$$L_\lambda = \sum_{j=1}^{k} (L_j - \lambda_j \delta)^* * (L_j - \lambda_j \delta)$$

gives rise to a left-invertible singular integral operator $\pi^1_{L_\lambda}$ on $L^2(N)$.

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The Mackey completions of some interpolation F-spaces

by

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Abstract. We characterize the Mackey completions of locally concave F-spaces which are interpolation spaces with respect to a special couple of Banach lattices. The results are applied to the interpolation spaces generated by the K method of interpolation.

1. Introduction. An *F-quasinorm* on a vector space X is a nonnegative function $\|\cdot\|$ on X which vanishes only at zero and has the following properties for every $x, y \in X$ and scalar t with $|t| \leqslant 1$:

(i) $\|tx\| \leqslant \|x\|$,
(ii) $\|x+y\| \leqslant C(\|x\|+\|y\|)$ for some $C > 0$,
(iii) $\|tx\| \to 0$ as $t \to 0$.

An F-quasinorm for which $C = 1$ is called an *F-norm*, and an F-norm which is *p-homogeneous* for some $0 < p \leqslant 1$,

(iv) $\|\lambda x\| = |\lambda|^p \|x\|$ whenever λ is scalar,

is called a *p-norm* (a *norm* if $p = 1$). An F-quasinorm which is 1-homogeneous is called a *quasinorm*.

A linear space equipped with a Hausdorff vector topology determined by an F-norm (p-norm, quasinorm) is called an *F^*-space* (*p-normed space*, *quasinormed space*, respectively). A topologically complete p-normed space (quasinormed space) X is called a *p-Banach space* (*quasi-Banach space*).

Two topological vector spaces (tvs) X and Y are considered as equal ($X = Y$) whenever $X = Y$ as sets and their topologies are equivalent. If τ is a topology on X and Z is a subspace of X, then $\tau_{|Z}$ is the topology induced on Z by τ.

A pair $\bar{A} = (A_0, A_1)$ of normed (Banach) spaces is called a *normed* (*Banach*) *couple* if A_0 and A_1 are both algebraically and topologically imbedded in some Hausdorff tvs.

For a normed (Banach) couple $\bar{A} = (A_0, A_1)$ we can form the *sum* $\Sigma(\bar{A}) = A_0 + A_1$ and the *intersection* $\Delta(\bar{A}) = A_0 \cap A_1$. They are both normed (Banach) spaces, in the natural norms $\|a\|_\Sigma = K(1, a; \bar{A})$ and $\|a\|_\Delta = J_1(1, a; \bar{A})$,

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respectively, where for any $t \in \mathbf{R}_+ = (0, \infty)$

$$K(t, a; \bar{A}) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1}: a = a_0 + a_1\}, \quad a \in \Sigma(\bar{A}),$$
$$J_1(t, a; \bar{A}) = \|a\|_{A_0} + t\|a\|_{A_1}, \quad a \in \Delta(\bar{A}).$$

A Hausdorff tvs A is called an *intermediate space* with respect to a normed couple $\bar{A}$ if $\Delta(\bar{A}) \subsetneq A \subsetneq \Sigma(\bar{A})$. Here and in the sequel we let the symbol $\subsetneq$ stand for *continuous inclusion*.

We denote by $\mathscr{L}(\bar{A})$ the normed (Banach) space of all linear operators $T: \Sigma(\bar{A}) \to \Sigma(\bar{A})$ such that the restriction of T to the space A_i is a bounded operator in A_i, $i = 0, 1$, with the norm

$$\|T\|_{\mathscr{L}(\bar{A})} = \max\{\|T\|_{A_0 \to A_0}, \|T\|_{A_1 \to A_1}\}.$$

A Hausdorff tvs (a quasinormed space) A intermediate with respect to a normed couple $\bar{A}$ is called an *interpolation space* (an *exact interpolation space*) with respect to $\bar{A}$ if every linear operator from $\mathscr{L}(\bar{A})$ maps continuously A into itself (respectively $\|Ta\|_A \leqslant \|T\|_{\mathscr{L}(\bar{A})}\|a\|_A$ for $a \in A$).

The *Mackey topology* of a tvs $X = (X, \tau)$ is the strongest locally convex topology μ on X which produces the same continuous linear functionals as the original topology τ of X. If X is metrizable, then μ coincides with the strongest locally convex topology τ^c on X which is weaker than τ (see [18]). Obviously, if $\mathscr{B}$ is a base of neighbourhoods of zero for τ, then the family $\{\operatorname{conv} V: V \in \mathscr{B}\}$ is a base of neighbourhoods of zero for τ^c. So if the dual space $(X, \tau)^*$ separates the points of X, then the Mackey topology μ is metrizable. The completion $\hat{X}$ of (X, μ) is an *F-space* (i.e., metrizable and complete) which we call the *Mackey completion* of X.

If $X = (X, \|\cdot\|)$ is a quasinormed space, then the Mackey topology μ of X is seminormable (normable, if X has a total dual and in consequence $\hat{X}$ is a Banach space). In this case the Mackey topology μ of X is generated by the Minkowski functional of the convex hull of the unit ball $\{x \in X: \|x\| \leqslant 1\}$ of X, which is called the *Mackey seminorm* (norm, if X has a total dual). If X is a concrete space, one may attempt to describe μ as the topology induced by another concrete space which is locally convex (or even Banach). This has been done e.g. for the Hardy spaces H^p $(0 < p < 1)$ and some other spaces of analytic and harmonic functions (see [18, 19]). M. Cwikel and C. Fefferman [5] have computed the Mackey seminorm of a Weak L^1 space. A. Haaker [9] has shown (under some assumption on the function ϕ) that the Mackey topology of a Lorentz space $L(\phi, q)$ $(0 < q < 1)$ coincides with the topology induced by $L(\phi, 1)$ (for a more general result see [15]). N. J. Kalton [10] has shown that the Mackey topology of a separable Orlicz sequence space coincides with the topology induced by another Orlicz sequence space (for the nonseparable case see [8]). In [13] it was shown that the Mackey completions of some quasinormed interpolation spaces with respect to a Banach couple $\bar{A}$ may be identified with other concrete Banach spaces which are interpolation spaces with respect to $\bar{A}$ (see Theorem 4.4). This result and [15] were the main motivation to write this paper, in which we describe the Mackey completions of some interpolation F^*-spaces. The results obtained are applied to the well-known interpolation quasi-Banach spaces generated by the real method of interpolation.

2. Technical results. In this section we give some technical results needed in the sequel. We give first some notation and definitions.

Let (Ω, ν) be a measure space with ν complete and σ-finite. Denote by $L^0 = L^0(\Omega, \nu)$ the F-space of all equivalence classes of ν-measurable real-valued functions defined and ν-a.e. finite on Ω, equipped with the topology of convergence in measure on ν-finite sets.

A subset U of L^0 is called *solid* (in L^0) if the conditions $x \in L^0$, $y \in U$, $|x| \leqslant |y|$ a.e. imply that $x \in U$. A vector topology on a subspace of L^0 is *solid* if there is a base of neighbourhoods of zero consisting of solid sets. In the sequel by a solid tvs we shall mean a solid space with a solid topology. Every solid tvs contained in L^0 is continuously imbedded in L^0 (see [17, Proposition 2.7.2]).

We say that an F-quasinorm $\|\cdot\|$ on a solid subspace of L^0 is *monotone* if it satisfies the condition

(v) $|x| \leqslant |y|$ a.e. implies $\|x\| \leqslant \|y\|$.

A solid subspace X of L^0 together with a monotone F-norm (p-norm, quasinorm) will be called an *F^*-lattice* (*p-normed lattice*, *quasinormed lattice*, respectively). A topologically complete F^*-lattice (p-normed lattice, quasinormed lattice) will be called an *F-lattice* (*p-Banach lattice*, *quasi-Banach lattice*, respectively). Recall that if X and Y are F-lattices in L^0, then $X \subset Y$ implies that the inclusion map is continuous (by the closed graph theorem and the fact $X, Y \subsetneq L^0$).

Denote by L^∞ respectively L_1^∞ the Banach lattice in $L^0(\mathbf{R}_+, dt/t)$ which consists of $f \in L^0$ such that $|f(t)|$, respectively $|f(t)|/t$ is essentially bounded. Put $\bar{L}^\infty = (L^\infty, L_1^\infty)$.

Let $\mathscr{P}$ denote the set of *quasiconcave* functions defined on $\mathbf{R}_+$, i.e., $\psi \in \mathscr{P}$ if $0 < \psi(s) \leqslant \max(1, s/t)\psi(t)$ for all $s, t > 0$. By $\mathscr{P}_c$ we denote the set of all nonnegative *concave* functions defined on $[0, \infty)$. We say that $\psi_1, \psi_2 \in \mathscr{P}$ are *equivalent* $(\psi_1 \approx \psi_2)$ if $c_1\psi_1(t) \leqslant \psi_2(t) \leqslant c_2\psi_1(t)$ for some $c_1, c_2 > 0$ and all $t > 0$.

For every $f \in \Sigma(\bar{L}^\infty)$ and all $t > 0$, we put

$$\tilde{f}(t) = \inf\{g(t): g \geqslant |f| \text{ a.e., } g \in \mathscr{P}_c\},$$

so $\tilde{f}$ is the minimal concave function which is a.e. greater than $|f|$. In [4] it was shown that $\tilde{f}(\cdot) = K(\cdot, f; \bar{L}^\infty)$.

For a Banach couple $\bar{A}$ and a quasiconcave function $\varphi \in \mathscr{P}$ we denote by $\Lambda_\varphi(\bar{A})$ the space of all $a \in \Sigma(\bar{A})$ which can be represented in the form

$$a = \sum_{\nu=-\infty}^{\infty} a_\nu, \quad a_\nu \in \Delta(\bar{A}) \quad (\text{convergence in } \Sigma(\bar{A})),$$

with $\sum_{\nu=-\infty}^{\infty} \varphi(2^\nu)^{-1} J_1(2^\nu, a_\nu) < \infty$, where $J_1(t, a) = J_1(t, a; \bar{A}) = \|a\|_{A_0} + t\|a\|_{A_1}$ for $a \in \Delta(\bar{A})$ and $t > 0$. The space $\Lambda_\varphi(\bar{A})$ with the norm

$$\|a\|_{\Lambda_\varphi(\bar{A})} = \inf\{\sum_{\nu=-\infty}^{\infty} \varphi(2^\nu)^{-1} J_1(2^\nu, a_\nu)\colon a = \sum_{\nu=-\infty}^{\infty} a_\nu,\ a_\nu \in \Delta\}$$

is an exact interpolation Banach space with respect to $\bar{A}$. Note that if in the above definition we replace J_1 by the well-known functional J, then $\Lambda_\varphi(\bar{A})$ is a special *J-space* $\bar{A}_{\varphi,1;J}$ (see [6]). Throughout the paper the space $\Lambda_\varphi(\bar{L}^\infty)$ is often denoted by Λ_φ.

The proof of the following proposition is similar to that of Theorem 3.5.2 of [3].

PROPOSITION 2.1. *If $\bar{A}$ is a Banach couple and A is a Banach space intermediate with respect to $\bar{A}$, then $\|a\|_A \leqslant c\|a\|_{\Lambda_\varphi(\bar{A})}$ for $a \in \Lambda_\varphi(\bar{A})$ if and only if $\|a\|_A \leqslant c\varphi(2^\nu)^{-1} J_1(2^\nu, a; \bar{A})$ for each $a \in \Delta(\bar{A})$ and $\nu \in \mathbf{Z}$.*

Now, we give some properties of the spaces $\Lambda_\varphi(\bar{L}^\infty)$ needed in the sequel. First we give some auxiliary results. The following interesting result is due to I. U. Asekritova (see [2]). For the sake of completeness and availability we give the proof.

LEMMA 2.2. *Let $f_0, f_1, f \in \mathscr{P}_c$ be such that $f \leqslant f_0 + f_1$. Then there exist $\bar{f}_0, \bar{f}_1 \in \mathscr{P}_c$ with $\bar{f}_0 \leqslant f_0$, $\bar{f}_1 \leqslant f_1$ and $f = \bar{f}_0 + \bar{f}_1$.*

Proof. We consider the set $\mathscr{A} = \mathscr{A}(f_0, f_1, f) = \{(g_0, g_1)\colon g_0, g_1 \in \mathscr{P}_c,\ g_0 \leqslant f_0,\ g_1 \leqslant f_1 \text{ and } g_0 + g_1 \geqslant f\}$. Since $(f_0, f_1) \in \mathscr{A}$, we have $\mathscr{A} \neq \emptyset$. Let $(g_0, g_1) \ll (h_0, h_1)$ if $g_0 \leqslant h_0$ and $g_1 \leqslant h_1$. Then $\mathscr{A}$ is partially ordered by $\ll$. Since the infimum of functions from $\mathscr{P}_c$ is concave, it follows that every chain in $\mathscr{A}$ has a lower bound in $\mathscr{A}$. Consequently, by Kuratowski–Zorn's lemma, $\mathscr{A}$ has a minimal element $(\bar{f}_0, \bar{f}_1)$. We show that $\bar{f}_0 + \bar{f}_1 = f$. Assume that $\bar{f}_0 + \bar{f}_1 \neq f$; then $\bar{f}_0(s_0), \bar{f}_1(s_0) > f(s_0)$ for some $s_0 > 0$. Let $a = \inf\{s > 0\colon \bar{f}_0(s_0) + \bar{f}_1(s) > l(s)\}$, where $l = l(s)$ is the tangent to f at the point $(s_0, f(s_0))$. Since $0 \leqslant a < s_0$, two cases are possible:

$$\bar{f}_0(a) + \bar{f}_1(a) < \bar{f}_0(s_0) + \bar{f}_1(s_0) \quad \text{or} \quad \bar{f}_0(a) + \bar{f}_1(a) = \bar{f}_0(s_0) + \bar{f}_1(s_0).$$

In the first case, we have either $\bar{f}_0(a) < \bar{f}_0(s_0)$ or $f_1(a) < f_1(s_0)$. Let, for example, $\bar{f}_0(a) < \bar{f}_0(s_0)$, and let l_0 be the line through $(a, \bar{f}_0(a))$ and $(s_0, \bar{f}_0(s_0) - \varepsilon)$, where $\varepsilon > 0$ is sufficiently small. Now if $g_0 = \min(l_0, \bar{f}_0)$, then obviously $g_0 \in \mathscr{P}_c$, $g_0 \leqslant \bar{f}_0$, $g_0 \neq \bar{f}_0$ and $(g_0, \bar{f}_1) \in \mathscr{A}$; a contradiction, since $(\bar{f}_0, \bar{f}_1)$ is the minimal element of $\mathscr{A}$.

Now consider the second case, $\bar{f}_0(a) + \bar{f}_1(a) = \bar{f}_0(s_0) + \bar{f}_1(s_0)$. From concavity of $\bar{f}_0 + \bar{f}_1$ we have $\bar{f}_0(s) + \bar{f}_1(s) = \text{const}$ for $s \geqslant a$, so by the definition of a, it follows that $a = 0$. This implies $\bar{f}_i(s) = \bar{f}_i(s_0)$ for all $s > 0$ $(i = 0, 1)$. Without loss of generality, we can assume that $\bar{f}_0$ is positive. Let l_0 be the line through the points $(0, \bar{f}_0(s_0) - \varepsilon/2)$ and $(s_0, \bar{f}_0(s_0))$, where $\varepsilon = \min\{\bar{f}_0(s_0) + \bar{f}_1(s_0) - l(s_0),\ \bar{f}_0(s_0)\}$. Then for $h_0 = \min\{l_0, \bar{f}_0\}$, we have $h_0 \in \mathscr{P}_c$, $h_0 \leqslant \bar{f}_0$, $h_0 \neq \bar{f}_0$ and $(h_0, \bar{f}_1) \in \mathscr{A}$, a contradiction. Consequently $\bar{f}_0 + \bar{f}_1 = f$, and the proof is complete.

PROPOSITION 2.3. *If E is an exact interpolation normed space with respect to $\bar{L}^\infty$ such that $\Delta(\bar{L}^\infty)$ is a dense subspace of E, then there exists a completion of E in $\Sigma(\bar{L}^\infty)$.*

Proof. This follows by Theorem 2 of [13].

PROPOSITION 2.4. *Let $\varphi \in \mathscr{P}$. Then for any f in $\Delta = \Delta(\bar{L}^\infty)$ $(\Delta \cap \mathscr{P}_c)$*

$$\|f\|_{\Lambda_\varphi} = \inf\{\sum_{\nu=-k}^{n} \varphi(2^\nu)^{-1} J_1(2^\nu, f_\nu)\colon f = \sum_{\nu=-k}^{n} f_\nu\},$$

where the infimum is taken over all finite representations $f = \sum_{\nu=-k}^{n} f_\nu$ with $f_\nu \in \Delta$ (respectively $f_\nu \in \Delta \cap \mathscr{P}_c$), $k, n \in \mathbf{N}$.

Proof. Let E be the space Δ with the norm

$$\|f\|_E = \inf\{\sum_{\nu=-k}^{n} \varphi(2^\nu)^{-1} J_1(2^\nu, f_\nu)\colon f = \sum_{\nu=-k}^{n} f_\nu,\ f_\nu \in \Delta,\ n, k \in \mathbf{N}\}$$

for $f \in \Delta$. Obviously E is an exact interpolation space with respect to $\bar{L}^\infty$. Thus there exists a completion $\tilde{E}$ of E in $\Sigma(\bar{L}^\infty)$, by Proposition 2.3. Since for any $f \in \Delta$ and $\nu \in \mathbf{Z}$

$$\|f\|_{\tilde{E}} = \|f\|_E \leqslant \varphi(2^\nu)^{-1} J_1(2^\nu, f)$$

and $\tilde{E}$ is an intermediate space with respect to $\bar{L}^\infty$, it follows by Proposition 2.1 that $\|f\|_{\tilde{E}} \leqslant \|f\|_{\Lambda_\varphi}$ for $f \in \Delta$. Since $\|f\|_E \geqslant \|f\|_{\Lambda_\varphi}$ for $f \in \Delta$, we get $\|f\|_E = \|f\|_{\Lambda_\varphi}$.

Now let $f \in \Delta \cap \mathscr{P}_c$. Fix $\varepsilon > 0$; then by the above there exists a finite sequence $\{g_\nu\}_{\nu=-k}^{n}$ with $g_\nu \in \Delta$, $f = \sum_{\nu=-k}^{n} g_\nu$ a.e. on $\mathbf{R}_+$ and

$$\sum_{\nu=-k}^{n} \varphi(2^\nu)^{-1} J_1(2^\nu, g_\nu) < \|f\|_{\Lambda_\varphi} + \varepsilon. \tag{1}$$

Hence $|f| \leqslant \sum_{\nu=-k}^{n} |g_\nu|$ a.e., so $f = \tilde{f} \leqslant \sum_{\nu=-k}^{n} \tilde{g}_\nu$. By Lemma 2.2, we obtain $f = \sum_{\nu=-k}^{n} f_\nu$, where $f_\nu \leqslant \tilde{g}_\nu$ and $f_\nu \in \mathscr{P}_c$, so $f_\nu \in \Delta \cap \mathscr{P}_c$. Since $J_1(2^\nu, g_\nu) = J_1(2^\nu, \tilde{g}_\nu) \geqslant J_1(2^\nu, f_\nu)$, we have

$$\varepsilon + \|f\|_{\Lambda_\varphi} > \sum_{\nu=-k}^{n} \varphi(2^\nu)^{-1} J_1(2^\nu, f_\nu) \geqslant \|f\|_E^*$$

$$= \inf\{\sum_{\nu=-k}^{n} \varphi(2^\nu)^{-1} J_1(2^\nu, f_\nu)\colon f = \sum_{\nu=-k}^{n} f_\nu,\ f_\nu \in \Delta \cap \mathscr{P}_c,\ n, k \in \mathbf{N}\}$$

by (1). Thus $\|f\|_{\Lambda_\varphi} \geqslant \|f\|_E^*$. The inequality $\|f\|_{\Lambda_\varphi} \leqslant \|f\|_E^*$ is obvious. Consequently $\|f\|_{\Lambda_\varphi} = \|f\|_E^*$ and the proof is finished.

In the sequel let $\mathscr{S}$ denote the subset of $\mathscr{P}_c$ defined by

$$\mathscr{S} = \{f \in \mathscr{P}_c\colon f = \sum_{i=1}^{n} c_i f_{t_i},\ c_i, t_i \in \mathbf{R}_+,\ i = 1, \ldots, n,\ n \in \mathbf{N}\},$$

where for $t > 0$, $f_t(s) = \min(1, s/t)$ for all $s > 0$. Let us remark that $\mathscr{S} \subset \Delta(\bar{L}^\infty)$.

PROPOSITION 2.5. *Let $\{t_i\}_{i=0}^n \subset [0, \infty)$ be a given sequence such that $0 = t_0 < t_1 < \ldots < t_n$.*

(a) *If f is a positive, nondecreasing, continuous function on $[0, \infty)$ with $f(0) = 0$, linear on the interval $[t_{i-1}, t_i]$, $i = 1, \ldots, n$, and constant on $[t_n, \infty)$, then there exists a unique representation of f of the form $\sum_{i=1}^n c_i f_{t_i}$ with $c_i > 0$.*

(b) *Let $\varphi_1, \ldots, \varphi_m$ be positive functions in $\mathscr{P}_c$. Then $\sum_{i=1}^n c_i f_{t_i} = \sum_{\nu=1}^m \varphi_\nu$ if and only if*

$$\varphi_\nu = \sum_{i=1}^{n} a_{i\nu} f_{t_i},$$

where $a_{i\nu} > 0$, $\sum_{\nu=1}^m a_{i\nu} = c_i$, $i = 1, \ldots, n$, $\nu = 1, \ldots, m$.

Proof. (a) is obvious. The proof of (b) easily follows from the following fact: if $\varphi_0, \varphi_1 \in \mathscr{P}_c$ and $\varphi_0(t) + \varphi_1(t) = at + b$ for $t \in I = [s_0, s_1]$, where $0 \leqslant s_0 < s_1 < \infty$, and $a \geqslant 0$, $b > 0$, then φ_0 and φ_1 are convex on I. Hence from the concavity and convexity of φ_i on I, $i = 0, 1$, we have $\varphi_i(t) = a_i t + b_i$ on I, where $a_i \geqslant 0$, $b_i > 0$, $i = 0, 1$. So (a) applies.

Now we observe that the norm of $\Lambda_\psi(\bar{L}^\infty)$ has a special property on $\mathscr{S}$:

PROPOSITION 2.6. *The norm of the space $E = \Lambda_\psi(\bar{L}^\infty)$ is additive on $\mathscr{S}$ for any $\psi \in \mathscr{P}$, i.e., $\|f+g\|_E = \|f\|_E + \|g\|_E$ for all $f, g \in \mathscr{S}$.*

Proof. It is sufficient to show that if $f = \sum_{i=1}^n c_i f_{t_i} \in \mathscr{S}$, where $0 < t_1 < \ldots < t_n < \infty$, then $\|f\|_E \geqslant \sum_{i=1}^n c_i \|f_{t_i}\|_E$. Fix $\varepsilon > 0$. From Proposition 2.4 it follows that there exists a sequence $\{\varphi_\nu\}_{\nu=-k}^m \subset \Delta(\bar{L}^\infty) \cap \mathscr{P}_c$, $k, m \in \mathbf{N}$, such that $f = \sum_{\nu=-k}^m \varphi_\nu$ and

$$\sum_{\nu=-k}^{m} \psi(2^\nu)^{-1} J_1(2^\nu, \varphi_\nu;\ \bar{L}^\infty) < \|f\|_E + \varepsilon. \tag{2}$$

By Proposition 2.5, we obtain $\varphi_\nu = \sum_{i=1}^n a_{i\nu} f_{t_i}$, where $a_{i\nu} > 0$, $\nu = -k, \ldots, m$, and

$$\sum_{\nu=-k}^{m} a_{i\nu} = c_i, \quad i = 1, \ldots, n. \tag{3}$$

Thus

$$J_1(2^\nu, \varphi_\nu) = J_1(2^\nu, \varphi_\nu;\ \bar{L}^\infty) = \sum_{i=1}^{n} a_{i\nu} + 2^\nu \sum_{i=1}^{n} a_{i\nu}/t_i = \sum_{i=1}^{n} J_1(2^\nu, a_{i\nu} f_{t_i}).$$

Hence by (2) and (3), we get

$$\begin{aligned} \varepsilon + \|f\|_E &> \sum_{\nu=-k}^{m} \sum_{i=1}^{n} \psi(2^\nu)^{-1} J_1(2^\nu, a_{i\nu} f_{t_i}) \\ &\geqslant \sum_{i=1}^{n} \sum_{\nu=-k}^{m} \|a_{i\nu} f_{t_i}\|_E = \sum_{i=1}^{n} c_i \|f_{t_i}\|_E. \end{aligned}$$

Since ε is arbitrary, $\|f\|_E \geqslant \sum_{i=1}^n c_i \|f_{t_i}\|_E$ and the proof is complete.

Define $\bar{\psi}_E(t) = \|f_t\|_E^{-1}$ for $t > 0$, where E is an F^*-space intermediate with respect to $\bar{L}^\infty$ equipped with an F-norm $\|\cdot\|_E$. If $E = \Lambda_\psi(\bar{L}^\infty)$ we write $\bar{\psi}$ instead of $\bar{\psi}_E$.

COROLLARY 2.7. (a) $\{f \in \mathscr{S}\colon \|f\|_{\Lambda_\psi(\bar{L}^\infty)} = 1\} = \operatorname{conv}\{x_t\colon t \in \mathbf{R}_+\}$, *where* $x_t = \bar{\psi}(t) f_t$.

(b) $4^{-1}\psi(t) \leqslant \bar{\psi}(t) \leqslant \psi(t)$ *for all* $t > 0$.

Proof. (a) easily follows from Proposition 2.6. Let $t > 0$. Choose $\nu \in \mathbf{Z}$ such that $2^{\nu-1} < t \leqslant 2^\nu$. Then

$$\begin{aligned} \bar{\psi}(t)^{-1} = \|f_t\|_{\Lambda_\varphi} &\leqslant \psi(2^\nu)^{-1} J_1(2^\nu, f_t;\ \bar{L}^\infty) \\ &\leqslant 2\psi(t)^{-1} \max(1, 2^\nu/t) \leqslant 4\psi(t)^{-1}. \end{aligned}$$

Now let $\varepsilon > 0$. From Proposition 2.4 it follows that there exists a sequence $\{\varphi_\nu\}_{\nu=-k}^m \subset \Delta(\bar{L}^\infty) \cap \mathscr{P}_c$, $k, m \in \mathbf{N}$, such that $f_t = \sum_{\nu=-k}^m \varphi_\nu$ and

$$\sum_{\nu=-k}^{m} \psi(2^\nu)^{-1} J_1(2^\nu, \varphi_\nu;\ \bar{L}^\infty) < \bar{\psi}(t)^{-1} + \varepsilon.$$

By Proposition 2.5, we obtain $\varphi_\nu = a_\nu f_t$, where $a_\nu > 0$ for $\nu = -k, \ldots, m$, and $\sum_{\nu=-k}^m a_\nu = 1$. Thus $J_1(2^\nu, \varphi_\nu;\ \bar{L}^\infty) \geqslant a_\nu \max(1, 2^\nu/t)$ and

$$\varepsilon + \bar{\psi}(t)^{-1} > \sum_{\nu=-k}^{m} a_\nu \psi(2^\nu)^{-1} \max(1, 2^\nu/t) \geqslant \sum_{\nu=-k}^{m} a_\nu \psi(t)^{-1} = \psi(t)^{-1}.$$

Consequently $\bar{\psi}(t)^{-1} \geqslant \psi(t)^{-1}$ for all $t > 0$, and the proof is finished.

3. Locally concave interpolation F-spaces. A subset U of $\Sigma(\bar{L}^\infty)$ is called *$\Sigma(\bar{L}^\infty)$-monotone* if $f \in U$, $g \in \Sigma(\bar{L}^\infty)$ and $\tilde{g} \leqslant \tilde{f}$ implies $g \in E$. Note that a $\Sigma(\bar{L}^\infty)$-monotone set is solid in $L^0 = L^0(\mathbf{R}_+, dt/t)$. Let X be a tvs set-theoretically contained in $\Sigma(\bar{L}^\infty)$. We say that X is monotone (in $\Sigma(\bar{L}^\infty)$) if there is a base of neighbourhoods of zero in X consisting of $\Sigma(\bar{L}^\infty)$-monotone sets. Let us remark that if an F-space X is monotone, then X is a complete solid tvs contained in L^0, so it is continuously imbedded in $\Sigma(\bar{L}^\infty)$. This implies that X has a separating dual. The following proposition gives more information about monotone and metrizable topological vector spaces.

PROPOSITION 3.1. (a) *A metrizable tvs X is monotone in $\Sigma(\overline{L}^\infty)$ if and only if there is a monotone F-norm $\|\cdot\|$ on X such that $\|\tilde f\| = \|f\|$ for all $f \in X$ and the original topology of X is induced by $\|\cdot\|$.*

(b) *A metrizable monotone tvs intermediate with respect to the couple $\overline{L}^\infty$ is an interpolation space with respect to $\overline{L}^\infty$.*

Proof. (a) Let $X = (X, \tau)$ be a metrizable tvs, monotone in $\Sigma(\overline{L}^\infty)$. So there is a base $\mathscr{B} = \{V_n \colon n \in \mathbf{N}\}$ of neighbourhoods of zero in X, consisting of $\Sigma(\overline{L}^\infty)$-monotone sets. Without loss of generality we can assume that $V_n + V_n + V_n \subset V_{n-1}$, $n \in \mathbf{N}$, where $V_0 = X$. Define on X a functional ϱ by

$$\varrho(f) = \begin{cases} 2^{-n} & \text{for } f \in V_{n-1} \setminus V_n, \\ 0 & \text{for } f \in \bigcap_{n=1}^{\infty} V_n. \end{cases}$$

Then the functional defined on X by

$$\|f\| = \inf\{\sum_{k=1}^{n} \varrho(f_k) \colon f = \sum_{k=1}^{n} f_k\}$$

is an F-norm on X which generates the original topology τ of X (see [14]). Since the sets V_n are solid, $\varrho(f) = \varrho(|f|)$ for all $f \in X$ and this implies that $\|f\| \leqslant \|g\|$ whenever $|f| \leqslant |g|$ a.e., $f, g \in X$. Thus $\|\cdot\|$ is a lattice F-norm on X. Moreover, by the $\Sigma(\overline{L}^\infty)$-monotonicity of V_n, we have $\varrho(f) = \varrho(\tilde f)$ for all $f \in X$ and in consequence we easily get $\|\tilde f\| = \|f\|$.

For the converse let $\|\cdot\|$ be a monotone F-norm on X such that $\|\tilde f\| = \|f\|$ for all $f \in X$. If the original topology of X is generated by $\|\cdot\|$, then the family of sets $U_n = \{f \in X \colon \|f\| \leqslant 1/n\}$, $n \in \mathbf{N}$, is a base of neighbourhoods of zero in X consisting of $\Sigma(\overline{L}^\infty)$-monotone sets.

(b) Let $T \in \mathscr{L}(\overline{L}^\infty)$. Then $\widetilde{Tf} \leqslant C\tilde f$ for all $f \in \Sigma(\overline{L}^\infty)$, where $C = \|T\|_{\mathscr{L}(\overline{L}^\infty)}$, whence $T(X) \subset X$ if X is a monotone F-lattice in $\Sigma(\overline{L}^\infty)$. Moreover, if $f_n \to 0$ in a topology of X, then $\|f_n\| \to 0$, and $\|Tf_n\| = \|\widetilde{Tf_n}\| \leqslant (C+1)\|\tilde f_n\| = (C+1)\|f_n\|$ by the above properties of the F-norm $\|\cdot\|$ given in (a). So $Tf_n \to 0$ in X, thus T is continuous in X and the proof is complete.

If X is a solid tvs ($\subset L^0(\mathbf{R}_+, dt/t)$) containing the set $\mathscr{S}$ and V is a solid and *absorbing* subset of X, then we define the function $\psi_V \colon \mathbf{R}_+ \to (0, \infty]$ by

$$\psi_V(t) = \sup\{\lambda > 0 \colon \lambda f_t \in V\} \quad \text{for } t > 0.$$

It is obvious that ψ_V is nondecreasing (quasiconcave if it is finite). If $\mathscr{B} = \{V_n \colon n \in \mathbf{N}\}$ is a fixed base of solid neighbourhoods of zero in X, we write ψ_n instead of ψ_{V_n}, $n \in \mathbf{N}$. It is easily seen that if X is continuously imbedded in $\Sigma(\overline{L}^\infty)$, then ψ_{n_0} is finite for some $n_0 \in \mathbf{N}$.

Throughout the remaining part of this paper we will be interested in special F-spaces which are interpolation spaces. Let X be a monotone, metrizable tvs, intermediate with respect to $\overline{L}^\infty$, and let U be a $\Sigma(\overline{L}^\infty)$-monotone subset of X. We say that U is *concave* (in X) if

$$U \cap \operatorname{conv}(\{cf_t \colon c, t \in \mathbf{R}_+\} \setminus U) = \emptyset.$$

The space X is called *locally concave* (in $\Sigma(\overline{L}^\infty)$) if there is a base $\mathscr{B} = \{V_n \colon n \in \mathbf{N}\}$ of neighbourhoods of zero in X consisting of concave sets such that every function ψ_n is finite. Note that if a base $\mathscr{B} = \{V_n \colon n \in \mathbf{N}\}$ of neighbourhoods of zero in X is such that $V_n \supset V_{n+1}$ and V_n are concave, $n \in \mathbf{N}$, then $\mathscr{B}_{n_0} = \{V_n \colon n \geqslant n_0\}$, where $n_0 = \min\{n \in \mathbf{N} \colon \psi_n \text{ is finite}\}$, is a base of neighbourhoods of zero in X equivalent to $\mathscr{B}$ such that ψ_n is finite for all $n \geqslant n_0$, so X is locally concave. The following proposition gives examples of monotone and locally concave F^*-lattices in $\Sigma(\overline{L}^\infty)$.

PROPOSITION 3.2. *Let E be a quasinormed exact interpolation lattice with respect to $\overline{L}^\infty$.*

(a) *E is monotone in $\Sigma(\overline{L}^\infty)$.*

(b) *If*

$$(*) \qquad \|\alpha f + \beta g\|_E \geqslant \alpha\|f\|_E + \beta\|g\|_E$$

for all $f, g \in \mathscr{S}$ and $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$, then E is locally concave.

Proof. (a) It is enough to establish that $\|\tilde f\|_E = \|f\|_E$ for all $f \in E$ (cf. [4] if E is a Banach space), by Proposition 3.1(a). To see this, define $p \colon \Sigma(\overline{L}^\infty) \to \Sigma(\overline{L}^\infty)$ by $p(x) = \tilde x$ for $x \in \Sigma(\overline{L}^\infty)$. Then obviously p is sublinear, i.e., $p(x+y) \leqslant p(x) + p(y)$, $p(\lambda x) = |\lambda| p(x)$ for all $x, y \in E$ and $\lambda \in \mathbf{R}$.

Fix $f \in E$. Then by the Hahn–Banach extension theorem (see [1, Theorem 2.1]) there exists a linear operator $T \colon \Sigma(\overline{L}^\infty) \to \Sigma(\overline{L}^\infty)$ such that

$$(4) \qquad |Tx| \leqslant p(x) \text{ a.e. for all } x \in \Sigma(\overline{L}^\infty) \text{ and } Tf = p(f) \text{ a.e.}$$

Hence $\|Tx\|_{L^\infty} \leqslant \|p(x)\|_{L^\infty}$ for all $x \in L^\infty$ and $\|Tx\|_{L_1^\infty} \leqslant \|p(x)\|_{L_1^\infty} = \|x\|_{L_1^\infty}$ for all $x \in L_1^\infty$, so $T \in \mathscr{L}(\overline{L}^\infty)$ and $\|T\|_{\mathscr{L}(\overline{L}^\infty)} \leqslant 1$. Thus

$$\|\tilde f\|_E = \|p(f)\|_E = \|Tf\|_E \leqslant \|f\|_E,$$

by (4) and by E being an exact interpolation space with respect to $\overline{L}^\infty$. On the other hand, $|f| \leqslant \tilde f$ a.e., so $\|f\|_E \leqslant \|\tilde f\|_E$. Consequently $\|f\|_E = \|\tilde f\|_E$.

(b) From (a) it follows that the family of sets $V_n = \{f \in E \colon \|f\|_E \leqslant 1/n\}$, $n \in \mathbf{N}$, is a base of neighbourhoods of zero in E consisting of $\Sigma(\overline{L}^\infty)$-monotone sets. Now let $f = \sum_{i=1}^{n} \alpha_i c_i f_{t_i}$, where $\sum_{i=1}^{n} \alpha_i = 1$, $\alpha_i \geqslant 0$, $c_i > 0$, $i = 1, \ldots, n$. Then

$$\|f\|_E \geqslant \sum_{i=1}^{n} \alpha_i \|c_i f_{t_i}\|_E$$

by (*). This implies that the V_n are concave sets. Since $\psi_n(t) = n^{-1}\bar{\psi}_E(t)$ for $t>0$, E is locally concave.

The following lemma easily follows from Proposition 6.2 in [4].

LEMMA 3.3. *For any $q>1$ and any concave function $f \in \Delta(\bar{L}^\infty)$ there is a function $g \in \mathscr{S}$ such that*

$$\frac{q-1}{q+1} g \leqslant f \leqslant qg.$$

THEOREM 3.4. *Let $\Delta(\bar{L}^\infty)$ be a dense subspace of an F-space $X = (X, \tau)$ contained in $\Sigma(\bar{L}^\infty)$ and let $\mathscr{B} = \{V_n\colon n \in \mathbf{N}\}$ be a base of neighbourhoods of zero in X consisting of concave sets. Then X is contained in $\Lambda_\psi(\bar{L}^\infty)$ if and only if $\sup\{\psi_n(t)/\bar{\psi}(t)\colon t \in \mathbf{R}_+\} < \infty$ for some $n \in \mathbf{N}$.*

Proof. Assume $X \subset \Lambda_\psi(\bar{L}^\infty)$. Since $X, \Lambda_\psi(\bar{L}^\infty) \subsetneq L^0$, by the closed graph theorem the inclusion mapping is continuous from X into $\Lambda_\psi(\bar{L}^\infty)$. This implies that there is $n \in \mathbf{N}$ such that $\|f\| = \|f\|_{\Lambda_\psi(\bar{L}^\infty)} \leqslant 1$ for every $f \in V_n$, whence it follows that $\psi_n(t) < \infty$ for every $t>0$. Put $x_t = \psi_n(t) f_t$ for $t>0$. Then $x_t \in V_n$ and

$$\psi_n(t)/\bar{\psi}(t) = \psi_n(t)\|f_t\| = \|x_t\| \leqslant 1 \quad \text{for all } t>0.$$

For the converse, assume that $\sup\{\psi_n(t)/\bar{\psi}(t)\colon t \in \mathbf{R}_+\} = C < \infty$ for some $n \in \mathbf{N}$. We shall show that $V_n \cap \Delta(\bar{L}^\infty) \subset \{f \in \Lambda_\psi(\bar{L}^\infty)\colon \|f\| \leqslant 6C\}$. We first establish that $V_n \cap \mathscr{S} \subset B = \{f \in \Lambda_\psi(\bar{L}^\infty)\colon \|f\| \leqslant C\}$. To see this, fix $f \in V_n \cap \mathscr{S}$. Thus by Corollary 2.7(a) we have

$$f = \sum_{i=1}^{k} \alpha_i \|f\| x_{t_i}, \tag{5}$$

where $\sum_{i=1}^k \alpha_i = 1$, $\alpha_i \geqslant 0$, $x_{t_i} = \bar{\psi}(t_i) f_{t_i}$, $i = 1, \ldots, k$. Suppose that $f \notin B$. Then $\|f\| > C$ and hence $\|f\|\bar{\psi}(t_i) > \psi_n(t_i)$ (otherwise $\|f\|\bar{\psi}(t_i) \leqslant \psi_n(t_i)$, so $C < \|f\| \leqslant \psi_n(t_i)/\bar{\psi}(t_i) \leqslant C$, a contradiction). Hence $\|f\| x_{t_i} \notin V_n$ for $i = 1, \ldots, k$, by the definition of ψ_n. Since V_n is concave, $f \notin V_n$ by (5), a contradiction with $f \in V_n \cap \mathscr{S}$. This shows that $V_n \cap \mathscr{S} \subset B$ as desired. Now assume that $f \in V_n \cap \Delta(\bar{L}^\infty)$. Then by Lemma 3.3 there is $g \in \mathscr{S}$ such that

$$\tfrac{1}{3} g \leqslant \tilde{f} \leqslant 2g. \tag{6}$$

Since $V_n \cap \Delta(\bar{L}^\infty)$ and B are $\Sigma(\bar{L}^\infty)$-monotone, $\tilde{f} \in 6B$, by $V_n \cap \mathscr{S} \subset B$ and (6). But $|f| \leqslant \tilde{f}$ a.e. and the set $6B$ is solid, so $f \in 6B$. In consequence

$$V_n \cap \Delta(\bar{L}^\infty) \subset \{f \in \Lambda_\psi(\bar{L}^\infty)\colon \|f\| \leqslant 6C\}.$$

Hence the inclusion mapping $\Delta(\bar{L}^\infty) \subset \Lambda_\psi(\bar{L}^\infty)$ is continuous if we equip $\Delta(\bar{L}^\infty)$ with the topology induced by τ. This and the density of $\Delta(\bar{L}^\infty)$ in X imply that $X \subsetneq \Lambda_\psi(\bar{L}^\infty)$ and the proof is complete.

4. Mackey completion. In this section we describe the Mackey completions of locally concave (in $\Sigma(\bar{L}^\infty)$) F-spaces in which $\Delta(\bar{L}^\infty)$ is a dense subspace. Next applying some results from [13] we give applications of our results. We need the following easily verified

PROPOSITION 4.1. *The convex hull of any solid ($\Sigma(\bar{L}^\infty)$-monotone) subset of $\Sigma(\bar{L}^\infty)$ is solid ($\Sigma(\bar{L}^\infty)$-monotone).*

For any quasiconcave function ψ let $\{U_n(\psi)\colon n \in \mathbf{N}\}$ denote the base of neighbourhoods of zero in $\Lambda_\psi(\bar{L}^\infty)$ formed by the sets

$$U_n(\psi) = \{f \in \Lambda_\psi(\bar{L}^\infty)\colon \|f\|_{\Lambda_\psi(\bar{L}^\infty)} \leqslant 1/n\}, \quad n \in \mathbf{N}.$$

THEOREM 4.2. *Let $\Delta(\bar{L}^\infty)$ be a dense subspace of a locally concave F-space $X = (X, \tau)$ and let $\mathscr{B} = \{V_n\colon n \in \mathbf{N}\}$ be a locally concave base of neighbourhoods of zero in X. Then the Mackey completion of X is the F-space*

$$E = \bigcap_{n \in \mathbf{N}} \Lambda_{\psi_n}(\bar{L}^\infty)$$

equipped with the natural projective topology π.

If V is a bounded concave neighbourhood of zero in X, then $\hat{X} = \Lambda_{\psi_V}(\bar{L}^\infty)$.

Proof. By Corollary 2.7(b) we have $\sup\{\psi_n(t)/\bar{\psi}_n(t)\colon t \in \mathbf{R}_+\} \leqslant 1/4$ for all $n \in \mathbf{N}$. Thus $X \subsetneq \Lambda_{\psi_n}(\bar{L}^\infty)$ (by Theorem 3.4) and $E \subsetneq \Lambda_{\psi_n}(\bar{L}^\infty)$ for all $n \in \mathbf{N}$ (by definition of the topology π) implies $X \subsetneq E$. Hence $\pi_{|X} \leqslant \tau$. So $\pi_{|X} \leqslant \mu$ since μ is the strongest locally convex topology on X which is weaker than τ. Now we show that $\mu \leqslant \pi_{|X}$. By density of $\Delta(\bar{L}^\infty)$ in (X, μ) ($\mu \leqslant \tau$ and $\Delta(\bar{L}^\infty)$ is dense in X) it suffices to show that $\mu_{|\Delta(\bar{L}^\infty)} \leqslant \pi_{|\Delta(\bar{L}^\infty)}$. To see this it is enough to establish that $\Delta(\bar{L}^\infty) \cap U_1(\psi_n) \subset 6 \operatorname{conv} V_n$ for all $n \in \mathbf{N}$. We show first that

$$\mathscr{S} \cap U_1(\psi_n) \subset \operatorname{conv} V_n \quad \text{for all } n \in \mathbf{N}. \tag{*}$$

Fix $f \in \mathscr{S} \cap U_1(\psi_n)$. Then by Corollary 2.7(a) it follows that

$$f = \sum_{i=1}^{n} \alpha_i \|f\|_{\Lambda_{\psi_n}(\bar{L}^\infty)} x_{t_i},$$

where $\sum_{i=1}^n \alpha_i = 1$, $\alpha_i \geqslant 0$, $x_{t_i} = \bar{\psi}_n(t_i) f_{t_i}$, $i = 1, \ldots, n$. Now, since $\operatorname{conv} V_n$ is solid and $x_{t_i} \in V_n$, $i = 1, \ldots, n$, and $f \leqslant \sum_{i=1}^n \alpha_i x_{t_i}$ it follows that $f \in \operatorname{conv} V_n$. Thus the inclusion $\Delta(\bar{L}^\infty) \cap U_1(\psi_n) \subset 6 \operatorname{conv} V_n$ follows by (*), Lemma 2.2 and Proposition 4.1 (cf. the proof of Theorem 4.2). Finally, $\mu = \pi_{|X}$. Since E is an F-space and the density of $\Delta(\bar{L}^\infty)$ in $\Lambda_{\psi_n}(\bar{L}^\infty)$ for all $n \in \mathbf{N}$ implies that X is a dense subspace of E, we have $\hat{X} = E$.

If V is a bounded concave neighbourhood of zero in X, then obviously $\hat{X} = \Lambda_{\psi_V}(\bar{L}^\infty)$ from the above, and the proof is finished.

Now we give an example showing that in general the Mackey completion of a locally concave F-space is not locally bounded. It follows that in Theorem 4.1 the assumption that X is locally bounded is essential.

EXAMPLE. Let $\psi_n(t) = \min(1, t^{\alpha_n})$ for $t > 0$ and $n \in \mathbf{N}$, where $\alpha_1 < \alpha_2 < \ldots$, and $\alpha_n \in (0, 1)$. Let $X_n = \Lambda_{\psi_n}(\bar{L}^\infty)$ for $n \in \mathbf{N}$. Then the space

$$X = \bigcap_{n\in\mathbf{N}} X_n$$

equipped with the natural projective topology π is locally concave (by Proposition 2.6 and Corollary 2.7(b)), with $\Delta(\bar{L}^\infty)$ a dense subspace, by density of $\Delta(\bar{L}^\infty)$ in X_n for all $n \in \mathbf{N}$. Since X is a Fréchet space, it is a Mackey space. Therefore the Mackey completion of X is (X, π). We have $\psi_n(t) \geqslant \psi_{n+1}(t)$ for every $t > 0$ and $n \in \mathbf{N}$, so $X_{n+1} \subsetneq X_n$. Hence (X, π) is not locally bounded, otherwise there exists $n_0 \in \mathbf{N}$ such that for some $\lambda_n > 0$, $\|x\|_{X_n} \leqslant \lambda_n \|x\|_{X_{n_0}}$ for all $x \in X$ and $n > n_0$, $n \in \mathbf{N}$. In consequence the norms of X_{n_0} and X_n are equivalent on X for $n > n_0$. Since $\|f_t\|_{X_n'}^{-1} \approx \psi_n(t)$, $n \in \mathbf{N}$ (by Corollary 2.7(b)) and ψ_n, ψ_{n+1} are not equivalent, we obtain a contradiction.

Now we give applications of our results. Recall the definition of a *real interpolation space*. Let $\bar{A}$ be a couple of normed spaces. For any F^*-lattice (p-normed lattice, quasinormed lattice) $E = (E, \|\cdot\|_E)$ intermediate with respect to $\bar{L}^\infty$, we define the real interpolation space $\bar{A}_E$ to consist of all $a \in \Sigma(\bar{A})$ such that $K(\cdot, a; \bar{A}) \in E$. $\bar{A}_E$ is an F^*-space (p-normed space, quasinormed space) with F-norm (p-norm, quasinorm) defined by

$$\|a\|_{\bar{A}_E} = \|K(\cdot, a; \bar{A})\|_E.$$

We say that a Banach couple $\bar{A} = (A_0, A_1)$ is *mutually closed* if $\bar{A}_{L^\infty} = A_0$ and $\bar{A}_{L_1^\infty} = A_1$ isometrically.

We give an example of E with $\bar{L}_E^\infty$ locally concave. Namely, let φ be a positive and concave function on $\mathbf{R}_+$ such that $\varphi(0) = 0$ and let a positive function $w \in L^0(\mathbf{R}_+, dt/t)$ be such that

$$\int_{\mathbf{R}_+} \varphi(\min(1, t)/w(t))\,dt/t < \infty.$$

Put

$$\tilde{L}_{\varphi,w} = \{f \in \Sigma(\bar{L}^\infty)\colon\ \|f\| = \int_{\mathbf{R}_+} \varphi(\tilde{f}(t)/w(t))\,dt/t < \infty\},$$

and write $\bar{A}_{w,\varphi}$ instead of $\bar{A}_E$ for $E = \tilde{L}_{\varphi,w}$. Observe that for $\varphi(t) = t^p$, $0 < p \leqslant 1$, $w(t) = t^\theta$, $0 < \theta < 1$, we obtain the $\bar{A}_{\theta,p}$ space of Lions–Peetre (see [3]).

If $\bar{A}$ is a Banach couple, then by $\Sigma(\bar{A})^0$ we denote the closure of $\Delta(\bar{A})$ in $\Sigma(\bar{A})$. An easy proof of the following proposition may be omitted.

PROPOSITION 4.3. *Let $E = (\tilde{L}_{\varphi,w}, \|\cdot\|)$. Then*

(a) *E is a locally concave F-lattice with the topology τ defined by the F-norm $\|\cdot\|$.*

(b) *$\Delta(\bar{L}^\infty)$ is a dense subspace of (E, τ) if and only if $E \subset \Sigma(\bar{L}^\infty)^0$.*

(c) *If there exists a constant $C > 2$ such that*

$$2\varphi(t) \leqslant \varphi(Ct) \quad \textit{for all } t > 0,$$

then $U = \{f \in E\colon\ \|f\| \leqslant 1\}$ is a bounded concave neighbourhood of zero in (E, τ) and the topology τ is generated by the monotone quasinorm

$$|f| = \inf\{\lambda > 0\colon\ \|f/\lambda\| \leqslant 1\}.$$

Moreover, if $E \subset \Sigma^0$, then the Mackey completion of E is $\hat{E} = \Lambda_\psi(\bar{L}^\infty)$, where $\psi(t) = |f_t|^{-1}$ for $t > 0$.

Remark. For any Banach couple $\bar{A}$, we have $a \in \Sigma(\bar{A})^0$ if and only if $\min(1, 1/t)K(t, a; \bar{A}) \to 0$ as $t \to 0, \infty$ (see [3, Chap. 3]). Hence $E \subset \Sigma(\bar{L}^\infty)^0$ is equivalent to

$$(*) \qquad \min(1, 1/t)\tilde{f}(t) \to 0 \quad \text{as } t \to 0, \infty \quad \text{for all } f \in E.$$

If $\min(1, 1/t)\,\bar{\psi}_E(t) \to 0$ as $t \to 0, \infty$, then $(*)$ holds, since $\tilde{f}(t) \leqslant \bar{\psi}_E(t)\|f\|_E$ for all $f \in E$ and $t > 0$.

In [13] the following theorem was shown:

THEOREM 4.4. *Let $\bar{A}$ be a Banach couple and let E be a quasinormed lattice which is an exact interpolation space with respect to $\bar{L}^\infty$ such that $\Delta(\bar{L}^\infty)$ is a dense subspace of E. Then the Mackey completion of $\bar{A}_E$ is $\bar{A}_{\hat{E}}$.*

In the sequel $(E, \|\cdot\|)$ is a quasinormed lattice as in Theorem 4.4.

COROLLARY 4.5. *Let $\bar{A}$ be a mutually closed Banach couple and let E satisfy the condition $(*)$ of Proposition 3.2(b). Then the Mackey completion of $\bar{A}_E$ is $\Lambda_\psi(\bar{A})$, where $\psi(t) = \|f_t\|^{-1}$ for $t > 0$.*

Proof. Since $\bar{A}$ is a mutually closed Banach couple, $\bar{A}_{\Lambda_\psi} = \Lambda_\psi(\bar{A})$ by Theorem 12.1 of [4] (see also Example 4.7(i) of [16]), so our statement follows by Proposition 3.2 and Theorems 4.2 and 4.4.

For a positive function f defined on $\mathbf{R}_+$ let $M_f(t) = \sup\{f(st)/f(s)\colon\ s \in \mathbf{R}_+\}$.

COROLLARY 4.6. *Let ψ be a quasiconcave function such that $\min(1, 1/t)M_\psi(t) \to 0$ as $t \to 0, \infty$ and let $\varphi \in \mathscr{P}_c$ be such that $M_\varphi(t) \to 0$ as $t \to 0$. Then for every Banach couple $\bar{A}$ the Mackey completion of $\bar{A}_{\psi,\varphi}$ is $\bar{A}_{\psi,1}$.*

Proof. Consider the space $\tilde{L}_{\varphi,\psi}$ with the quasinorm $|\cdot|$ defined in Proposition 4.3(c). Applying Lemmas 1.4 and 1.5 of [12] it is easy to check that $|f_t|^{-1} \approx \psi(t)$. Since $M_\varphi(t) \to 0$, $2\varphi(t) \leqslant \varphi(Ct)$ for some $C > 2$ and all $t > 0$. Thus by the above remark, $E = \tilde{L}_{\varphi,\psi} \subset \Sigma(\bar{L}^\infty)^0$. In consequence $\hat{E} = \Lambda_\psi$ by Proposition 4.3. Since $\Lambda_\psi = \bar{L}_{\psi,1}^\infty$, the Mackey completion of $\bar{A}_E$ is $\bar{A}_{\psi,1}$ by Theorem 4.4, and the proof is complete.

Recall that any concave function φ generates the *symmetric Lorentz space* $\Lambda(\varphi)$ on $\mathbf{R}_+$, defined by

$$\Lambda(\varphi)=\{f\in L^0(\mathbf{R}_+,m):\ \|f\|_{\Lambda(\varphi)}=\int_{\mathbf{R}_+} f^*(s)\,d\varphi(s)<\infty\},$$

where f^* is the nonincreasing rearrangement of f with respect to the Lebesgue measure m. The symmetric Lorentz spaces are important in the theory of interpolation of linear operators in symmetric spaces (see [12] for more details).

Consider the Banach couple $(L^1, L^\infty)=(L^1(\mathbf{R}_+), L^\infty(\mathbf{R}_+))$, where $\mathbf{R}_+$ is equipped with the Lebesgue measure. As an application of Corollary 4.5 we obtain

COROLLARY 4.7. *If E satisfies the condition $(*)$ of Proposition* 3.2(b) *and $\psi(t)=\|f_t\|^{-1}$ for $t>0$, then the Mackey completion of $(L^1, L^\infty)_E$ is the Lorentz space $\Lambda(\tilde{\psi}_*)$, where $\psi_*(t)=t/\psi(t)$ for $t>0$.*

Proof. Since for any quasiconcave function ψ, we have $\tilde{\psi}\approx\psi$ (see [12, p. 70]), thus $\Lambda_\psi(L^1, L^\infty)=\Lambda_{\tilde{\psi}}(L^1, L^\infty)$. Since $\Lambda(\tilde{\psi}_*)=\Lambda_{\tilde{\psi}}(L^1, L^\infty)$ (see [7]) and (L^1, L^∞) is mutually closed, Corollary 4.5 applies.

Remark. If $\bar{A}$ is a Banach couple such that $K(t, b;\ \bar{A})\leqslant K(t, a;\ \bar{A})$ for all $t>0$ implies that there exists $T\in\mathscr{L}(\bar{A})$ with $b=Ta$ (such a couple is called a *Calderón couple*) and A is a p-Banach interpolation space with respect to $\bar{A}$, then $A=\bar{A}_E$ for some interpolation p-Banach lattice E with respect to $\bar{L}^\infty$ (see [16]). The Banach couple (L^1, L^∞) is well known to be a Calderón couple.

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