue measure null. The Plancherel measure is the ordinary Lebesgue measure on $\boldsymbol{R}^d$ with the density function equal to (cf. [9] p. 281)

$$(2.2a)\qquad [(2\pi)^d (2\pi)^{n/2}]^{-1} |P_1|\,|P_3| \ldots |P_{n-1}|$$

in the coordinates a_{kl}, $k < l$.

The representation corresponding to the orbit of f as in (2.1a) acts on $L^2(\boldsymbol{R}^\nu)$ and is given by

$$(2.3a)\qquad \big(T_f(u)\varphi\big)\big((\xi_h)_{h=1}^{\nu}\big)$$
$$= \exp\Big(i \sum_{k<l} a_{kl} u_{kl}\Big) \exp\Big(-i \sum_{h=1} P_{2h-1}(\xi_h - \tfrac{1}{2} z_{2h}) z_{2h-1}\Big)\, \varphi\big((\xi_h - z_{2h})_{h=1}^{\nu}\big),$$
$$\varphi \in L^2(\boldsymbol{R}^\nu),\ (\xi_h)_{h=1}^{\nu} \in \boldsymbol{R}^\nu \ \text{and} \ z_{2h} = 2^{-1/2}[({}^t\Omega U)_{2h-1} - ({}^t\Omega U)_{2h}],$$
$$z_{2h-1} = 2^{-1/2}[({}^t\Omega U)_{2h-1} + ({}^t\Omega U)_{2h}],$$

$({}^t\Omega U)_j$ denoting the jth component of the vector ${}^t\Omega U \in \boldsymbol{R}^n$. Consequently

$$(2.4a)\qquad dT_f(X_j) = -2^{-1/2} \sum_{h=1}^{\nu} \Big[(\omega_{j2h-1} - \omega_{j2h}) \frac{\partial}{\partial \xi_h} + i\,\xi_h P_{2h-1}(\omega_{j2h-1} + \omega_{j2h})\Big],$$
$$j = 1, \ldots, n$$

and

$$(2.5a)\qquad dT_f(\Delta_{n,2}) = \sum_{h=1}^{\nu} \Big(\frac{\partial^2}{\partial \xi_h^2} - P_{2h-1}^2 \xi_h^2\Big).$$

Remark. The representation T_f in (2.3a) is induced from the character χ_f of the subgroup H of $N_{n,2}$ (cf. e.g. [1], p. 155) where

$$H = \{(U, u_{kl})_{k<l}\colon\ U = 2^{-1/2}\Omega Z,\ Z = (z_1, z_1, z_3, z_3, \ldots, z_{n-1}, z_{n-1}),$$
$$z_{2h-1} \in \boldsymbol{R},\ h = 1, \ldots, \nu,\ u_{kl} \in \boldsymbol{R},\ k < l\},$$

i.e. its Lie algebra $\mathfrak{h}$ is spanned by ν vectors, each taken from a different

2-dimensional eigenspace of A corresponding to the hth diagonal block in ${}^t\Omega A\,\Omega$, and by X_{kl}'s.

$$\chi_f(U, u_{kl}) = \exp\Big(i \sum_{1 \leqslant k < l \leqslant n} a_{kl} u_{kl}\Big)$$

for $(U, u_{kl}) = (u_j, u_{kl})$ in H. $N_{n,2}/H$ is identified with the orthogonal complement $\mathfrak{h}^\perp$ of $\mathfrak{h}$ in $\mathcal{N}_{n,2}$, i.e. with

$$\{(U, 0)\colon\ U = 2^{-1/2}\Omega Z',\ Z' = (z_2, -z_2, z_4, -z_4, \ldots, z_n, -z_n),$$
$$z_{2h} \in \boldsymbol{R},\ h = 1, \ldots, \nu\}$$

(b) $n = 2\nu + 1$. Orbits of maximal dimension are parameterized by the functionals of the form

$$(2.1b)\qquad f = (a_j, a_{kl}) \quad \text{with} \quad a_j = a\omega_{jn},\ a\text{—real and } \operatorname{rank} A = n-1;$$

so the Plancherel measure lives on $\boldsymbol{R}^{d+1}$ with $d = \binom{n}{2}$, and its density function with respect to the Lebesgue measure on $\boldsymbol{R}^{d+1}$ in the coordinates a, a_{kl}, $k < l$, is equal to

$$(2.2b)\qquad [(2\pi)^d (2\pi)^{(n+1)/2}]^{-1} |P_1|\,|P_3| \ldots |P_{n-2}|.$$

According to the notations of (2.3a) the representation T_f acting on $L^2(\boldsymbol{R}^\nu)$ is given by

$$(2.3b)\qquad \big(T_f(u)\varphi\big)\big((\xi_h)_{h=1}^{\nu}\big) =$$
$$\exp\Big[i\Big(a({}^t\Omega U)_n + \sum_{k<l} a_{kl} u_{kl}\Big)\Big] \exp\Big[-i \sum_{h=1}^{\nu} P_{2h-1}(\xi_h - \tfrac{1}{2} z_{2h}) z_{2h-1}\Big]\, \varphi\big((\xi_h - z_{2h})_{h=1}^{\nu}\big)$$

and we have

$$(2.4b)\qquad dT_f(X_j)$$
$$= -2^{-1/2} \sum_{h=1}^{\nu} \Big[(\omega_{j2h-1} - \omega_{j2h}) \frac{\partial}{\partial \xi_h} + i\,\xi_h P_{2h-1}(\omega_{j2h-1} + \omega_{j2h})\Big] + i a \omega_{jn},$$

$$(2.5b)\qquad dT_f(\Delta_{n,2}) = \sum_{1}^{\nu} \Big(\frac{\partial^2}{\partial \xi_h^2} - P_{2h-1}^2 \xi_h^2\Big) - a^2.$$

Remark. As before, $T_f = \mathrm{ind}_H \chi_f$ with

$$H = \{(U, u_{kl})\colon\ U = 2^{-1/2}\Omega Z,$$
$$Z = (z_1, z_1, z_3, z_3, \ldots, z_{n-2}, z_{n-2}, z_0),\ z_{2h-1}, z_0 \in \boldsymbol{R},\ h = 1, \ldots, \nu\},$$
$$\chi_f(U, u_{kl}) = \exp\Big[i\Big(a({}^t\Omega U)_n + \sum_{k<l} a_{kl} u_{kl}\Big)\Big] \quad \text{for} \quad (U, u_{kl}) \in H,$$

$$N_{n,2}/H \cong \{(U, 0)\colon\ U = 2^{-1/2}\Omega Z',$$
$$Z' = (z_2, -z_2, \ldots, z_{n-1}, -z_{n-1}, 0),\ z_{2h} \in \boldsymbol{R},\ h = 1, \ldots, \nu\}.$$

3. Solution of the operator equation. To solve (1_T) and (2_T) we note that the functions (cf. e.g. [11], pp. 77–79)

$$\psi_k^\lambda(x) = \left[\sqrt{\frac{\pi}{\lambda}}\, 2^k k!\right]^{-1/2} \exp\left(-\frac{\lambda x^2}{2}\right) H_k(\sqrt{\lambda}x)$$

where H_k is the kth Hermite polynomial, $\lambda > 0$, $k = 0, 1, \ldots, x \in \boldsymbol{R}$, form the total orthonormal system in $L^2(\boldsymbol{R})$ and are the eigenfunctions of the harmonic oscillator $\frac{d^2}{dx^2} - \lambda^2 x^2$ with eigenvalues $\lambda(2k+1)$.

Now we fix f, i.e. the representation $T \doteq T_f$, and perform all the calculations for this T. The functions

$$\psi_m(\xi) = \psi_{m_1}^{\lambda_1}(\xi_1) \ldots \psi_{m_\nu}^{\lambda_\nu}(\xi_\nu) \tag{3.1}$$

with $\xi = (\xi_1, \ldots, \xi_\nu) \in \boldsymbol{R}^\nu$, $\lambda_h = |P_{2h-1}|$, $1 \leqslant h \leqslant \nu$, $m = (m_1, \ldots, m_\nu) \in \boldsymbol{N}_\nu$ form the total orthonormal system in $L^2(\boldsymbol{R}^\nu)$ and are the eigenfunctions of $dT_f(\Delta_{n,2})$ with eigenvalues

$$\theta_m = -\sum_{h=1}^{\nu} |P_{2h-1}|(2m_h+1) \quad \text{for } n = 2\nu, \tag{3.2a}$$

$$\theta_m = -\sum_{h=1}^{\nu} |P_{2h-1}|(2m_h+1) - \alpha^2 \quad \text{for } n = 2\nu+1, \tag{3.2b}$$

cf. (2.5a) and (2.5b). Moreover, $\psi_m \in V^\infty$. Therefore from (1_T) we obtain

$$\langle P_s \psi_m, \psi_r\rangle = c_{m,r} \exp(s\,\theta_m) \tag{3.3}$$

with θ_m and ψ_m, ψ_r as in (3.2) and (3.1), $m, r \in \boldsymbol{N}^\nu$.

Since the operator P_s is supposed to be bounded, to determine it completely it suffices to find the constants $c_{m,r}$, which is done by the use of (2_T) as follows:

$$\exp(s\,\theta_m)\langle \mathscr{F}\varphi C\psi_m, \psi_r\rangle \to \langle \mathscr{F}\varphi\psi_m, \psi_r\rangle$$

as $s \to 0^+$, where $C\psi_m = \sum_p c_{m,p}\psi_p$, the series being convergent in $L^2(\boldsymbol{R}^\nu)$. So $\langle \mathscr{F}\varphi C\psi_m, \psi_r\rangle = \langle \mathscr{F}\varphi\psi_m, \psi_r\rangle$, $\varphi \in L^1(N_{n,2})$, hence $\langle C\psi_m, \psi_r\rangle = \langle \psi_m, \psi_r\rangle$ and consequently

(3.4) $c_{m,r}$ is equal to 1 if $m = r$ and to 0 otherwise.

The operator P_s is plainly characterized by (3.3) and (3.4).

4. Retransformation of the solution. In order to apply the reciprocal formula (R) one has to compute the trace of the operator $T_f(-u)P_s$.

The following form of *Mehler's formula* (the heat kernel for the harmonic oscillator)

$$\sum_{k=0}^{\infty} e^{-s|\lambda|(2k+1)} \psi_k^\lambda(x-A)\psi_k^\lambda(x) \tag{4.1}$$
$$= [\lambda(2\pi\,\mathrm{sh}(2s\lambda))^{-1}]^{1/2} \exp\left[-\frac{A^2}{4}\lambda\,\mathrm{cth}(s\lambda) - \left(x - \frac{1}{2}A\right)^2 \lambda\,\mathrm{th}(s\lambda)\right]$$

(cf. e.g. [11], p. 79, (3.7.3) with $\psi_k^\lambda(x) = |\lambda|^{1/4}\psi_k(|\lambda|^{1/2}x)$) combined with the integral

$$\int_{-\infty}^{\infty} e^{-i\lambda\Theta x} \exp\left[-\left(x - \frac{1}{2}A\right)^2 \lambda\,\mathrm{th}(s\lambda)\right] dx \tag{4.2}$$
$$= (\pi^{-1}\lambda\,\mathrm{th}(s\lambda))^{-1/2} \exp\left(-i\frac{\lambda}{2}\Theta A - \frac{\Theta^2}{4}\lambda\,\mathrm{cth}(s\lambda)\right)$$

gives the main formula for our calculations of the trace:

$$\sum_{k=0}^{\infty} e^{-s|\lambda|(2k+1)} \int_{-\infty}^{\infty} \exp\left[i\lambda\left(\frac{1}{2}\Theta A - x\Theta\right)\right] \psi_k^\lambda(x-A)\psi_k^\lambda(x)\,dx \tag{4.3}$$
$$= (2\,\mathrm{sh}(s|\lambda|))^{-1} \exp\left(-\frac{A^2+\Theta^2}{4s}\,\frac{s\lambda}{\mathrm{th}(s\lambda)}\right).$$

We observe that $(T_f(-u)\psi_m)((\xi_h)_{h=1}^{\nu})$ is equal to some function $\chi = \chi(u$ multiplied by the product of ν functions of the form

$$\exp[i\lambda(\tfrac{1}{2}\Theta A - x\Theta)]\psi_k^\lambda(x-A) \tag{4.4}$$

with $\lambda = |P_{2h-1}|$, $\Theta = -z_{2h-1}$, $A = -z_{2h}$, $x = \xi_h$, $h = 1, \ldots, \nu$.

Thus

$$\mathrm{tr}(T_f(-u)P_s) = \sum_{m \in \boldsymbol{N}^\nu} \langle T_f(-u)P_s\psi_m, \psi_m\rangle \tag{4.5}$$

(we recall that P_s is given by (3.3) and (3.4)) and, distinguishing again two cases, we get the following formulas for the trace (4.5):

(a) For $n = 2\nu$, according to (2.3a) and (3.2a),

$$\exp\Big(-i\sum_{k<l} a_{kl}u_{kl}\Big) \times \tag{4.6a}$$
$$\times \prod_{h=1}^{\nu}\Big\{\sum_{m=0}^{\infty} e^{-s|P_{2h-1}|(2m_h+1)} \int_{-\infty}^{+\infty} \exp\left[i|P_{2h-1}|\left(\frac{1}{2}z_{2h-1}z_{2h} + \xi_h z_{2h-1}\right)\right] \times$$
$$\times \psi_{m_h}^{|P_{2h-1}|}(\xi_h + z_{2h})\psi_{m_h}^{|P_{2h-1}|}(\xi_h)\,d\xi_h\Big\}$$

and in virtue of (4.3) and (4.4) this is equal to

(4.7a)

$$\exp\Big(-i\sum_{k<l}a_{kl}u_{kl}\Big)\prod_{h=1}^{\nu}\Big[\big(2\,\mathrm{sh}(s\,|P_{2h-1}|)\big)^{-1}\exp\Big(-\frac{z_{2h-1}^2+z_{2h}^2}{4s}\,\frac{sP_{2h-1}}{\mathrm{th}(sP_{2h-1})}\Big)\Big].$$

(b) For $n=2\nu+1$, according to (2.3b) and (3.2b), similar calculations give

(4.7b) $$\mathrm{tr}\big(T_f(-u)P_s\big)=\exp\Big[-i\Big(a({}^t\Omega\,U)_n+\sum_{k<l}a_{kl}u_{kl}\Big)\Big]\,e^{-a^2s}\prod_{h=1}^{\nu}\ldots$$

where the dots mean that the remaining part of the formula is the same as in (4.7a).

(4.8) We denote by E_h $(h=1,\ldots,\nu)$ the orthogonal projection on the eigenspace of A in $\boldsymbol{R}^n$ corresponding to the hth diagonal block in ${}^t\Omega A\,\Omega$, see (2.0), and by E_0 the projection corresponding to the 1-dimensional zero block in ${}^t\Omega A\,\Omega$ if n is an odd number.

We note that $z_{2h-1}^2+z_{2h}^2=({}^t\Omega\,U)_{2h-1}^2+({}^t\Omega\,U)_{2h}^2=|E_hU|^2$, see the notations in (2.3a). $|\cdot|$ is the Euclidean norm on $\boldsymbol{R}^n$.

Using the formula (R) with μ as in (2.2a) and (2.2b), and integrating over a in case (b), we obtain

(4.9) Proposition (cf. Gaveau [6], and [13], p. 125). *The heat kernel for the $N_{n,2}$ group is given by*

$$p_s(u_j,u_{kl})=c_n\int_{\boldsymbol{R}^{\binom{n}{2}}}\exp\Big(-i\sum_{k<l}a_{kl}u_{kl}\Big)K_n(s,U,f)\prod_{k<l}da_{kl}$$

with

$$K_n(s,U,f)=\prod_{h=1}^{\nu}\Big[\Big(\frac{sP_{2h-1}}{\mathrm{sh}(sP_{2h-1})}\Big)\exp\Big(-\frac{|E_hU|^2}{4s}\,\frac{sP_{2h-1}}{\mathrm{th}(sP_{2h-1})}\Big)\Big]\qquad \textit{for}\quad n=2\nu,$$

and

$$K_n(s,U,f)$$
$$=\exp\Big(-\frac{|E_0U|^2}{4s}\Big)\prod_{h=1}^{\nu}\Big[\Big(\frac{sP_{2h-1}}{\mathrm{sh}(sP_{2h-1})}\Big)\exp\Big(-\frac{|E_hU|^2}{4s}\,\frac{sP_{2h-1}}{\mathrm{th}(sP_{2h-1})}\Big)\Big]\qquad \textit{for}\quad n=2\nu+1,$$

$$c_n^{-1}=(2\pi)^{\binom{n}{2}}(4\pi s)^{n/2},\quad U=(u_1,\ldots,u_n),\ f=(0,a_{kl})_{k<l}$$

and $E_h=E_h(f)$, $P_{2h-1}=P_{2h-1}(f)$ *are as in* (4.8) *and* (2.0).

(4.10) Remark. The function $K_n(s,U,\cdot)$ is well defined almost everywhere on $\boldsymbol{R}^d$, $d=\binom{n}{2}$, cf. (2.1a), (2.1b). However, since eigenvalues iP_{2h-1} and eigenprojections E_h depend continuously on the skew-symmetric operator A (this is due to Rellich [10], Satz 5, modulo reduction of the skew-symmetric case to Hermitian one), the function $K_n(s,U,\cdot)$ may be extended to the continuous function on $\boldsymbol{R}^d$. We shall denote this extension by $\mathscr{K}_n(s,U,\cdot)$.

In fact, it is not difficult to see that

$$\mathscr{K}_n(s,U,f)=\det{}^{1/2}\Big(\frac{sA}{\sin(sA)}\Big)\exp\Big(-\frac{1}{4s}\Big\langle\frac{sA}{\mathrm{tg}(sA)}U,U\Big\rangle\Big)$$

where $U=(u_1,\ldots,u_n)$, A is as in (2.0), $\langle\cdot,\cdot\rangle$ denotes the ordinary scalar product in $\boldsymbol{R}^n$ and the (matrix) functions $\dfrac{X}{\sin X}$ and $\dfrac{X}{\mathrm{tg}\,X}$ are given by the usual power series. Hence we may rewrite (4.9) in the following form:

(4.9′) $$p_s(u_j,u_{kl})=(4\pi s)^{-r/2}\int_{\boldsymbol{R}^{\binom{n}{2}}}\exp\Big(-i\sum_{k<l}a_{kl}\frac{u_{kl}}{2s}\Big)\times$$
$$\times\det{}^{1/2}\Big(\frac{\frac12A}{\sin\frac12A}\Big)\exp\Big(-\frac{1}{4s}\Big\langle\frac{\frac12A}{\mathrm{tg}\frac12A}U,U\Big\rangle\Big)\prod_{k<l}da_{kl}$$

with $r=n+2\binom{n}{2}$.

5. General case. Let $\mathfrak{g}$ be the nilpotent Lie algebra of class 2 with vector space decomposition $\mathfrak{g}=V_1+V_2$ such that $[V_1,V_1]=V_2$ and $[V_1,V_2]=\{0\}$. Let $X_1,\ldots,X_n,X_{n+1},\ldots,X_m$ be a basis in $\mathfrak{g}$ such that $X_1,\ldots,X_n$ span V_1 and $X_{n+1},\ldots,X_m$ span V_2, the product being given by $[X_k,X_l]=\sum_{r=n+1}^{m}c_{kl}^rX_r$.

Let G be the associated Lie group with the exponential map

$$\exp\Big(\sum_{i=1}^{m}u_iX_i\Big)=(u_j,u_r)$$

(we shall write (u_j,u_r) instead of $g=(u_1,\ldots,u_n,u_{n+1},\ldots,u_m)$ for the points of G) and the group law

$$(u_j,u_r)\cdot(v_j,v_r)=\Big(u_j+v_j,\ u_r+v_r+\tfrac12\sum_{k<l}c_{kl}^r(u_kv_l-v_ku_l)\Big).$$

We denote by $\mathscr{L}$ the sub-Laplacian $X_1^2+\ldots+X_n^2$ on G. The ordinary Lebesgue measure on $\boldsymbol{R}^m\approx G$ is a bi-invariant measure. We normalize it as in the $N_{n,2}$ case (cf. Section 2). The mapping $Q^{\sim}$ from $N_{n,2}$ onto G given by

(5.1) $$Q^{\sim}(u_j, u_{kl}) = \Big(u_j, \sum_{k<l} c_{kl}^r u_{kl}\Big)$$

is a Lie group homomorphism as well as a linear map of $\mathscr{N}_{n,2}\approx\boldsymbol{R}^{n+d}$, $d=\binom{n}{2}$ onto $\mathfrak{g}\approx\boldsymbol{R}^m$. We denote by Q the canonical extension of this map to the linear isomorphism of the spaces $\boldsymbol{R}^{n+d}\approx N_{n,2}$ and $\boldsymbol{R}^m\times\boldsymbol{R}^k\approx G\times\boldsymbol{R}^k$, $k=n+d-m$.

We note that Q may differ from the identity mapping only on the orthogonal complement of

$$\{(u_j, v)\colon\ (0, v)\in\ker Q^{\sim},\ u_j\in\boldsymbol{R},\ j=1,\ldots,n\}$$

in $\boldsymbol{R}^{n+d}$.

Now, using Q, we read off the equation

$$(\partial/\partial s-\Delta_{n,2})p=\delta_0 \quad\text{on}\quad \boldsymbol{R}\times N_{n,2}$$

in the coordinates of $\boldsymbol{R}\times G\times\boldsymbol{R}^k$, where we have extended p to $\boldsymbol{R}\times N_{n,2}$ by setting $p(s,u)=0$ for $s\leqslant 0$, cf. [3]. Proposition (3.3). Namely we obtain the equation

$$[(\partial/\partial s-\mathscr{L})+\mathscr{L}_1]q=\delta_0 \quad\text{on}\quad \boldsymbol{R}\times G\times\boldsymbol{R}^k$$

with

(5.2) $$q(s,g,\xi)=(p\circ Q^{-1})(s,g,\xi)\,|\det Q|^{-1}$$

where $(s,g,\xi)\in\boldsymbol{R}\times G\times\boldsymbol{R}^k$ and $\mathscr{L}_1$ is the differential operator containing the derivatives in the directions transversal to $\boldsymbol{R}\times G$. By the method of descent (cf. e.g. [12], p. 195) we conclude that the function h given by

(5.3) $$h(s,g)=\int_{\boldsymbol{R}^k} q(s,g,\xi)\,d\xi$$

is the fundamental solution for $\partial/\partial s-\mathscr{L}$.

Using (5.1), (5.2), (4.9) and Remark (4.10), we can state (5.3) as follows,

(5.4) PROPOSITION. *The heat kernel h for the simply connected nilpoten Lie group G of class 2, parameterized as at the beginning of this section is given by*

$$h(s,g)=c_G(s)\int_{(\ker Q^{\sim})^{\perp}} e^{-i\langle f,u\rangle}\mathscr{K}_n(s,U,f)\,df$$

where $u=Q^{-1}(g)$, $U=(u_1,\ldots,u_n)$, $(\ker Q\ )^{\perp}$ denotes the functionals $f=(0,\alpha_{kl})\in\mathscr{N}_{n,2}^*$ which annihilate $\ker Q^{\sim}$, df stands for the normalized Lebesgue measure on $(\ker Q^{\sim})^{\perp}$—the linear subspace of $\boldsymbol{R}^{\binom{n}{2}}$, or for $\delta_0(f)$ if $(\ker Q^{\sim})^{\perp}=\{0\}$, $c_G(s)^{-1}=(4\pi s)^{n/2}(2\pi)^{\dim G-n}\,|\det Q|$, $\mathscr{K}_n(s,U,\cdot)$ is as in (4.10).

(5.5) COROLLARY. *According to the notations above and those of* (4.10) *we have*

$$h(s,g)=(4\pi s)^{-R/2}|\det Q|^{-1}\int_{(\ker Q^{\sim})^{\perp}}\exp\Big(-i\Big\langle f,\frac{u}{2s}\Big\rangle\Big)\times$$
$$\times\det{}^{1/2}\Big(\frac{\frac12 A}{\sin\frac12 A}\Big)\exp\Big(-\frac{1}{4s}\Big\langle\frac{\frac12 A}{\operatorname{tg}\frac12 A}U, U\Big\rangle\Big)\,df$$

with $R=n+2(\dim G-n)$.

EXAMPLE (cf. Gaveau [5] and Hulanicki [8]). Let $\mathfrak{h}_d$ be the Lie algebra spanned as a vector space by $m=2d+1$ elements $X_1, Y_1; X_2, Y_2, \ldots, X_d, Y_d, T$ with the bracket product $[X_i, Y_i]=-4T$ for $i=1,\ldots,d$, all other commutators being equal to 0. The corresponding group is the *Heisenberg group* H_d (cf. e.g. [2]). The mapping $Q^{\sim}$ from $N_{2d,2}$ to H_d is then given by

$$Q^{\sim}(u_j, u_{kl})=\Big(u_j, -4\sum_{i=1}^d u_{2i-1,2i}\Big)\qquad(=(u_j,t)\in H_d)$$

and $\det Q=-4$, $(\ker Q^{\sim})^{\perp}=\{f=(0,\alpha_{kl})\colon \alpha_{12}=\alpha_{34}=\ldots=\alpha_{2d-1,2d}$, the other α_{kl}'s being $0\}$. This may be shown as follows: (1) it is obvious that α_{kl}'s with k, l other than $1,2; 3,4;\ldots; 2d-1, 2d$ are zero. (2) Suppose that $\langle f,u\rangle=\sum_{i=1}^d \alpha_{2i-1,2i}\,u_{2i-1,2i}$ and $\langle f,u\rangle=0$ for all $u\in N_{2d,2}$ with $\sum_{i=1}^d u_{2i-1,2i}=0$; then (2) has the unique solution f (up to a constant factor) and we may simply put $\alpha_{2i-1,2i}=\lambda$ for $i=1,\ldots,d$, λ being the arbitrary constant.

For such an f all the corresponding P_{2h-1}, $h=1,\ldots,d$ are equal to λ and we get

$$h(s,(u_j,t))$$
$$=c(s)\int_{-\infty}^{\infty}\Big(\frac{s\lambda}{\operatorname{sh}(s\lambda)}\Big)^d\exp\Big[-i\lambda\Big(\sum_{i=1}^d u_{2i-1,2i}\Big)\Big]\exp\Big(-\frac{U^2}{4s}\,\frac{s\lambda}{\operatorname{th}(s\lambda)}\Big)\,d\lambda$$

with $\sum_{i=1}^d u_{2i-1,2i}=-t/4$, $U^2=\sum_{j=1}^{2d}u_j^2$, $c(s)^{-1}=(4\pi s)^d(2\pi)4$.

Added in proof. It seems that in the case of odd n our formula (4.9) on p. 234 differs form that in [6] and [13] p. 125, (Théorème 1) by the additional factor $\exp\left(-\frac{|E_0 U|^2}{4s}\right)$ under the integral sing. However, in the particular case $n = 3$, these formulas agree up to the sign before $\frac{|u|^2}{s}$ in the argument of the exp function, cf. [13] p. 126.

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Quelques propriétés de l'espace des opérateurs compacts

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Résumé. Soient $\boldsymbol{K}(W, V)$ (resp. $\boldsymbol{F}(W, V)$) l'espace des opérateurs compacts (resp. faiblement compacts) d'un espace de Banach W dans V. On donne des conditions suffisantes pour que $\boldsymbol{K}(W, V)$ ne contienne pas de sous-espace isomorphe à $l^1(\boldsymbol{N})$ ou bien soit faiblement séquentiellement complet. Le bidual de $\boldsymbol{K}(W, V)$ est canoniquement isométrique à un sous-espace de $\boldsymbol{L}(W'', V'')$ l'espace des opérateurs bornés de W'' dans V''. Si V est un espace $C(X)$ ou bien si W est un espace $L^1(\mu)$, on montre que $\boldsymbol{F}(W, V)$ est canoniquement isométrique à un sous-espace du bidual de $\boldsymbol{K}(W, V)$. Si X est le support d'une mesure ou bien si la mesure μ est σ-finie, l'espace $\boldsymbol{F}(W, V)$ s'injecte dans $\boldsymbol{K}_1''(W, V)$ l'adhérence séquentielle de $\boldsymbol{K}(W, V)$ dans son bidual.

Introduction et notations. Soient W et V deux espaces de Banach; on se propose d'établir les résultats annoncés plus haut en se basant sur un résultat assez simple qui permet de représenter les opérateurs compacts ou faiblement compacts de W dans V comme des fonctions continues sur des compacts adéquats. L'isométrie de $\boldsymbol{K}''(W, V)$ dans $\boldsymbol{L}(W'', V'')$ en résultera de façon naturelle. L'injection de $\boldsymbol{F}(W, V)$ dans $\boldsymbol{K}''(W, V)$ ou même $\boldsymbol{K}_1''(W, V)$ quand W ou bien V vérifient les hypothèses citées plus haut provient de cette représentation et de l'étude des fonctions séparément continues sur le produit de deux compacts. En particulier, on retrouve que l'image d'un espace $L^1(\mu)$ relatif à une mesure σ-finie par un opérateur faiblement compact est un espace séparable.

Soit V un espace de Banach; on note $B(V)$ sa boule unité fermée et son dual V' sera, sauf mention du contraire muni de $\sigma(V', V)$. Si A est une partie de V on note $\operatorname{conv}(A)$ son enveloppe convexe et on désigne par $E(X)$ l'ensemble des points extrémaux d'un convexe X. Rappelons qu'un espace vérifie la propriété de Radon–Nikodym si pour tout espace mesuré (X, Σ, μ) où μ est une mesure positive et toute mesure σ-additive m définie sur (X, Σ) à valeurs dans V, à variation bornée et qui est absolument continue par rapport à μ, il existe une fonction f, intégrable au sens de Bochner de X dans V, telle que $m = f\cdot\mu$. Il résulte des travaux de Stegall [8] qu'un espace dual V' possède cette propriété si est seulement si tout sous-espace séparable de V a un dual séparable.

Si W et V sont deux espaces de Banach on note $\boldsymbol{K}(W, V)$ (resp.