STUDIA MATHEMATICA, T. LXXXVII. (1987)

# Ultrapowers of unbounded selfadjoint operators

by

ANDRZEJ KRUPA and BOGDAN ZAWISZA (Warszawa)

**Abstract.** The paper extends the notion of ultrapower of an operator to the case of unbounded selfadjoint operators and examines it. The results are applied to the study of almost periodic Schrödinger operators $-\Delta+V$ in Hilbert spaces of almost periodic functions.

**Introduction.** The study of ultrapowers of unbounded operators which we present here arose from the search for an "abstract framework" for the analysis of Schrödinger operators in nonseparable Hilbert spaces [17]. There are various Hilbert function spaces $\mathfrak{H}$, different from $L^2$, where the formal expression $L=-\Delta+V$ defines a selfadjoint operator $L_{\mathfrak{H}}$. Such operators appear for instance in quantum theories of crystals of disordered systems, where $V$ is almost periodic (a.p.) and $\mathfrak{H}=B^2(\boldsymbol{R}^n)$, the space of Besicovitch a.p. functions (see Burnat [5–9], Romerio [27], Shubin [28, 29], Chojnacki [10]; the papers of Shubin treat a more general case of pseudodifferential operators with spatially a.p. symbols). Other nonseparable Hilbert spaces such as $B^2(\boldsymbol{R}^k)\otimes L^2(\boldsymbol{R}^l)$ appear in the papers of Burnat [8, 9] and Herczyński [14–17]. This "nonseparable approach" is motivated by the belief that examining the operator (and its spectral resolution) in various spaces should provide a more complete information about its properties than the sole $L^2$-analysis. One can prove, for instance, that the Bloch waves (generalized eigenfunctions) for $-\Delta+V$, with $V$ periodic, constitute a complete system of eigenvectors of the corresponding operator in $B^2(\boldsymbol{R}^n)$ (see Burnat [5–7]; more on the results and motivations for the "nonseparable analysis" of the Schrödinger operators can be found in [17]).

There are some common features of the operator $L$ in $L^2(\boldsymbol{R}^n)$ and its formal analogues $L_{\mathfrak{H}}$ in certain nonseparable spaces $\mathfrak{H}$, such as the equality of spectra (see [28] and [15]) or of the integral kernels of the same functions of $L$ and $L_{\mathfrak{H}}$ ([6], [18]), which suggest that the relation between $L$ and $L_{\mathfrak{H}}$ does not consist solely in the formal identity of the defining differential expression, but is of a functional-analytic nature. This intuition leads to the search for a more general scheme, which would allow to interpret this relation in terms of the abstract operator theory in Hilbert spaces. (The theory of Gelfand triples (cf. [13])—in view of which $L_{\mathfrak{H}}$ can be interpreted as a distributional extension of $L$, restricted to a certain subspace—is insufficient for this purpose since it does not define, in the spaces of distributions, any natural scalar product which would leave the extended operator symmetric.) Such a scheme can be found by means of the theory of ultrapowers.

An ultrapower of a Hilbert space is a particular case of a Banach space ultraproduct, introduced by D. Dacunha-Castelle and J. L. Krivine in [11] (1). The theory of ultraproducts, enriched with the concept of ultraproduct of a family of bounded operators (A. Pietsch [24]) was employed by numerous authors, e.g. in the study of operator ideals and of local properties of Banach spaces.

In this paper we introduce a new notion in the theory: a partial ultrapower (p.u.) of a linear (not necessarily bounded) operator in a Hilbert space, which can be regarded as a generalization of A. Pietsch's ultrapower (p.u.'s of a bounded operator are just the restrictions of its ultrapower to invariant subspaces). It turns out that, for certain pairs of Hilbert function spaces $H$ and $\mathfrak{H}$ (e.g. for $H = L^2(\boldsymbol{R}^n)$ and $\mathfrak{H} = B^2(\boldsymbol{R}^n)$), with two selfadjoint operators, $L_H$ in $H$ and $L_{\mathfrak{H}}$ in $\mathfrak{H}$, given by the same differential expression $L$, there exist isometrical embeddings $J$ of $\mathfrak{H}$ into certain ultrapowers of $H$ such that $JL_{\mathfrak{H}}J^{-1}$ is a partial ultrapower of $L_H$. The spectral theory of p.u.'s of an abstract selfadjoint operator and examples of its applications are the subject of this paper.

The paper is organized as follows. In the first chapter we recall the rudiments of the theory of bounded ultrapowers. In Chapter 2 we define a partial ultrapower of a selfadjoint operator and prove a theorem on the existence of a maximal partial ultrapower, called just ultrapower. The spectral properties of ultrapowers are analysed in Chapter 3: the functional calculus and spectral measures of ultrapowers are examined. The last chapter is devoted to the applications of our theory: without giving detailed proofs we describe the relations between the spectra of Schrödinger operators defined in various function spaces and formulate a theorem on integral kernels of functions of some almost periodic differential operators in $B^2(\boldsymbol{R}^n)$. Relations between Gelfand triples and some partial ultrapowers are also briefly discussed.

The paper is based on more general expositions given in Ph.D. dissertations of B. Zawisza [30] and A. Krupa [20] (a related approach, not employing the ultrapower theory, was proposed earlier by A. Krupa in [19]). Most of the results presented below were announced in [21].

(1) From the point of view of nonstandard analysis the Banach space ultraproduct coincides with the *nonstandard hull* of Banach spaces, constructed within a certain particular model of analysis (nonstandard hulls of normed spaces were introduced by W. A. J. Luxemburg in [22]; implicitly they appear already in A. Robinson's treatise [26]). This paper, however, exploits the explicit construction of Dacunha-Castelle and Krivine and can be read with no model-theoretical prerequisites.



We would like to express our gratitude to Professor M. Burnat of the Warsaw University, who inspired our search for an abstract version of nonseparable analysis of differential operators and whose valuable suggestions and constant encouragement were of much help in our work.

**1. Preliminaries.** We shall use the following notation. If $A$ is a linear operator in a Hilbert space $H$ and $\lambda \in \varrho(A)$ (the resolvent set) we put $R(\lambda, A) = (A-\lambda)^{-1}$. If $E$ is a linear subspace of $H$ then $A \upharpoonright E$ denotes the restriction of $A$ to $D(A) \cap E$. We say that $E$ is *invariant* for $A$ if $\operatorname{Ran}(A \upharpoonright E) \subset E$. If $E$ is closed then we say that $E$ *reduces* $A$ if $P_E A \subset AP_E$ where $P_E$ denotes the orthogonal projection onto $E$. $B(H)$ denotes the space of all bounded linear operators defined in $H$.

We recall that:

– $A \in B(H) \Rightarrow E$ reduces $A$ iff $E$ is invariant for both $A$ and $A^*$.

– $A$ is selfadjoint $\Rightarrow E$ reduces $A$ iff $E$ reduces $R(i, A)$, and then $A \upharpoonright E$ is selfadjoint in $E$ and $f(A \upharpoonright E) = f(A) \upharpoonright E$ for any Borel function $f\colon \sigma(A) \to \boldsymbol{C}$ ($\boldsymbol{C}$ is the complex field).

Now we recall the basic ideas of the theory of ultrapowers. Let $J$ be an infinite set. A nonempty family $\mathscr{F}$ of its subsets is called a *filter* iff

(i) $A \in \mathscr{F}, B \in \mathscr{F} \Rightarrow B \cap A \in \mathscr{F}$.

(ii) $A \in \mathscr{F}, B \supset A \Rightarrow B \in \mathscr{F}$.

(iii) $\emptyset \notin \mathscr{F}$.

Example: the Fréchet filter in the set $N$ of natural numbers – the family of all $A \subset N$ such that $N \setminus A$ is finite.

If $\mathscr{F}, \mathscr{F}'$ are filters in $J$ then we say that $\mathscr{F}'$ is *finer* than $\mathscr{F}$ iff $\mathscr{F}' \supset \mathscr{F}$.

A filter $\mathscr{U}$ is called an *ultrafilter* iff there are no filters finer than $\mathscr{U}$ (but $\mathscr{U}$ itself). For any filter $\mathscr{F}$ there exists an ultrafilter finer than $\mathscr{F}$. A filter $\mathscr{U}$ is an ultrafilter iff for any $A \subset J$ either $A \in \mathscr{U}$ or $\setminus A \in \mathscr{U}$.

Let now $X$ be a topological Hausdorff space, $x\colon J \to X$ a function and $\mathscr{F}$ a filter in $J$. $x_0 \in X$ is called an *$\mathscr{F}$-limit* of $x$ iff for any neighbourhood $\mathcal{O}$ of $x_0$, $x^{-1}(\mathcal{O}) \in \mathscr{F}$. An $\mathscr{F}$-limit is unique and we write

$$x_0 = \lim_{\mathscr{F}} x$$

(if $J = N$ and $\mathscr{F}$ is the Fréchet filter then $\lim_{\mathscr{F}} x = \lim_{n\to\infty} x(n)$). If $\mathscr{F}' \supset \mathscr{F}$ then $\lim_{\mathscr{F}'} x = \lim_{\mathscr{F}} x$ if the latter exists. If $X$ is compact and $\mathscr{U}$ is an ultrafilter then $\lim_{\mathscr{U}} x$ exists for any $x\colon J \to X$.

Let now $H$ be a Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$, let $J$ be a set of indices and $\mathscr{U}$ an ultrafilter in $J$. By $l^\infty(J, H)$ we denote the space of all (norm-) bounded functions $x\colon J \to H$ (we shall sometimes use the symbols $x_j$ instead of $x(j)$ and $(x_j)$ instead of $x$; then $\lim_{\mathscr{U}} x$ is replaced by $\lim_{\mathscr{U}} x_j$). Then $|||x||| \stackrel{\mathrm{df}}{=} \lim_{\mathscr{U}} \|x_j\|$ exists for any $x \in l^\infty(J, H)$, $|||\cdot|||$ is a seminorm and

$$c_{\mathscr{U}} = \{x \in l^\infty(J, H):\ \lim_{\mathscr{U}} \|x_j\| = 0\}$$

is its kernel. $(H)_{\mathscr{U}} = l^\infty(J, H)/c_{\mathscr{U}}$, the *ultrapower* of $H$ with respect to $\mathscr{U}$, is then a unitary space—and, in fact, a Hilbert space—with inner product

$$\langle [x], [y] \rangle = \lim_{\mathscr{U}} (x_j, y_j),$$

where $[x]$ denotes the equivalence class of $x \in l^\infty(J, H)$.

Let $A$: $H \to H$ be a bounded linear operator. We define the *ultrapower* $(A)_{\mathscr{U}}$ of $A$, $(A)_{\mathscr{U}}$: $(H)_{\mathscr{U}} \to (H)_{\mathscr{U}}$, by

$$(1.1)\qquad (A)_{\mathscr{U}}[(x_j)] = [(Ax_j)].$$

The correctness of the definition follows from the continuity of $A$. The following theorem holds:

(1.2) THEOREM. $(\cdot)_{\mathscr{U}}$ *is a $*$-isometrical isomorphism of the algebra $B(H)$ onto its range in $B((H)_{\mathscr{U}})$.* ∎

For the definitions and statements concerning ultrafilters and ultrapowers see [4], [24].

An ultrafilter is called *free* if it is not trivial, i.e. if $\bigcap_{A \in \mathscr{U}} A = \emptyset$. If $\mathscr{U}$ is trivial then $\bigcap_{A \in \mathscr{U}} A$ consists of exactly one point, say $j_0$, and then there exists a unitary mapping $U$: $H \to (H)_{\mathscr{U}}$ such that $UAU^{-1} = (A)_{\mathscr{U}}$ for any $A \in B(H)$ ($U$ is defined by the formula $Uh = [(h_j)]$, $h_j = h$, $j \in J$, and the inverse formula is $U^{-1}[(h_j)] = h_{j_0}$). For this reason we shall consider free ultrafilters only. Moreover, we shall assume that $J = \boldsymbol{N}$. The reason is that countable infinite sets are the smallest ones that admit nontrivial ultrafilters. It should be emphasized that the assumption $J = \boldsymbol{N}$ does not cause essential restrictions in our theory: all definitions and most of theorems (e.g. the theorems on functional calculus) remain valid for arbitrary $J$. We stress that if $\mathscr{U}$ is a free ultrafilter in $\boldsymbol{N}$ then it is finer then the Fréchet filter and thus $\lim_{\mathscr{U}} x_n = \lim_{n\to\infty} x_n$ whenever the latter exists.

**2. Definitions.** Let $H$ denote a Hilbert space, $A$ a selfadjoint (possibly unbounded) operator in $H$ and $\mathscr{U}$ a fixed free ultrafilter in $\boldsymbol{N}$. Extending the definition of ultrapower to the case of $A$ unbounded meets essential difficulties. First of all, the relation $[(x_j)] = 0$ does not imply that $[(Ax_j)] = 0$. Indeed, there exists a sequence $x_j \in D(A)$ such that $\lim_{\mathscr{U}} \|x_j\| = 0$, but $\|Ax_j\|$ is not even bounded. Thus Definition (1.1) would not be correct. However, we would like to preserve the property that $(A)_{\mathscr{U}}\alpha = [(Ax_j)]$ for some $(x_j)$ such that $[(x_j)] = \alpha$. This implies that the domain of $(A)_{\mathscr{U}}$ must be contained in the linear space

$$(2.1)\qquad \mathscr{D} = \{\alpha \in (H)_{\mathscr{U}}:\ \exists (x_n) \in \alpha:\ x_n \in D(A),\ n \in \boldsymbol{N},\ \text{and}\ (Ax_n) \in l^\infty(H)\},$$



which turns out not to be dense in $(H)_{\mathscr{U}}$. Reason: since $A$ is unbounded, $0 \in \sigma(R(i, A))$. Take $y_n$ such that $\|y_n\| = 1$ and $R(i, A)y_n \to 0$ as $n \to \infty$. Then

$$\lim_{\mathscr{U}} (x_n, y_n) = \lim_{\mathscr{U}} ((A+i)x_n, R(i, A)y_n) = 0$$

for any $(x_n)$ such that $((A+i)x_n)$ is norm-bounded. Thus $(A)_{\mathscr{U}}$ cannot be densely defined.

We first introduce the following notion.

(2.2) DEFINITION. Let $\mathscr{H} \subset (H)_{\mathscr{U}}$ be a closed linear subspace and let $\mathscr{A}$ be a densely defined linear operator in $\mathscr{H}$. Then $\mathscr{A}$ is called a *partial ultrapower* (*p.u.*) of $A$ in $\mathscr{H}$ if

$$(2.3)\qquad \forall \alpha \in D(\mathscr{A})\ \exists (x_n) \in \alpha:\quad x_n \in D(A),\ n \in \boldsymbol{N},\ \text{and}\ \mathscr{A}\alpha = [(Ax_n)].$$

All selfadjoint operators have partial ultrapowers; here are the simplest examples:

(2.4) *Trivial p.u.*: Let $V$: $H \to (H)_{\mathscr{U}}$ be given by the formula $Vx = [(x_n)]$, where $x_n = x$, $n \in \boldsymbol{N}$. Then $V$ maps $H$ isometrically onto its range $\mathscr{H}$, and $\mathscr{A} = VAV^{-1}$ is a p.u. of $A$ in $\mathscr{H}$.

(2.5) *One-dimensional p.u.*: Let $\lambda \in \sigma(A)$ and let $x_n \in D(A)$ be such that $\|x_n\| = 1$ and $Ax_n - \lambda x_n \to 0$. Put $\mathscr{H} \coloneqq \{a[(x_n)]:\ a \in \boldsymbol{C}\}$ and $\mathscr{A}$: $\mathscr{H} \to \mathscr{H}$, $\mathscr{A}\alpha = \lambda\alpha$. Then $\mathscr{A}$ is a p.u. of $A$ in $\mathscr{H}$ and $\lambda$ is its eigenvalue.

It is natural to ask to what extent partial ultrapowers are determined by the condition (2.3). The following theorem gives the precise answer.

(2.6) THEOREM. *Put*

$$(2.7)\qquad \mathscr{H}(A) = \bar{\mathscr{D}}.$$

*Then*:

(a) *There exists a (unique) linear operator $(A)_{\mathscr{U}}$ defined on $\mathscr{D}$ with values in $\mathscr{H}(A)$ such that for $\alpha \in \mathscr{D}$ and $\beta \in \mathscr{H}(A)$*

$$(A)_{\mathscr{U}}\alpha = \beta \Leftrightarrow \{\exists (x_n) \in \alpha:\ x_n \in D(A),\ n \in \boldsymbol{N},\ \text{and}\ \beta = [(Ax_n)]\}.$$

(b) *$(A)_{\mathscr{U}}$ is the maximal p.u. of $A$, i.e. if $\mathscr{H} \subset (H)_{\mathscr{U}}$ and $\mathscr{A}$ is a p.u. of $A$ in $\mathscr{H}$ then $D(\mathscr{A}) \subset \mathscr{D}$ and $\mathscr{A} = (A)_{\mathscr{U}} \upharpoonright D(\mathscr{A})$.*

(c) *$(A)_{\mathscr{U}}$ is selfadjoint in $\mathscr{H}(A)$, $\sigma((A)_{\mathscr{U}}) = \sigma(A)$, and $R(i, (A)_{\mathscr{U}}) = (R(i, A))_{\mathscr{U}} \upharpoonright \mathscr{H}(A)$.*

Proof. (a) The uniqueness is obvious. Put $R = R(i, A)$, $\mathscr{R} = (R)_{\mathscr{U}} \upharpoonright \mathscr{H}(A)$. Let us notice that

$$\mathscr{D} = \{\alpha \in (H)_{\mathscr{U}}:\ \exists (x_n) \in \alpha:\ x_n \in D(A),\ n \in \boldsymbol{N},\ \text{and}\ ((A+i)x_n) \in l^\infty(H)\}.$$

From this we derive that $\mathscr{D} = \operatorname{Ran}(R)_{\mathscr{U}}$. But $(R)_{\mathscr{U}}$ is normal (cf. (1.2)) so $\mathscr{H}(A) = (\ker(R)_{\mathscr{U}})^\perp$ and $\mathscr{H}(A)$ reduces $(R)_{\mathscr{U}}$. Hence $\mathscr{D} = \operatorname{Ran}\mathscr{R}$. Define

$$(A)_{\mathscr{U}}:\ \mathscr{D} \to \mathscr{H}(A),\qquad (A)_{\mathscr{U}} = \mathscr{R}^{-1} + i.$$

Let $\alpha\in\mathscr{D}$ and $\beta=(A)_{\mathscr{U}}\alpha$. Then $\mathscr{R}(\beta-i\alpha)=\alpha$, so for some $(x_n),(y_n)\in\alpha$ and $(z_n)\in\beta$ we have $Rz_n-iRy_n=x_n$, $n\in N$. Hence $z_n+i(x_n-y_n)=Ax_n$ and, finally, $[(Ax_n)]=[(z_n)]=\beta$. Conversely, let $\beta\in\mathscr{H}(A)$ and $\beta=[(Ax_n)]$ for some $[(x_n)]\in\mathscr{D}$. To prove that $\beta=(A)_{\mathscr{U}}[(x_n)]$ it suffices to show $\mathscr{R}(\beta-i[(x_n)])=[(x_n)]$, which is obvious.

(b) Let $\mathscr{A}$ be a p.u. of $A$ in $\mathscr{H}$. Then (2.3) implies that $D(\mathscr{A})\subset\mathscr{D}$ and $\mathscr{H}=\overline{D(\mathscr{A})}\subset\overline{\mathscr{D}}=\mathscr{H}(A)$. The assertion $\mathscr{A}\alpha=(A)_{\mathscr{U}}\alpha$ now follows from the characterization of $(A)_{\mathscr{U}}$ given in (a).

(c) Let $R$, $\mathscr{R}$ be as in (a). Then

$$\sigma(\mathscr{R})=\overline{\sigma((R)_{\mathscr{U}})\setminus\{0\}}=\overline{\sigma(R)\setminus\{0\}}=\sigma(R)$$

where the second equality comes from (1.2). Put $f(\lambda)=1/\lambda+i$; then $(A)_{\mathscr{U}}=f(\mathscr{R})$ and $A=f(R)$, which implies that $\sigma(A)=\sigma((A)_{\mathscr{U}})$. So, in particular, $\sigma((A)_{\mathscr{U}})\subset\boldsymbol{R}$ and $(A)_{\mathscr{U}}$, being a function of a normal operator, is normal itself, so $(A)_{\mathscr{U}}$ is selfadjoint. The remaining part of (c) is a reformulation of the definition of $(A)_{\mathscr{U}}$. ∎

(2.8) Definition. $(A)_{\mathscr{U}}$ defined in Theorem (2.6) is called the *ultrapower* of $A$ with respect to $\mathscr{U}$.

Notice that if $A\in B(H)$, then Definition (2.8) coincides with (1.1). We stress that $\mathscr{H}(A)\neq(H)_{\mathscr{U}}$ if $A$ is unbounded. It should also be noticed that, although $\sigma((A)_{\mathscr{U}})=\sigma(A)$, the character of the spectra can be entirely different. Example (2.5) shows for instance that the whole spectrum of $(A)_{\mathscr{U}}$ is covered by its eigenvalues, no matter whether $A$ has any eigenvalue or not.

Theorem (2.6) implies that all partial ultrapowers of $A$ are symmetric closable operators and that their closures are also p.u.'s of $A$. When are they selfadjoint? The general theory ensures that if a subspace $\mathscr{H}$ of $\mathscr{H}(A)$ reduces $(A)_{\mathscr{U}}$ then $(A)_{\mathscr{U}}\restriction\mathscr{H}$ is selfadjoint. It is not difficult to prove that the converse is also true:

(2.9) Proposition. *Let $\mathscr{A}$ be an essentially selfadjoint p.u. of $A$ in a subspace $\mathscr{H}$ of $(H)_{\mathscr{U}}$. Then $\mathscr{H}\subset\mathscr{H}(A)$, $\mathscr{H}$ reduces $(A)_{\mathscr{U}}$ and $\bar{\mathscr{A}}=(A)_{\mathscr{U}}\restriction\mathscr{H}$. ($\bar{\mathscr{A}}$ denotes the unique selfadjoint extension of $\mathscr{A}$.)*

Proof. The assertion $\mathscr{H}\subset\mathscr{H}(A)$ is a part of Theorem (2.6). Since $\mathscr{A}$ satisfies (2.3), we have $(R(\pm i,A))_{\mathscr{U}}(\mathscr{A}\mp i)\alpha=\alpha$ for all $\alpha\in D(\mathscr{A})$. Thus

$$(R(\pm i,A))_{\mathscr{U}}(\operatorname{Ran}(\mathscr{A}\mp i))\subset\mathscr{H}.$$

But $\operatorname{Ran}(\mathscr{A}\mp i)$ is assumed to be dense in $\mathscr{H}$, so $(R(\pm i,A))_{\mathscr{U}}(\mathscr{H})\subset\mathscr{H}$, which implies that $\mathscr{H}$ reduces

$$(R(i,A))_{\mathscr{U}}\restriction\mathscr{H}(A)=R(i,(A)_{\mathscr{U}}).$$

So $\mathscr{H}$ reduces $(A)_{\mathscr{U}}$. $(A)_{\mathscr{U}}\restriction\mathscr{H}$ is then a selfadjoint extension of $\mathscr{A}$ and so $(A)_{\mathscr{U}}\restriction\mathscr{H}=\bar{\mathscr{A}}$. ∎



**3. Functional calculus and spectral decomposition.** In the present chapter $H$ denotes again a Hilbert space and $\mathscr{U}$ a fixed free ultrafilter in $N$.

Let $A_1,\ldots,A_m$ be a family of normal operators (not necessarily bounded) in $H$. By their *joint spectrum* $\sigma(A_1,\ldots,A_m)$ we mean the set of all points $(\lambda_1,\ldots,\lambda_m)\in\boldsymbol{C}^m$ such that

$$\sum_{k=1}^{m}(A_k-\lambda_k)^*(A_k-\lambda_k)$$

has no inverse in $B(H)$. If we additionally assume that the $A_k$ are bounded and commute, then Theorem (1.2) implies that the $(A_k)_{\mathscr{U}}$ have the same properties, and moreover,

$$\sigma((A_1)_{\mathscr{U}},\ldots,(A_m)_{\mathscr{U}})=\sigma(A_1,\ldots,A_m),\tag{3.1}$$

$$f((A_1)_{\mathscr{U}},\ldots,(A_m)_{\mathscr{U}})=(f(A_1,\ldots,A_m))_{\mathscr{U}}\tag{3.2}$$

for any continuous function $f\colon\sigma(A_1,\ldots,A_m)\to\boldsymbol{C}$.

Do (3.1) and (3.2) hold for $A_k$ unbounded? It is obvious that both statements must be reformulated. First of all, $A_k$ must be selfadjoint and $f$ real-valued if unbounded (we have not defined an ultrapower of an unbounded nonselfadjoint operator). Further, $(A_k)_{\mathscr{U}}$, $k=1,\ldots,m$, and $(f(A_1,\ldots,A_m))_{\mathscr{U}}$ are now defined in different spaces. So in order to give the proper restatement of (3.1)–(3.2) we should examine in detail the relations between ultrapowers of a family of commuting selfadjoint operators.

Let $A$ be a selfadjoint operator in $H$ and $B\in B(H)$. We say that $A$ and $B$ *commute* if $BA\subset AB$. It is well known that the above condition is equivalent to $B$ and $R(\lambda,A)$ commuting for some (and hence for all) $\lambda\in\varrho(A)$. Another equivalent condition is that $BE(\Delta)=E(\Delta)B$, where $E$ denotes the projection-valued measure determined by $A$ and $\Delta$ runs over all Borel subsets of $\boldsymbol{R}$ (we shall always consider $E$ to be defined everywhere, assuming $E(\Delta)=0$ when $\Delta\cap\sigma(A)=\emptyset$). We say that two selfadjoint operators $A_1$, $A_2$ *commute* if so do $A_k$ and $R(i,A_j)$, $j,k=1,2$; $k\neq j$. It is obvious that if $B_1,B_2\in B(H)$ commute then so do $(B_1)_{\mathscr{U}}$ and $(B_2)_{\mathscr{U}}$. So if $A$ is selfadjoint and $B\in B(H)$ commutes with $A$ then $(B)_{\mathscr{U}}$ and $(R(i,A))_{\mathscr{U}}$ commute. Denote by $\mathscr{E}$ the spectral measure of $(R(i,A))_{\mathscr{U}}$. Then $\mathscr{H}(A)=\operatorname{Ran}\mathscr{E}(\boldsymbol{R}\setminus\{0\})$, so $\mathscr{H}(A)$ reduces $(B)_{\mathscr{U}}$ and $(B)_{\mathscr{U}}\restriction\mathscr{H}(A)$ commutes with $(R(i,A))_{\mathscr{U}}\restriction\mathscr{H}(A)=R(i,(A)_{\mathscr{U}})$. Thus we have proved:

(3.3) Proposition. *Let $A$ be selfadjoint and suppose $B\in B(H)$ commutes with $A$. Then $\mathscr{H}(A)$ reduces $(B)_{\mathscr{U}}$ and $(B)_{\mathscr{U}}\restriction\mathscr{H}(A)$ commutes with $(A)_{\mathscr{U}}$.* ∎

Taking for $B$ the resolvents of $A_k$, $k=1,\ldots,m$, and reasoning by induction we obtain

(3.4) Corollary. *Let $A_1, \ldots, A_m$ be a family of commuting selfadjoint operators in $H$. Then*

$$\mathscr{H} = \bigcap_{k=1}^{m} \mathscr{H}(A_k)$$

*reduces each $(A_k)_{\mathscr{U}}$ and $\mathscr{A}_k = (A_k)_{\mathscr{U}} \upharpoonright \mathscr{H}$, $k = 1, \ldots, m$, form a family of commuting selfadjoint operators in $\mathscr{H}$.* ∎

Now we can give the proper equivalent of (3.1)–(3.2):

(3.5) Theorem. *Let $A_1, \ldots, A_m, \mathscr{H}, \mathscr{A}_1, \ldots, \mathscr{A}_m$ be as in Corollary (3.4). Then $\sigma(\mathscr{A}_1, \ldots, \mathscr{A}_m) = \sigma(A_1, \ldots, A_m)$ and given any continuous function $f\colon \sigma(A_1, \ldots, A_m) \to \boldsymbol{C}$ ($f$ real-valued if unbounded) we have*

(i) $\mathscr{H} \subset \mathscr{H}(f(A_1, \ldots, A_m))$.

(ii) $\mathscr{H}$ *reduces* $(f(A_1, \ldots, A_m))_{\mathscr{U}}$.

(iii) $f(\mathscr{A}_1, \ldots, \mathscr{A}_m) = (f(A_1, \ldots, A_m))_{\mathscr{U}} \upharpoonright \mathscr{H}$.

Proof. The equality of spectra can be proved as in Theorem (2.6). For the sake of simplicity we shall now assume that $m = 1$, we put $A = A_1$ and then $\mathscr{H} = \mathscr{H}(A)$. The proof for $m$ arbitrary needs no essential modifications.

First let us note that for $f$ bounded (i) and (ii) have already been proved (Proposition (3.3)). So it remains to demonstrate (iii) for $f$ bounded and (i)–(iii) for $f$ unbounded and real-valued. This will be done in two steps.

*The first step*: we take $f$ with a finite limit at infinity:

$$\lim_{\lambda\to\infty} f(\lambda) = f(\infty).$$

Then

$$f(A) = (f \circ g)(R(i, A))$$

where $g(\lambda) = 1/\lambda + i$ is a homeomorphism of $\sigma(R(i, A))$ onto $\sigma(A) \cup \{\infty\}$. Then in view of (1.2)

$$(f(A))_{\mathscr{U}} = (f \circ g)((R(i, A))_{\mathscr{U}}).$$

Put $\mathscr{R} = (R(i, A))_{\mathscr{U}}$ and let $\mathscr{E}$ denote the spectral measure of $\mathscr{R}$. Then

$$(f(A))_{\mathscr{U}} = (f \circ g)(\mathscr{R})(1 - \mathscr{E}\{0\}) + f(\infty)\,\mathscr{E}\{0\}.$$

But $\mathscr{H} = \operatorname{Ran}(1 - \mathscr{E}\{0\})$, so

$$(f(A))_{\mathscr{U}} \upharpoonright \mathscr{H} = (f \circ g)(\mathscr{R} \upharpoonright \mathscr{H}) = (f \circ g)(R(i, (A)_{\mathscr{U}})) = f((A)_{\mathscr{U}}).$$

*The second step*: $f$ arbitrary.

With no loss of generality we can assume that $f$ is real-valued (even if bounded). For $M > 0$, let $\varphi_M$ be a continuous function, $\varphi_M\colon \boldsymbol{R} \to [0, 1]$, such that $\varphi_M(\lambda) = 1$ for $|\lambda| \leqslant M$ and $\varphi_M(\lambda) = 0$ for $|\lambda| \geqslant M+1$. Put

$$\mathscr{G} = \bigcup_{M>0} \operatorname{Ran} \varphi_M((A)_{\mathscr{U}}).$$



Then $\mathscr{G}$ is an essential domain of $f((A)_{\mathscr{U}})$. Take $\alpha = [(x_n)] \in \mathscr{G}$. Then, for some $M$, $\alpha = \varphi_M((A)_{\mathscr{U}})\alpha = (\varphi_M(A))_{\mathscr{U}}\alpha$ (the second equality comes from the first step). So we have $\alpha = [(y_n)]$, where $y_n = \varphi_M(A)\, x_n$. Now

$$\begin{aligned} f((A)_{\mathscr{U}})\alpha &= f((A)_{\mathscr{U}})\,\varphi_M((A)_{\mathscr{U}})\alpha = (f\varphi_M)((A)_{\mathscr{U}})\alpha \\ &= ((f\varphi_M)(A))_{\mathscr{U}}\alpha = [(f(A)\,\varphi_M(A)\, x_n)] = [(f(A)\, y_n)]. \end{aligned}$$

So $f((A)_{\mathscr{U}}) \upharpoonright \mathscr{G}$ is an essentially selfadjoint partial ultrapower of $f(A)$. The assertion now follows from Proposition (2.9). ∎

(3.6) Remark. Let $A_1, \ldots, A_m, \mathscr{H}, \mathscr{A}_1, \ldots, \mathscr{A}_m$ be as in Corollary (3.4) and let us additionally assume that $\sigma(A_1, \ldots, A_m)$ is not compact (i.e. at least one of $A_1, \ldots, A_m$ is unbounded). Let $f\colon \sigma(A_1, \ldots, A_m) \to \boldsymbol{R}$ be a continuous function such that

$$\lim_{|\lambda|\to\infty} f(\lambda) = +\infty.$$

Then $\mathscr{H} = \mathscr{H}(f(A_1, \ldots, A_m))$ and thus $(f(A_1, \ldots, A_m))_{\mathscr{U}} = f(\mathscr{A}_1, \ldots, \mathscr{A}_m)$.

Proof. Let $B = f(A_1, \ldots, A_m)$. We have to prove that $\mathscr{H}(B) \subset \mathscr{H}$, i.e.

$$\ker(R(i, B))_{\mathscr{U}} \supset \bigcup_{k=1}^{m} \ker(R(i, A_k))_{\mathscr{U}},$$

i.e. $\lim_{\mathscr{U}}(R(i, B)\, x_j) = 0$ for any bounded sequence $(x_j)$ such that $\exists\, k \in \{1, \ldots, m\}$: $\lim_{\mathscr{U}}(R(i, A_k)\, x_j) = 0$.

Let $(x_j)$ and $k$ be as above. Let $E$ be the joint spectral projector family for $(A_1, \ldots, A_m)$, $E_M = E(\{\lambda \in \boldsymbol{R}^m\colon |\lambda| \leqslant M\})$ and $S_M = (A_k - i)\, E_M$. Then $S_M$ is bounded and $E_M = S_M R(i, A_k)$. Let

$$g(M) = \|(1 - E_M)\, R(i, B)\|^2 = \sup\left\{\frac{1}{1 + f^2(\lambda)}\colon |\lambda| \geqslant M,\ \lambda \in \sigma(A_1, \ldots, A_m)\right\}.$$

Then for $j \in N$

$$\begin{aligned} \|R(i, B)\, x_j\|^2 &= \|R(i, B)\, E_M x_j\|^2 + \|R(i, B)(1 - E_M)\, x_j\|^2 \\ &\leqslant \|E_M x_j\|^2 + g(M)\,\|x_j\|^2 \\ &= \|S_M R(i, A_k)\, x_j\|^2 + g(M)\,\|x_j\|^2. \end{aligned}$$

Thus for any $M \geqslant 0$

$$\lim_{\mathscr{U}} \|R(i, B)\, x_j\|^2 \leqslant g(M) \lim_{\mathscr{U}} \|x_j\|^2 = g(M)\, \|[(x_j)]\|^2.$$

But $\inf g = 0$ so $\lim_{\mathscr{U}}(R(i, B)\, x_j) = 0$. ∎

(3.7) Corollary. *Let $A$ be a selfadjoint operator in $H$. Let $\mathscr{H} \subset (H)_{\mathscr{U}}$ and let $\mathscr{A}$ be a selfadjoint partial ultrapower of $A$ in $\mathscr{H}$. Then $\sigma(\mathscr{A}) \subset \sigma(A)$ and $f(\mathscr{A}) = f((A)_{\mathscr{U}}) \upharpoonright \mathscr{H}$ for any continuous function $f\colon \sigma(A) \to \boldsymbol{C}$ (real-valued if $f$ is unbounded).*

Proof. This is an immediate consequence of Theorem (3.5) and Proposition (2.9). ■

Corollary (3.7) will be exploited in Chapter 3 (applications). Here we note an interesting consequence concerning essential spectra. We say that $\lambda \in C$ belongs to the *discrete spectrum* of a selfadjoint operator $A$, $\lambda \in \sigma_{\text{disc}}(A)$, if $\lambda$ is an isolated point of $\sigma(A)$ and the corresponding eigenspace is finite-dimensional. The remaining $\lambda$'s constitute the *essential spectrum* of $A$: $\sigma_{\text{ess}}(A) = \sigma(A)\setminus\sigma_{\text{disc}}(A)$.

Let us return to Example (2.4). The space $\mathscr{H}$, consisting of all classes of constant sequences, can be naturally identified with $H$; we can then say that $A$ is a p.u. of itself. Hence, by virtue of (2.9), $H$ reduces $(A)_{\mathscr{U}}$. It is not difficult to compute the orthogonal projection $P_H$ of $(H)_{\mathscr{U}}$ onto $H$:

$$P_H[(x_n)] = w\text{-}\lim_{\mathscr{U}} x_n,$$

where $w\text{-}\lim_{\mathscr{U}} x_n$ denotes the limit with respect to $\mathscr{U}$ in the weak topology in $H$ (we recall that $\lim_{\mathscr{U}} a_n$ always exists if $a_n$'s vary in a compact set). Thus

$$H^{\perp} = \{[(x_n)]:\ w\text{-}\lim_{\mathscr{U}} x_n = 0\}.$$

Put $\mathscr{H}_0 = \mathscr{H}(A)\cap H^{\perp}$. Then obviously $\mathscr{H}_0$ reduces $(A)_{\mathscr{U}}$ and $(A)_{\mathscr{U}}\restriction\mathscr{H}_0$ is a selfadjoint p.u. of $A$ in $\mathscr{H}_0$; one may call it the *essential ultrapower* of $A$.

(3.8) PROPOSITION. *Let $A$ be a selfadjoint operator in $H$ and let $\mathscr{A} = (A)_{\mathscr{U}}\restriction\mathscr{H}_0$ where $\mathscr{H}_0 = \mathscr{H}(A)\cap H^{\perp}$. Then $\sigma(\mathscr{A}) = \sigma_{\text{ess}}(A)$.*

Proof. Let $\lambda \in \sigma_{\text{ess}}(A)$. Then there exist $x_n \in D(A)$, $n\in N$, such that $\|x_n\| = 1$, $(x_n, x_m) = 0$ for $n\neq m$ and $\|Ax_n - \lambda x_n\| \to 0$ as $n\to\infty$. Obviously

$$[(x_n)] \in D\big((A)_{\mathscr{U}}\big)\cap H^{\perp} = D(\mathscr{A}) \quad\text{and}\quad \mathscr{A}[(x_n)] = \lambda[(x_n)],$$

so $\lambda\in\sigma(\mathscr{A})$.

Now let $\lambda\in\sigma_{\text{disc}}(A)$. Denote by $E$ the spectral measure of $A$ and by $\mathscr{E}$ the spectral measure of $\mathscr{A}$. Then (3.7) yields $\mathscr{E}(\{\lambda\}) = \big(E(\{\lambda\})\big)_{\mathscr{U}}\restriction\mathscr{H}_0$. But for any $[(x_n)]\in\mathscr{H}_0$, $[(E(\{\lambda\})x_n)] = 0$, since $w\text{-}\lim_{\mathscr{U}} x_n = 0$ and $E(\{\lambda\})$ has finite-dimensional range. Thus $\mathscr{E}(\{\lambda\}) = 0$ and $\lambda\notin\sigma(\mathscr{A})$. ■

One may ask whether Theorem (3.5) holds for all Borel-measurable $f$. The answer is negative. Indeed, suppose that $A$ is a selfadjoint operator in $H$ and that $\lambda\in\sigma(A)$ is not an eigenvalue. Denote by $E$ the spectral measure of $A$ and by $\mathscr{E}$ the spectral measure of $(A)_{\mathscr{U}}$. Then $E(\{\lambda\}) = 0$, but $\mathscr{E}(\{\lambda\}) \neq 0$ since the whole spectrum of $(A)_{\mathscr{U}}$ is covered by eigenvalues.

Following this example one can show that if $\lambda$ is an accumulation point of $\sigma(A)$ and, at the same time, a point of discontinuity of $f$ then there exists an eigenvector $\alpha$ of $(A)_{\mathscr{U}}$ ($(A)_{\mathscr{U}}\alpha = \lambda\alpha$) such that either $\alpha\notin D\big((f(A))_{\mathscr{U}}\big)$ or $(f(A))_{\mathscr{U}}\alpha \neq f\big((A)_{\mathscr{U}}\big)\alpha$. This leads to a supposition that the failure of (3.5) for $f$ discontinuous is caused by the existence of too many eigenvectors of $(A)_{\mathscr{U}}$ and that

$$(f(A))_{\mathscr{U}}\restriction\mathscr{H}_c = f\big((A)_{\mathscr{U}}\restriction\mathscr{H}_c\big)$$

where $\mathscr{H}_c$ consists of all elements of $\mathscr{H}(A)$ which are orthogonal to all eigenvectors of $(A)_{\mathscr{U}}$.



The last equality, although in general false, holds for $f$'s whose points of discontinuity form a countable set [2]. For the proof we shall need some notions and facts from measure theory.

Let $X$ be a separable metric space, $J$ a set of indices and $\{\mu_j\}_{j\in J}$ a uniformly bounded family of nonnegative Borel measures on $X$. Let $\mathscr{W}$ be an ultrafilter in $J$. We say that a measure $\mu$ on $X$ is the *weak $\mathscr{W}$-limit* of $\{\mu_j\}$ if

$$\lim_{\mathscr{W}}\int_X f\,d\mu_j = \int_X f\,d\mu$$

for $f\colon X\to C$ bounded and continuous.

(3.9) LEMMA. *Let $X, J, \mu_j$ and $\mathscr{W}$ be as above. Suppose that $\mu = w\text{-}\lim_{\mathscr{W}}\mu_j$. Then*

*(a) $\lim_{\mathscr{W}}\mu_j(G) \geqslant \mu(G)$ for any open subset $G$ of $X$.*

*(b) $\lim_{\mathscr{W}}\int_X f\,d\mu_j = \int_X f\,d\mu$ for any bounded Borel-measurable function $f\colon X\to C$ whose points of discontinuity form a $\mu$-null set.* ■

The proof can be found e.g. in [3] (to be exact: [3] treats ordinary limits and assumes that $\mu_j$'s are probabilistic; the above lemma can be proved similarly).

(3.10) THEOREM. *Let $\mathscr{H}_p$ denote the closed linear span of the set of all eigenvectors of $(A)_{\mathscr{U}}$ and let $\mathscr{H}_c = \mathscr{H}(A)\ominus\mathscr{H}_p$. Put $\mathscr{A} = (A)_{\mathscr{U}}\restriction\mathscr{H}_c$ and denote by $\mathscr{E}$ the spectral measure of $\mathscr{A}$. Let $f$ be a Borel function $f\colon\sigma(A)\to C$ ($f$ real-valued if unbounded) whose points of discontinuity form a set of measure $\mathscr{E}$-null (e.g. a countable set). Then $\mathscr{H}_c\subset\mathscr{H}(f(A))$, $\mathscr{H}_c$ reduces $(f(A))_{\mathscr{U}}$ and $f(\mathscr{A}) = (f(A))_{\mathscr{U}}\restriction\mathscr{H}_c$.*

Proof. With no loss of generality we can assume that $f$ is real.

Assume first that $f$ is bounded. Then Proposition (3.3) implies that $(f(A))_{\mathscr{U}}\restriction\mathscr{H}(A)$ commutes with $(A)_{\mathscr{U}}$, thus $(f(A))_{\mathscr{U}}(\mathscr{H}_p)\subset\mathscr{H}_p$. So, since $(f(A))_{\mathscr{U}}$ is selfadjoint, $\mathscr{H}_p$ reduces $(f(A))_{\mathscr{U}}$ and so does $\mathscr{H}_c$. Thus it suffices to prove that

$$\langle (f(A))_{\mathscr{U}}\alpha, \alpha\rangle = \langle f(\mathscr{A})\alpha, \alpha\rangle \quad\text{for any } \alpha\in\mathscr{H}_c.$$

So let us take $\alpha = [(x_n)]\in\mathscr{H}_c$. Denote by $E$ the spectral measure of $A$. We have

$$(*)\qquad \langle (g(A))_{\mathscr{U}}\alpha, \alpha\rangle = \lim_{\mathscr{U}}\big(g(A)x_n, x_n\big) = \lim_{\mathscr{U}}\int g(\lambda)\big(E(d\lambda)x_n, x_n\big),$$

$$(**)\qquad \langle g(\mathscr{A})\alpha, \alpha\rangle = \int g(\lambda)\langle\mathscr{E}(d\lambda)\alpha, \alpha\rangle$$

[2] So Theorem 4.3 of [21] is false as stated. Theorem (3.10) in this paper is its correct version.

for any Borel-measurable bounded function $g$. If we take $g$ continuous then Corollary (3.7) implies that the left-hand sides of (*) and (**) coincide. Hence $\langle\mathscr{E}(\cdot)\alpha, \alpha\rangle = w\text{-}\lim_{\mathscr{U}}(E(\cdot)x_n, x_n)$. Thus by Lemma (3.9)(b) the right-hand sides of (*) and (**), with $g$ replaced by $f$, are equal, so $\langle f(\mathscr{A})\alpha, \alpha\rangle = \langle(f(A))_{\mathscr{U}}\alpha, \alpha\rangle$.

Let now $f$ be arbitrary as in the assumptions. Let

$$F_n = \{\lambda\in\sigma(A): |f(\lambda)| \geqslant n\}, \quad G_n = \{\lambda\in\sigma(A): \operatorname{dist}(\lambda, F_n) < 1/n\},$$

$$\mathscr{G} = \bigcup_{n=1}^{\infty} \operatorname{Ran}\mathscr{E}(\setminus G_n).$$

No point of continuity of $f$ belongs to $\bigcap_{n=1}^{\infty} G_n$, which implies that $\mathscr{E}(\bigcap_{n=1}^{\infty} G_n) = 0$, so $\mathscr{G}$ is dense in $\mathscr{H}_c$ and serves as an essential domain for $f(\mathscr{A})$. Let $\varphi_n$ be a continuous function, $\varphi_n$: $\sigma(A)\to[0, 1]$, such that $\varphi_n(\lambda) = 0$ for $\lambda\in F_n$ and $\varphi_n(\lambda) = 1$ for $\lambda\in G_n$, $n = 1, 2, \ldots$ Then arguing as in the proof of Theorem (3.5) we show that $f(\mathscr{A})\restriction\mathscr{G}$ is an essentially selfadjoint p.u. of $f(A)$, which ends the proof. ■

We have already mentioned that the ultrapower of the spectral measure of a selfadjoint operator need not coincide with the spectral measure of its ultrapower. We can ask, however, whether there exists a simple formula which expresses the spectral measure $\mathscr{E}(\Delta)$ of an ultrapower in terms of the ultrapowers of spectral projectors of the "initial" operator. The answer is affirmative for $\Delta$ closed (and thus for $\Delta$ open too).

(3.11) Proposition. *Let $A$ be a selfadjoint operator in $H$, $E$ its spectral measure and $\mathscr{E}$ the spectral measure of $(A)_{\mathscr{U}}$. Then for any closed subset $F$ of $\boldsymbol{R}$ we have*

$$(*) \qquad \mathscr{E}(F) = s\text{-}\lim_{k\to\infty}(E(G_k))_{\mathscr{U}}\restriction\mathscr{H}(A)$$

*where $G_k$ is any decreasing sequence of open neighbourhoods of $F$ such that $\bigcap_{k=1}^{\infty}\bar{G}_k = F$.*

Proof. First let us notice that ultrapowering is monotonous, i.e. for any $B_1, B_2\in B(H)$, selfadjoint and such that $(B_1x, x)\leqslant(B_2x, x)$ for all $x\in H$, we have $\langle(B_1)_{\mathscr{U}}\alpha, \alpha\rangle \leqslant \langle(B_2)_{\mathscr{U}}\alpha, \alpha\rangle$ for all $\alpha\in(H)_{\mathscr{U}}$. Thus the $(E(G_k))_{\mathscr{U}}$ form a decreasing sequence of selfadjoint projections, so $s\text{-}\lim_{k\to\infty}(E(G_k))_{\mathscr{U}}$ exists and defines a selfadjoint projection in $(H)_{\mathscr{U}}$. By virtue of (3.3), $\mathscr{H}(A)$ reduces $(E(G_k))_{\mathscr{U}}$, so in order to prove (*) it suffices to verify

$$(**) \qquad \langle\mathscr{E}(F)\alpha, \alpha\rangle = \lim_{k\to\infty}\langle(E(G_k))_{\mathscr{U}}\alpha, \alpha\rangle \quad \text{for all } \alpha\in\mathscr{H}(A).$$

Let $\alpha\in\mathscr{H}(A)$, $\alpha = [(x_n)]$. As in the proof of Theorem (3.10) we show that $\langle\mathscr{E}(\cdot)\alpha, \alpha\rangle = w\text{-}\lim_{\mathscr{U}}\langle E(\cdot)x_n, x_n\rangle$. So, Lemma (3.9)(a) yields for each $k\in N$

$$\langle(E(G_k))_{\mathscr{U}}\alpha, \alpha\rangle = \lim_{\mathscr{U}}(E(G_k)x_n, x_n) \geqslant \langle\mathscr{E}(G_k)\alpha, \alpha\rangle$$



and similarly

$$\langle\mathscr{E}(\bar{G}_k)\alpha, \alpha\rangle \geqslant \langle(E(\bar{G}_k))_{\mathscr{U}}\alpha, \alpha\rangle.$$

But $(E(\bar{G}_k))_{\mathscr{U}} \geqslant (E(G_k))_{\mathscr{U}}$ since $(\cdot)_{\mathscr{U}}$ is monotonous; thus

$$\langle\mathscr{E}(\bar{G}_k)\alpha, \alpha\rangle \geqslant \langle(E(G_k))_{\mathscr{U}}\alpha, \alpha\rangle \geqslant \langle\mathscr{E}(F)\alpha, \alpha\rangle.$$

The left-hand side converges, according to the assumptions, to $\langle\mathscr{E}(F)\alpha, \alpha\rangle$, which proves (**). ■

Both (3.10) and (3.11) can be extended to the case of a family of commuting selfadjoint operators, as in Theorem (3.5).

Proposition (3.11) gives us, in particular, an explicit expression for the orthogonal projection on the space of all eigenvectors of $(A)_{\mathscr{U}}$ corresponding to a fixed $\lambda\in\sigma(A)$:

$$\mathscr{E}(\{\lambda\}) = s\text{-}\lim_{\varepsilon\downarrow 0}(E((\lambda-\varepsilon, \lambda+\varepsilon)))_{\mathscr{U}}.$$

This formula applied to $(R(-1, A^2))_{\mathscr{U}}$ and $\lambda = 0$ allows to compute the projection onto $\ker(R(-1, A^2))_{\mathscr{U}} = \mathscr{H}(A^2)^{\perp} = \mathscr{H}(A)^{\perp}$ (see Remark (3.6)), and thus also the projector onto $\mathscr{H}(A)$ itself:

$$P_{\mathscr{H}(A)} = s\text{-}\lim_{T\to\infty}(E([-T, T]))_{\mathscr{U}}.$$

It also shows that the family of projections $(E(\Delta))_{\mathscr{U}}$ is not only different from $\mathscr{E}(\Delta)$ but even does not define a strongly $\sigma$-additive measure:

$$s\text{-}\lim_{\varepsilon\downarrow 0}(E((\lambda-\varepsilon, \lambda+\varepsilon)))_{\mathscr{U}} = \mathscr{E}(\{\lambda\}),$$

which is, in general, larger than $(E(\{\lambda\}))_{\mathscr{U}}$ (see the example above).

**4. Applications.** In this chapter we demonstrate how ultrapowers and their properties can be used in the spectral analysis of differential operators. The main object of interest is the Schrödinger operator with an almost periodic potential; however, most of the results can be extended to a wider class of uniformly elliptic selfadjoint operators with almost periodic coefficients. First let us present the function spaces we are going to use.

Let $e_\lambda(x) = e^{i\lambda x}$, for $\lambda, x\in\boldsymbol{R}^k$ ($\lambda x$ denotes the scalar product of $\lambda$ and $x$), and write

$$\operatorname{Trig}(\boldsymbol{R}^k) = \Big\{\sum_{j=1}^{m} a_j e_{\lambda_j}:\ m\in\boldsymbol{N},\ a_j\in\boldsymbol{C},\ \lambda_j\in\boldsymbol{R}^k\Big\}.$$

The space of trigonometric polynomials can be embedded in various function spaces with translation-invariant norms; the closures of $\operatorname{Trig}(\boldsymbol{R}^k)$ with respect to these norms constitute spaces of almost periodic functions of various types. We have:

– $\operatorname{CAP}(\boldsymbol{R}^k)$ – the closure of $\operatorname{Trig}(\boldsymbol{R}^k)$ in $C_b(\boldsymbol{R}^k)$, the space of all bounded continuous complex functions on $\boldsymbol{R}^k$ with sup-norm. If $f\in\operatorname{CAP}(\boldsymbol{R}^k)$ satisfies

$\partial^\alpha f \in C_b(\mathbf{R}^k)$, $\alpha \in \mathbf{Z}_+^k$, then, in fact, $\partial^\alpha f \in \mathrm{CAP}(\mathbf{R}^k)$ for all multiindices $\alpha$. We then write $f \in \mathrm{CAP}^\infty(\mathbf{R}^k)$. Both CAP and $\mathrm{CAP}^\infty$ are closed under pointwise products: $f, g \in \mathrm{CAP}$ (resp. $\mathrm{CAP}^\infty$) implies $f \cdot g \in \mathrm{CAP}$ ($\mathrm{CAP}^\infty$).

$-\mathrm{S}^p\mathrm{AP}(\mathbf{R}^k)$ – the closure of $\mathrm{Trig}(\mathbf{R}^k)$ with respect to the uniform $L^p_{\mathrm{loc}}$-norm:

$$|||f|||_p = \sup_{x \in \mathbf{R}^k} \Big( \int_{|x-y| \leqslant 1} |f(y)|^p \, dy \Big)^{1/p}, \qquad 1 \leqslant p < +\infty .$$

$\mathrm{S}^p\mathrm{AP}$ is called the space of *Stepanov a.p. functions*; we have obviously $\mathrm{CAP} \subset \mathrm{S}^p\mathrm{AP}$. The Stepanov space admits also unbounded elements, e.g. periodic locally $L^p$-functions.

Let

$$\mathcal{M}^2(\mathbf{R}^k) = \Big\{ f \in L^2_{\mathrm{loc}}(\mathbf{R}^k)\colon \, \mathbf{|} f \mathbf{|} = \overline{\lim_{T \to \infty}} \Big( \frac{1}{\omega_k T^k} \int_{|x| \leqslant T} |f(x)|^2 \, dx \Big)^{1/2} < +\infty \Big\}$$

where $\omega_k$ denotes the volume of the unit ball in $\mathbf{R}^k$, and let

$$M^2(\mathbf{R}^k) = \mathcal{M}^2(\mathbf{R}^k) / \{ f\colon \mathbf{|} f \mathbf{|} = 0 \}.$$

$M^2(\mathbf{R}^k)$ is a Banach space with norm $\mathbf{|}\cdot\mathbf{|}$ (see [23]). For $f \in \mathcal{M}^2(\mathbf{R}^k)$ the equivalence class represented by $f$ is denoted by $[\![ f ]\!]$. The formula $\mathrm{CAP} \ni f \to [\![ f ]\!] \in M^2$ defines a continuous mapping, which is injective, since $\mathbf{|} f \mathbf{|} = 0$ implies $f = 0$ if $f \in \mathrm{CAP}$. Thus we shall sometimes treat CAP as a subspace of $M^2$; this also refers to subspaces of CAP: Trig and $\mathrm{CAP}^\infty$. The closure of $\mathrm{CAP}(\mathbf{R}^k)$ in $M^2(\mathbf{R}^k)$ is denoted by $B^2(\mathbf{R}^k)$ and called the *Besicovitch space*. $B^2(\mathbf{R}^k)$ is a Hilbert space with inner product

$$\langle [\![ f ]\!], [\![ g ]\!] \rangle = \lim_{T \to \infty} \frac{1}{\omega_k T^k} \int_{|x| \leqslant T} f(x) \overline{g(x)} \, dx;$$

the limit exists for all $[\![ f ]\!]$, $[\![ g ]\!]$ (see [2]). The classes $[\![ e_\lambda ]\!]$, $\lambda \in \mathbf{R}^k$, form an orthonormal basis in $B^2(\mathbf{R}^k)$, so $B^2(\mathbf{R}^k)$ is not separable.

Now, given $f \in \mathrm{CAP}^\infty(\mathbf{R}^k)$, we define

(4.1) $$\mathfrak{A}_0 [\![ f ]\!] = [\![ -\Delta f ]\!].$$

It is not difficult to verify that $\mathfrak{A}_0$ is a correctly defined symmetric operator; moreover, $\mathfrak{A}_0 [\![ e_\lambda ]\!] = |\lambda|^2 [\![ e_\lambda ]\!]$, so $\mathfrak{A}_0$ is essentially selfadjoint in $B^2(\mathbf{R}^k)$.

Let $p \geqslant 2$ and let $V \in \mathrm{S}^p\mathrm{AP}(\mathbf{R}^k)$ be real-valued. Then

(4.2) $$\mathcal{V} [\![ f ]\!] = [\![ Vf ]\!], \qquad f \in \mathrm{CAP}^\infty(\mathbf{R}^k),$$

is a well-defined symmetric operator in $B^2(\mathbf{R}^k)$. Thus $\mathfrak{A}_0 + \mathcal{V}$ is symmetric and so closable. We define

(4.3) $$\mathfrak{A} = \overline{\mathfrak{A}_0 + \mathcal{V}}.$$

$\mathfrak{A}$ is called the *Schrödinger operator* with potential $V$ in $B^2(\mathbf{R}^k)$ (see also [7]).

$-\Delta$ and $V$ also define selfadjoint operators in $L^2(\mathbf{R}^k)$ with $C_0^\infty(\mathbf{R}^k)$ (the space of smooth compactly supported functions) as their essential domains. We shall denote them simply by $-\Delta$ and $V$. If we assume $p = 2$ for $k \leqslant 3$ and $p > k/2$ for $k \geqslant 4$, then $V$ is $(-\Delta)$-bounded, with relative bound 0 (briefly $V \ll -\Delta$, see [25], vol. 4). By virtue of Kato–Rellich's theorem ([25], vol. 2), $-\Delta + V$ is selfadjoint.

Let now $[\![ f ]\!] \in B^2(\mathbf{R}^k)$ and put

$$f_n(x) = n^{-k/2} f(x) \varphi(x/n)$$

where $\varphi \in C_0^\infty$ is a fixed function such that $0 \leqslant \varphi \leqslant 1$, $\varphi(0) = 1$ and $\int \varphi^2 = 1$. We have

$$\|f_n\|_{L^2}^2 \leqslant n^{-k} \int_{x/n \in \operatorname{supp}\varphi} |f(x)|^2 \, dx$$

so $(f_n) \in l^\infty(L^2(\mathbf{R}^k))$ and, given a free ultrafilter $\mathcal{U}$ in $\mathbf{N}$,

(4.4) $$\mathcal{J}_{\mathcal{U}} [\![ f ]\!] = [(f_n)]$$

is a correctly defined continuous mapping, $\mathcal{J}_{\mathcal{U}}\colon B^2(\mathbf{R}^k) \to (L^2(\mathbf{R}^k))_{\mathcal{U}}$. In fact, $\mathcal{J}_{\mathcal{U}}$ is an isometry:

$$\langle \mathcal{J}_{\mathcal{U}} e_\lambda, \mathcal{J}_{\mathcal{U}} e_\mu \rangle = \lim_{\mathcal{U}} \frac{1}{n^k} \int e^{i(\lambda-\mu)x} \varphi^2(x/n) \, dx = \lim_{\mathcal{U}} \int e^{in(\lambda-\mu)x} \varphi^2(x) \, dx$$
$$= \lim_{\mathcal{U}} (\varphi^2)\hat{}\,(n(\mu-\lambda)) = \delta_\mu^\lambda,$$

where $\hat{}$ denotes the Fourier transform and $\delta_\mu^\lambda$ is the Kronecker symbol.

Put $\mathcal{H} = \operatorname{Ran} \mathcal{J}_{\mathcal{U}}$, $\mathcal{A}_0 = \mathcal{J}_{\mathcal{U}} \mathfrak{A}_0 \mathcal{J}_{\mathcal{U}}^{-1}$, $\mathcal{W} = \mathcal{J}_{\mathcal{U}} \mathcal{V} \mathcal{J}_{\mathcal{U}}^{-1}$, $\mathcal{A} = \mathcal{J}_{\mathcal{U}} \mathfrak{A} \mathcal{J}_{\mathcal{U}}^{-1}$ where $\mathfrak{A}_0$, $\mathcal{V}$, $\mathfrak{A}$ are defined by (4.1)–(4.3). $\mathcal{A}_0$ (resp. $\mathcal{A}$, $\mathcal{W}$) is easily seen to be a p.u. of $-\Delta$ (resp. $-\Delta + V$, $V$) in $\mathcal{H}$. This allows us to prove the following:

(4.5) Theorem. *Let $V \in \mathrm{S}^p\mathrm{AP}(\mathbf{R}^k)$ be real-valued, $p \geqslant 2$ if $k \leqslant 3$, $p > k/2$ if $k \geqslant 4$. Then the Schrödinger operator $\mathfrak{A}$ (with potential $V$) in $B^2(\mathbf{R}^k)$ is selfadjoint and $\sigma(\mathfrak{A}) = \sigma(-\Delta + V)$.*

Proof. Let $\mathcal{A}_0$, $\mathcal{W}$, $\mathcal{A}$ be as above. Then $\mathcal{A}_0$ is essentially selfadjoint, since so is $\mathfrak{A}_0$. Moreover, $\mathcal{W} \ll \mathcal{A}_0$, since $V \ll -\Delta$ and for $f \in \mathrm{CAP}^\infty$, $\alpha = [(f_n)]$, we have $\mathcal{W}\alpha = [(Vf_n)]$ and $\mathcal{A}_0 \alpha = [(-\Delta f_n)]$. So $\mathcal{A} = \mathcal{A}_0 + \mathcal{W}$ is selfadjoint, which proves the first statement and also yields (see Corollary (3.7))

$$\sigma(\mathfrak{A}) = \sigma(\mathcal{A}) \subset \sigma(-\Delta + V).$$

The converse inclusion can be proved in a similar manner. In [15] J. Herczyński proved (following an idea of Shubin [28]) that there exists a sequence of functions $\psi_n \in \mathrm{CAP}(\mathbf{R}^k)$ such that, given any $\varphi \in C_0^\infty(\mathbf{R}^k)$, we have

$\psi_n * \varphi \in \mathrm{CAP}^\infty(\boldsymbol{R}^k)$ and

$$\lim_{n\to\infty} |\!|\psi_n * \varphi|\!| = \|\varphi\|_{L^2}, \qquad \lim_{n\to\infty} |\!|V(\psi_n * \varphi) - \psi_n * (V\varphi)|\!| = 0.$$

This proves that, given any free ultrafilter $\mathscr{U}$ in $N$, there exists a unitary mapping $I_{V,\mathscr{U}}$: $L^2(\boldsymbol{R}^k) \to (B^2(\boldsymbol{R}^k))_{\mathscr{U}}$ such that $I_{V,\mathscr{U}}(-\Delta + V) I_{V,\mathscr{U}}^{-1}$ is a selfadjoint p.u. of $\mathfrak{A}$ (see also [30]). ■

Theorem (4.5) was first discovered by Burnat for the case $k = 3$ and $V$ periodic ([7]). For the case of strongly elliptic selfadjoint pseudodifferential operators in $\boldsymbol{R}^k$ it was proved by Shubin ([5]). Shubin's theorem applied to $-\Delta + V$ gives (4.5) for $V \in \mathrm{CAP}^\infty(\boldsymbol{R}^k)$; in the present form it was proved by Herczyński ([14]).

Let us notice that Ran $\mathscr{J}_{\mathscr{U}}$, where $\mathscr{J}_{\mathscr{U}}$ is defined by (4.4), is contained in $(L^2(\boldsymbol{R}^k))^\perp$, the space of sequences weakly converging to 0, so Proposition (3.8) together with Theorem (4.5) gives

(4.6) Corollary (cf. Avron–Simon [1]). *The spectrum of* $-\Delta + V$ *in* $L^2(\boldsymbol{R}^k)$, *with* $V$ *satisfying the assumptions of Theorem* (4.5), *is essential.* ■

Here are further applications of unbounded ultrapowers. We do not give detailed proofs – they can be found in [20] or [30].

"*Crystal layers*". Let $X$ be a Banach (Hilbert) space. As at the beginning of the present chapter, we define $S^p\mathrm{AP}(\boldsymbol{R}^k, X)$ $(B^2(\boldsymbol{R}^k, X))$, the spaces of a.p. functions with values in $X$ (cf. [14]). Let

$$X = \{f \in L^p_{\mathrm{loc}}(\boldsymbol{R}^l): \lim_{|x|\to\infty} \int_{|x-y|\leq 1} |f(y)|^p\, dy = 0, f \text{ real-valued}\},$$

be equipped with the norm $|\!|\!|\cdot|\!|\!|_p$ and let $V \in S^p\mathrm{AP}(\boldsymbol{R}^k, X)$, with $p = 2$ if $k + l \leq 3$ and $p > (k+l)/2$ if $k + l \geq 4$ (we always assume $k \geq 1$). Then, as in $B^2(\boldsymbol{R}^k)$ $(= B^2(\boldsymbol{R}^k, \boldsymbol{C}))$, we construct a symmetric operator $\mathfrak{A}$ in $B^2(\boldsymbol{R}^k, L^2(\boldsymbol{R}^l))$ corresponding to the formal Schrödinger operator $-\Delta + V$ (for the physical interpretation, see [14]). Denote the corresponding selfadjoint operator in $L^2(\boldsymbol{R}^{k+l})$ also by $-\Delta + V$. In the manner analogous to that just presented we prove that $\mathfrak{A}$ is unitarily equivalent to a certain selfadjoint p.u. of $-\Delta + V$ and conversely. Consequences: $\mathfrak{A}$ is selfadjoint, $\sigma(\mathfrak{A}) = \sigma(-\Delta + V)$, $\sigma_{\mathrm{disc}}(-\Delta + V) = \emptyset$.

*Spectral cutting*. A real function $\psi \in C^2(\boldsymbol{R}^l)$ is called a *spectral cut-off function* if it is a smoothened characteristic function of a subset of $\boldsymbol{R}^k$ containing an open cone (for details see [6]). Let $V \in \mathrm{CAP}(\boldsymbol{R}^k)$ be real-valued and let $\mathfrak{A}$ be the operator in $B^2(\boldsymbol{R}^k)$ defined by (4.3). Put $V_{\mathrm{cut}} = \psi V$. Then there exist a free ultrafilter $\mathscr{U}$, a closed subspace $\mathscr{H}$ of $(L^2(\boldsymbol{R}^k))_{\mathscr{U}}$ and a unitary mapping $J$: $B^2(\boldsymbol{R}^k) \to \mathscr{H}$ such that $J\mathfrak{A}J^{-1}$ is a selfadjoint p.u. of $-\Delta + V_{\mathrm{cut}}$ in $\mathscr{H}$. Consequence: $\sigma(-\Delta + V) \subset \sigma(-\Delta + V_{\mathrm{cut}})$. For the construction of $\mathscr{U}$ and $J$, see [30].

All results presented can be (and have been) proved without the theory of ultrapowers; however, once the theory is established, it provides concise and elegant proofs (which can be compared to profits gained by employing nonstandard analysis in the proofs of classical theorems). The machinery of the functional calculus of ultrapowers also yields results which seem to be new:



(4.7) Theorem. *Let* $A$ *and* $\mathfrak{A}$ *be selfadjoint operators in* $L^2(\boldsymbol{R}^k)$ *and* $B^2(\boldsymbol{R}^k)$ *respectively such that for every free ultrafilter* $\mathscr{U}$ *in* $N$, $\mathscr{J}_{\mathscr{U}}\mathfrak{A}\mathscr{J}_{\mathscr{U}}^{-1}$ *is a partial ultrapower of* $A$ (*with* $\mathscr{J}_{\mathscr{U}}$ *defined by* (4.4)). *Suppose that* $\varphi$ *is a bounded continuous function on* $\sigma(A)$ *such that* $\varphi(A)$ *is an integral operator with kernel* $K$ *satisfying*

$$(*) \qquad |K(x, y)| \leq \psi(x - y)$$

*where* $\psi$ *is a function such that*

$$\int_{\boldsymbol{R}^k} \psi(x)(1 + |x|^k)\, dx < +\infty.$$

*Then*

$$\varphi(\mathfrak{A})[\![f]\!] = [\![\int K(\cdot, y) f(y)\, dy]\!]$$

*for any* $[\![f]\!] \in B^2(\boldsymbol{R}^k)$.

Proof. We only give an outline; the details can easily be filled in by the reader, or found in [20].

The formula $\mathfrak{B}[\![f]\!] = [\![\int K(\cdot, y) f(y)\, dy]\!]$ defines a bounded linear operator $\mathfrak{B}$: $M^2(\boldsymbol{R}^k) \to M^2(\boldsymbol{R}^k)$; we prove this by means of inequality $(*)$. Given a free ultrafilter $\mathscr{U}$ in $N$ we extend $J_{\mathscr{U}}$ to a bounded mapping $\tilde{J}_{\mathscr{U}}$: $M^2(\boldsymbol{R}^k) \to (L^2(\boldsymbol{R}^k))_{\mathscr{U}}$ by the same formula (4.4). Then $\tilde{J}_{\mathscr{U}}$ fails to be injective, but we have for $F, G \in M^2(\boldsymbol{R}^k)$

$$F = G \quad \text{iff} \quad (\tilde{J}_{\mathscr{U}} F = \tilde{J}_{\mathscr{U}} G \text{ for all free ultrafilters } \mathscr{U} \text{ in } N).$$

The latter fact can be proved from the general equality

$$\overline{\lim_{n\to\infty}}\, a_n = \sup_{\mathscr{U}} \lim a_n$$

where $a_n$ is a bounded numerical sequence and the supremum is taken over all free ultrafilters $\mathscr{U}$ in $N$. Applying Corollary (3.7) and again $(*)$, we deduce that

$$\tilde{J}_{\mathscr{U}}\varphi(\mathfrak{A})[\![e_\lambda]\!] = \tilde{J}_{\mathscr{U}}\mathfrak{B}[\![e_\lambda]\!]$$

for all $\mathscr{U}$ and all $\lambda \in \boldsymbol{R}^k$. Both $\varphi(\mathfrak{A})$ and $\mathfrak{B}$ are bounded, so we have $\varphi(\mathfrak{A}) = \mathfrak{B} \restriction B^2(\boldsymbol{R}^k)$. ■

(4.8) Corollary. *Let* $p > \max\{2, k/2\}$ *and* $V \in S^p\mathrm{AP}(\boldsymbol{R}^k)$, $V$ *real and bounded from below. Let* $\mathfrak{A}$ *be the Schrödinger operator with potential* $V$

*defined in $B^2(\mathbf{R}^k)$. Then for $t > 0$ and $[\![f]\!] \in B^2(\mathbf{R}^k)$*

$$e^{-t\mathfrak{A}}[\![f]\!] = [\![\int K_t(\cdot, y) f(y)\, dy]\!]$$

*where $K_t(x, y)$ is the integral kernel of $e^{-t(-\Delta+V)}$ in $L^2(\mathbf{R}^k)$.*

For the proof we need only an estimate of the type described in Theorem (4.7). It can be found in [12]. ∎

Theorem (4.7) can be applied also in the case of $\mathfrak{A}$ and $A$ generated by the same uniformly elliptic formally selfadjoint differential operator with $\mathrm{CAP}^\infty$ coefficients; the function $\varphi$ is assumed to be rapidly decreasing at $\infty$ along with all its derivatives (the required estimates have recently been found by Kozlov and Shubin [18] and used to prove a theorem analogous to (4.7)). Analogous results hold for operators in $B^2(\mathbf{R}^k, L^2(\mathbf{R}^l))$.

Let us return to the operator $\mathfrak{A}_0$ defined by (4.1). If we identify $f \in \mathrm{CAP}^\infty$ with $[\![f]\!]$ then $\mathfrak{A}_0$ becomes the "true" Laplacian, acting in a certain subspace of the space of distributions in $\mathbf{R}^k$. This shows that certain partial ultrapowers of $-\Delta$ can be identified with its distributional extension restricted to suitable subspaces. The same holds for selfadjoint uniformly elliptic differential operators with $\mathrm{CAP}^\infty$ coefficients. It turns out that a similar observation can be made for a comparatively wide class of abstract selfadjoint operators.

Let $\Phi \subset H \subset \Phi'$ be a Gelfand triple (see [13]), i.e. $H$ is a Hilbert space, $\Phi \subset H$ a dense linear subspace, equipped with its own topology which is stronger than that of $H$ and which makes $\Phi$ nuclear, and $\Phi'$ is the adjoint space. Let $A$ be a selfadjoint operator in $H$ such that $\Phi \subset D(A)$ and $A(\Phi) \subset \Phi$. Denote by $A'$ the operator adjoint to $A \restriction \Phi$, $A'\colon \Phi' \to \Phi'$, which can be regarded as an extension of $A$. Then, given any free ultrafilter $\mathscr{U}$ in $N$, we can find a $*$-weakly dense subspace $\Psi'$ of $\Phi'$ spanned by a certain family of generalized eigenvectors of $A$ (i.e. eigenvectors of $A'$) and a linear mapping $J\colon \Psi' \to (H)_{\mathscr{U}}$ with the following properties:

(i) $J$ is injective.

(ii) $\mathscr{A} = JAJ^{-1}$ is an essentially selfadjoint partial ultrapower of $A$ in $\mathscr{H} = \overline{\operatorname{Ran} J}$. Almost all elements of $\sigma(A)$ (with respect to the spectral measure of $A$) are eigenvalues of $\mathscr{A}$ (so $\sigma(\bar{\mathscr{A}}) = \sigma(A)$) and the multiplicities of the spectra are equal.

(iii) If $A$ has absolutely continuous spectrum then there is a sequence $(r_n)$ of positive numbers, $r_n \to +\infty$, such that

$$\forall f \in \Psi'\ \exists (f_n) \in Jf\colon \quad r_n f_n \to f, \quad r_n A f_n \to A'f$$

(the convergence in the $*$-weak topology in $\Psi'$). For details see [20].



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INSTYTUT MATEMATYKI UNIWERSYTETU WARSZAWSKIEGO
INSTITUTE OF MATHEMATICS, WARSAW UNIVERSITY
PKiN, IX p., 00-901 Warszawa, Poland
and
INSTYTUT PODSTAWOWYCH PROBLEMÓW TECHNIKI PAN
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH
POLISH ACADEMY OF SCIENCES
Świętokrzyska 21, 00-049 Warszawa, Poland

*Received August 1, 1985*
*Revised version June 26, 1986* (2081)

STUDIA MATHEMATICA, T. LXXXVII. (1987)

# Generalized Nash–Moser smoothing operators and the structure of Fréchet spaces

by

VINCENZO B. MOSCATELLI* (Lecce) and
MARILDA A. SIMÕES (Niteroi)

**Abstract.** In [3] E. Dubinsky related the Nash–Moser Inverse Function Theorem to the structure theory of Fréchet spaces via the smoothing operators of Nash–Moser type. Motivated by this, we introduce very general families of smoothing operators and show what implications their existence has on the structure of a Fréchet space.

**Introduction.** In recent times, some quite unexpected connections between two apparently unrelated topics in Functional Analysis, namely, the Inverse Function Theorem and the structure theory of Fréchet spaces, have begun to be noted (cf. [3]). The unexpectedness is due to the fact that, as everyone knows, the Inverse Function Theorem and linear analysis do not mix well. A crucial point of contact comes from the so-called Nash–Moser Theorem, which is an Inverse Function Theorem in Fréchet spaces based on a refinement of the old Newton's iteration method. (As is well known, the usual Banach space theorem does not go over to Fréchet spaces.) The technique, invented by J. Nash [12] in his solution of the isometric embedding problem for Riemannian manifolds, assumes the existence of an appropriate one-parameter family of smoothing operators on the space. The method was later fashioned by J. Moser [11] into an Inverse Function Theorem in Fréchet spaces which became known as the Nash–Moser Theorem, and wide applicability of the method and its subsequent generalizations (cf. e.g. [8]) was claimed by various authors over the years (see the survey article [5] by R. S. Hamilton). However, the impressive results of D. Vogt [20] (cf. also [4]) show that in the nuclear case, which is the most important in the applications, only a very small class of Fréchet spaces can support a family of smoothing operators of Nash–Moser type. In particular, the nuclear space $H(D)$ of analytic functions on the open unit disc $D$ of the complex plane does not belong to such a class and in [3] smoothing operators supported by this space were found and, through their use, an Inverse Function Theorem valid

* The research of this author was financed by the Italian Ministero della Pubblica Istruzione.