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Some counterexamples to subexponential growth of orthogonal polynomials

by

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Abstract. We give examples of polynomials $p(n)$ orthonormal with respect to a measure μ on $\mathbb{R}$ such that the sequence $\{p(n,x)\}$ has exponential lower bound for some points x of $\operatorname{supp}\mu$. Moreover, the set of such points is dense in the support of μ.

1. Introduction. Let μ be a probability measure on $\mathbb{R}$ with all moments finite. Applying the Gram–Schmidt procedure to $\{x^n\}$ with respect to the inner product $(f,g) = \int f\overline{g}\,d\mu$ we get a system of polynomials $\{p(n,x)\}$ satisfying

$$xp(n,x) = \lambda_{n+1}p(n+1,x) + \beta_n p(n,x) + \lambda_n p(n-1,x),$$

where $\lambda_n > 0$, $\beta_n \in \mathbb{R}$.

In [5] J. Zhang has shown that for λ_n and β_n asymptotically periodic and λ_n bounded away from 0 one has

$$\limsup_{n\to\infty} |p(n,x)|^{1/n} \le 1$$

uniformly for $x \in \operatorname{supp}\mu$. Zhang's proof is a refinement of methods used in [3] where the case of convergent coefficients was considered (see also [1], [2]).

There were suggestions that asymptotic periodicity of λ_n and β_n is essential in the result above. In [4] R. Szwarc has constructed examples where for a point x in $\operatorname{supp}\mu$,

$$\liminf_{n\to\infty} |p(n,x)|^{1/n} > 1 \tag{1}$$

with λ_n and β_n bounded, λ_n bounded away from 0. One of his examples is $\lambda_n = 1/2$, $n \ge 0$, and $\beta_n = 0$ for $n \in \mathbb{N}\setminus A$, where A is a lacunary subset of $\mathbb{N}$.

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In this paper we show that the set of points satisfying (1) can be dense in the support of μ. Moreover, we are able to compute $\operatorname{supp}\mu$ explicitly.

Our results are the following:

THEOREM 1. *Let $\{p(n,x)\}$ be the system of orthogonal polynomials satisfying*

$$xp(n,x) = \tfrac{1}{2}p(n+1,x) + b_n p(n,x) + \tfrac{1}{2}p(n-1,x),$$

where

$$b_n = \begin{cases} 0, & n_{2k} \le n < n_{2k+1}, \\ \pi, & n_{2k+1} \le n < n_{2k+2}, \end{cases}$$

and $n_k = 2^k, k > 0$, $n_0 = 0$, and μ be the corresponding orthogonality measure. Then $\operatorname{supp}\mu = [-1,1]\cup[\pi-1,\pi+1]$ and the set of points satisfying (1), *i.e.*

$$\{x \in \operatorname{supp}\mu \mid \liminf_{n\to\infty} |p(n,x)|^{1/n} > 1\},$$

is dense in $\operatorname{supp}\mu$.

THEOREM 2. *Let $\{p(n,x)\}$ and μ be as above, where*

$$n_k = \begin{cases} j2^j, & k = 2j, \\ (j+1)2^j, & k = 2j+1. \end{cases}$$

Then

$$\limsup_{n\to\infty} |p(n,x)|^{1/n} \le 1$$

uniformly for $x \in [\pi-1,\pi+1]$ and the set of points satisfying (1) *is dense in $[-1,1]$.*

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2. Some useful lemmas. Denote by $p(n,x)$ polynomials satisfying

$$\begin{gathered} xp(n,x) = \tfrac{1}{2}p(n+1,x) + b_n p(n,x) + \tfrac{1}{2}p(n-1,x), \\ p(-1,x) = 0, \quad p(0,x) = 1, \end{gathered} \tag{2}$$

where

$$b_n = \begin{cases} r_1, & n_{2k} \le n < n_{2k+1}, \\ r_2, & n_{2k+1} \le n < n_{2k+2}, \end{cases}$$

$\{n_k\}$ is an increasing sequence of integers, $n_0 = 0$, and r_1, r_2 are real numbers. We will write $p(n) = p(n,x)$ for fixed x.

With the polynomials $p(n,x)$ we associate the *Jacobi matrix* J

$$J = \begin{pmatrix} b_0 & \frac{1}{2} & & \\ \frac{1}{2} & b_1 & \frac{1}{2} & \\ & \frac{1}{2} & b_2 & \\ & & & \ddots \end{pmatrix}.$$

If the coefficients b_n are bounded then the orthogonality measure μ coincides with the spectral measure of J, hence $\operatorname{supp}\mu$ and the spectrum $\sigma(J)$ are equal.

Let

$$J_0 = \begin{pmatrix} 0 & \frac{1}{2} & & \\ \frac{1}{2} & 0 & \frac{1}{2} & \\ & \frac{1}{2} & 0 & \\ & & & \ddots \end{pmatrix}.$$

It is well known that $\sigma(J_0) = [-1,1]$.

LEMMA 1. *Let $\{p(n,x)\}$ be the system of orthogonal polynomials satisfying* (2), *and*

$$\limsup_{k\to\infty}(n_{2k+1} - n_{2k}) = \limsup_{k\to\infty}(n_{2k+2} - n_{2k+1}) = \infty. \tag{3}$$

Then $\operatorname{supp}\mu = [r_1-1, r_1+1] \cup [r_2-1, r_2+1]$, where μ is the orthogonality measure.

Proof. Without loss of generality we can assume $r_1 < r_2$. Hence we have

$$J_0 + r_1 I \le J \le J_0 + r_2 I.$$

This gives $\operatorname{supp}\mu = \sigma(J) \subset [r_1-1, r_2+1]$. From (3) we get

$$[r_1-1, r_1+1] \cup [r_2-1, r_2+1] \subset \sigma(J)$$

(for example the sequence $x_k^\psi \in \ell^2(\mathbb{N})$, $x_k^\psi(n) = e^{in\psi} + r_1$ for $n_{2k} \le n < n_{2k+1}$ and 0 otherwise, is an approximate eigenvector corresponding to the number $r_1 + \cos\psi \in \sigma(J)$).

Now if $r_2 - r_1 \le 2$, then $\operatorname{supp}\mu = [r_1-1, r_2+1]$. Assume $r_2 - r_1 > 2$. Fix $x \in (r_1+1, r_2-1)$. There exists a constant $c > 1$ such that $|x - r_i| > c$. Note that $p(1) = 2(x - b_0) > 2c > c = cp(0)$, so $p(1) - cp(0) > 0$. From (2) we have

$$|p(n+1)| + |p(n-1)| \ge 2c|p(n)|,$$

hence

$$|p(n+1)| - c|p(n)| \ge c|p(n)| - |p(n-1)| \ge c|p(n)| - c^2|p(n-1)|.$$

By induction we get $|p(n+1)| - c|p(n)| \geq 0$, hence $|p(n)| \geq c^n$. Therefore
$$\forall x \in (r_1+1, r_2-1) \quad \lim_{n\to\infty} |p(n,x)| = \infty.$$
Hence $(r_1+1, r_2-1) \cap \operatorname{supp}\mu = \emptyset$. ∎

Throughout the rest of this section we use the following notation:
$$r = r_2 - r_1, \tag{4}$$
$$x = r_2 + \cos\psi, \qquad \psi \in [0,\pi], \tag{5}$$
$$p(n) = p(n,x). \tag{6}$$
We assume that $|r| > 2$. Then there is a unique real number γ satisfying
$$\tfrac12(\gamma+\gamma^{-1}) = r + \cos\psi, \qquad |\gamma| > 1. \tag{7}$$

LEMMA 2. *Let $r = r_2 - r_1$ be a transcendental number. For any $\psi \in \mathbb{Q}\pi$,*
$$\frac{p(n+1,x)}{p(n,x)} \neq \gamma^{-1}. \tag{8}$$

Proof. By substituting $x' = x - r_1$ we can reduce ourselves to the case $r_1 = 0$, $r_2 = r$. Moreover, without loss of generality we can take $r > 0$. Assume now *a contrario* that we have equality in (8), i.e.
$$\frac{p(n+1, r+\cos\psi)}{p(n, r+\cos\psi)} = \gamma^{-1}, \tag{9}$$
for some n and ψ. From (7) we get
$$\gamma^{-1} = r + \cos\psi - \sqrt{(r+\cos\psi)^2 - 1}.$$
The left hand side of (9) is a rational function in r, as opposed to the right hand side. Hence (9) cannot be satisfied identically.

We can transform (9) to a polynomial equation in r with algebraic coefficients because $\cos\psi$ is algebraic. This implies that r is algebraic, which contradicts the assumptions. ∎

LEMMA 3. *We have*
$$p(n) = \frac{\gamma p(n_{2k}) - p(n_{2k}-1)}{\gamma^2-1}\gamma^{n-n_{2k}+1} + \frac{\gamma p(n_{2k}-1) - p(n_{2k})}{\gamma^2-1}\gamma^{-(n-n_{2k})} \tag{10}$$
for $n_{2k} < n \leq n_{2k+1}$, and
$$p(n) = \begin{cases} -\frac{\sin(n-n_{2k+1})\psi}{\sin\psi} p(n_{2k+1}-1) \\ \quad + \frac{\sin(n-n_{2k+1}+1)\psi}{\sin\psi} p(n_{2k+1}), & \psi \neq 0,\pi, \\ (n-n_{2k+1}+1)\,p(n_{2k+1}) - (n-n_{2k+1})\,p(n_{2k+1}-1), & \text{otherwise}, \end{cases} \tag{11}$$

for $n_{2k+1} < n \leq n_{2k+2}$ (ψ and γ are given by (5), (7)). Moreover, if $\psi = (q/q_0)\pi$, $\psi \neq 0,\pi$, and $q_0 \mid (n_{2k+2} - n_{2k+1})$ for any $k > k_0$, then for $k > k_0$ and $n_{2k} < n \leq n_{2k+1}$ one has
$$p(n) = \frac{\gamma p(n_{2k_0}) - p(n_{2k_0}-1)}{\gamma^2-1}\gamma^{m_n+1} + \frac{\gamma p(n_{2k_0}-1) - p(n_{2k_0})}{\gamma^2-1}\gamma^{-m_n}, \tag{12}$$
where
$$m_n = n - n_{2k} + \sum_{i=k_0}^{k-1}(n_{2i+1} - n_{2i}). \tag{13}$$

Proof. From (2) we get
$$\tfrac12 p(n) + r_1 p(n-1) + \tfrac12 p(n-2) = xp(n-1) = \left(\tfrac12(\gamma+\gamma^{-1}) + r_1\right)p(n-1),$$
so
$$p(n) = (\gamma+\gamma^{-1})\,p(n-1) - p(n-2) \quad \text{for } n_{2k} < n \leq n_{2k+1}, \tag{14}$$
and
$$\tfrac12 p(n) + r_2 p(n-1) + \tfrac12 p(n-2) = xp(n-1) = (r_2+\cos\psi)\,p(n-1),$$
so
$$p(n) = 2p(n-1)\cos\psi - p(n-2) \quad \text{for } n_{2k+1} < n \leq n_{2k+2}.$$
Now we can get (10) and (11) by induction.

To prove (12) observe that by (11) we get
$$p(n_{2k}) = p(n_{2k-1}), \quad p(n_{2k}-1) = p(n_{2k-1}-1), \tag{15}$$
for $k > k_0$. Arrange the numbers n satisfying
$$n \leq n_{2k_0+1} \quad \text{or} \quad n_{2k} < n \leq n_{2k+1} \quad \text{for } k > k_0$$
into an increasing sequence. A number n will occupy the position $\widetilde{n}$ given by
$$\widetilde{n} = \begin{cases} n & \text{for } n \leq n_{2k_0+1}, \\ n - \sum_{i=k_0}^{k-1}(n_{2i+2} - n_{2i+1}) = m_n + n_{2k_0} & \text{for } k > k_0, n_{2k} < n \leq n_{2k+1}. \end{cases}$$
Define a sequence $\widetilde{p}(n)$ by $\widetilde{p}(\widetilde{n}) = p(n)$. By (14) and (15) we get
$$\widetilde{p}(\widetilde{n}) = (\gamma+\gamma^{-1})\,\widetilde{p}(\widetilde{n}-1) - \widetilde{p}(\widetilde{n}-2) \quad \text{for } \widetilde{n} > n_{2k_0}.$$
By induction we obtain
$$\widetilde{p}(\widetilde{n}) = \frac{\gamma\widetilde{p}(\widetilde{n}_{2k_0}) - \widetilde{p}(\widetilde{n}_{2k_0}-1)}{\gamma^2-1}\gamma^{\widetilde{n}-\widetilde{n}_{2k_0}+1} + \frac{\gamma\widetilde{p}(\widetilde{n}_{2k_0}-1) - \widetilde{p}(\widetilde{n}_{2k_0})}{\gamma^2-1}\gamma^{-(\widetilde{n}-\widetilde{n}_{2k_0})}.$$
Taking into account that $\widetilde{n} = n$ for $n \leq n_{2k_0+1}$ and that $\widetilde{p}(\widetilde{n}) = p(n)$ gives the conclusion. ∎

Proposition 1. *Let $\psi = (q/q_0)\pi$, $\psi \neq 0, \pi$ and $q, q_0 \in \mathbb{N}$. Assume $q_0 \mid (n_{2k+2} - n_{2k+1})$ for all k starting from some k_0. Then there are a constant $A > 0$ and an integer N such that*

$$(16) \qquad |p(n)| > A|\gamma|^m \quad \text{for } N \le n_{2k} < n \le n_{2k+2},$$

where

$$m = \min\Big(n - n_{2k} + \sum_{i=k_0}^{k-1}(n_{2i+1} - n_{2i}), \sum_{i=k_0}^{k}(n_{2i+1} - n_{2i})\Big).$$

Proof. From (12) (Lemma 3) we get for $n_{2k} < n \le n_{2k+1}$, $k > k_0$,

$$(17) \qquad |p(n)| \ge \left|\frac{\gamma p(n_{2k_0}) - p(n_{2k_0} - 1)}{\gamma^2 - 1}\right| |\gamma|^{m_n + 1} - \left|\frac{\gamma p(n_{2k_0} - 1) - p(n_{2k_0})}{\gamma^2 - 1}\right| |\gamma|^{-m_n},$$

where m_n is given by (13). By Lemma 2,

$$\left|\frac{\gamma p(n_{2k_0+1}) - p(n_{2k_0+1} - 1)}{\gamma^2 - 1}\right| \neq 0.$$

Since $|\gamma| > 1$ there exist constants $B > 0$ and k_1 such that

$$(18) \qquad |p(n)| > B|\gamma|^{m_n},$$

where $k > k_1$ and $n_{2k} < n \le n_{2k+1}$.

From (11) we have for $n_{2k+1} < n \le n_{2k+2}$,

$$(19) \qquad |p(n)| = \left|\frac{p(n_{2k+1} - 1)}{\sin\psi}\right| \left|\frac{p(n_{2k+1})}{p(n_{2k+1} - 1)}\sin(n - n_{2k+1} + 1)\psi - \sin(n - n_{2k+1})\psi\right|.$$

Examine now $p(n_{2k+1})/p(n_{2k+1} - 1)$. From (12) we have

$$(20) \qquad p(n) = \frac{\gamma p(n_{2k_0+1}) - p(n_{2k_0+1} - 1)}{\gamma^2 - 1}\gamma^{m_n + 1} + \frac{\gamma p(n_{2k_0+1} - 1) - p(n_{2k_0+1})}{\gamma^2 - 1}\gamma^{-m_n}$$

for $k > k_0$, $n = n_{2k+1}, n_{2k+1} - 1$. Lemma 2 states that

$$\forall_{n \in \mathbb{N}} \quad \gamma p(n_{2k_0+1}) - p(n_{2k_0+1} - 1) \neq 0,$$

so from (20) we get

$$(21) \qquad \frac{p(n_{2k+1})}{p(n_{2k+1} - 1)} \to \gamma \quad \text{as } k \to \infty.$$

Now return to (19). We have assumed that r is transcendental, hence so is γ. Since $\sin k\psi$ is an algebraic number,

$$\forall_{k \in \mathbb{N}} \quad \gamma \sin k\psi - \sin(k - 1)\psi \neq 0.$$

Moreover, there exists a constant $C > 0$ (depending on ψ) such that

$$\forall_{k \in \mathbb{N}} \quad |\gamma \sin k\psi - \sin(k - 1)\psi| > C.$$

By (21) there exists k_2 such that $|p(n_{2k+1})/p(n_{2k+1} - 1) - \gamma| < C/2$ for $k > k_2$. Hence for $n_{2k+1} < n \le n_{2k+2}$, $k > k_2$,

$$\begin{aligned}\left|\frac{p(n_{2k+1})}{p(n_{2k+1} - 1)}\sin(n - n_{2k+1} + 1)\psi - \sin(n - n_{2k+1})\psi\right| &\ge |\gamma \sin(n - n_{2k+1} + 1)\psi - \sin(n - n_{2k+1})\psi| \\ &\quad - \left|\frac{p(n_{2k+1})}{p(n_{2k+1} - 1)} - \gamma\right| |\sin(n - n_{2k+1} + 1)\psi| \ge \frac{C}{2}.\end{aligned}$$

By (19) we thus get

$$|p(n)| \ge \left|\frac{p(n_{2k+1} - 1)}{\sin\psi}\right| \frac{C}{2},$$

and by (18),

$$(22) \qquad |p(n)| > \left|\frac{BC}{2}\right| |\gamma|^m,$$

where $n_{2k+1} < n \le n_{2k+2}$, $k > k_1, k_2$, and

$$m = n_{2k+1} - 1 - n_{2k} + \sum_{i=k_0}^{k-1}(n_{2i+1} - n_{2i}) < \sum_{i=k_0}^{k}(n_{2i+1} - n_{2i}).$$

Combining (18) and (22) gives the conclusion with

$$A = \min\left(B, \left|\frac{BC}{2\sin\psi}\right|\right) \quad \text{and} \quad N = \max(n_{2k_0}, n_{2k_1}, n_{2k_2}). \ ■$$

Proposition 2. *Let $n_{2k} < n \le n_{2k+2}$. Then*

$$(23) \qquad |p(n)| \le M 2^{k-1} A^k |\gamma|^m,$$

where

$$m = \min\Big(n - n_{2k} + \sum_{i=0}^{k-1}(n_{2i+1} - n_{2i}), \sum_{i=0}^{k}(n_{2i+1} - n_{2i})\Big),$$

$$M = \prod_{i=0}^{k}(n_{2i+2} - n_{2i+1} + 1) \quad \textit{and} \quad A = \left|\frac{\gamma^2}{\gamma^2 - 1}\right|.$$

Proof. (10) gives

$$p(n)=\frac{\gamma^2\gamma^m-\gamma^{-m}}{\gamma^2-1}\,p(n_{2k})+\frac{\gamma\gamma^m-\gamma\gamma^{-m}}{\gamma^2-1}\,p(n_{2k}-1)$$

for $n_{2k}<n\le n_{2k+1}$, where $m=n-n_{2k}$. Observe that $\gamma^2\gamma^m$ and γ^{-m} have the same sign, so

$$\left|\frac{\gamma^2\gamma^m-\gamma^{-m}}{\gamma^2-1}\right|\le\left|\frac{\gamma^2}{\gamma^2-1}\right||\gamma|^m.$$

In the same way we get

$$\left|\frac{\gamma\gamma^m-\gamma\gamma^{-m}}{\gamma^2-1}\right|\le\left|\frac{\gamma^2}{\gamma^2-1}\right||\gamma|^m.$$

Hence

$$|p(n)|\le\left|\frac{\gamma^2}{\gamma^2-1}\right||\gamma|^{n-n_{2k}}(|p(n_{2k}-1)|+|p(n_{2k})|). \tag{24}$$

For $n_{2k+1}<n\le n_{2k+2}$ we have from (11) (note that $|\sin n\psi|\le n|\sin\psi|$)

$$\begin{aligned}|p(n)|&\le(n-n_{2k+1})|p(n_{2k+1}-1)|+(n-n_{2k+1}+1)|p(n_{2k+1})|\\&\le(n_{2k+2}-n_{2k+1}+1)(|p(n_{2k+1}-1)|+|p(n_{2k+1})|).\end{aligned}$$

Hence

$$\begin{aligned}&|p(n)|\\&\quad\le(n_{2k+2}-n_{2k+1}+1)\left|\frac{\gamma^2}{\gamma^2-1}\right||\gamma|^{n_{2k+1}-n_{2k}}(|p(n_{2k}-1)|+|p(n_{2k})|).\end{aligned} \tag{25}$$

Combining (24) and (25) gives

$$|p(n)|\le(n_{2k+2}-n_{2k+1}+1)\left|\frac{\gamma^2}{\gamma^2-1}\right||\gamma|^m(|p(n_{2k}-1)|+|p(n_{2k})|),$$

for $n_{2k}<n\le n_{2k+2}$, with $m=\min(n-n_{2k},n_{2k+1}-n_{2k})$. Now we get the conclusion by induction. ■

Clearly the above two propositions can be proved for $x=r_1+\cos\psi$ using the same arguments. So set now $r=r_1-r_2$. Moreover, define x, ψ, γ as in (5), (7). In this notation we have:

PROPOSITION 3. *Let $q_0\mid(n_{2k+1}-n_{2k})$ for all k starting from some k_0 and $\psi=(q/q_0)\pi$, $\sin\psi\ne0$, $q,q_0\in\mathbb{N}$. Then there are a constant $A>0$ and an integer N such that*

$$|p(n)|>A|\gamma|^m \quad \textit{for } N\le n_{2k+1}<n\le n_{2k+3}, \tag{26}$$

where

$$m=\min\Big(n-n_{2k+1}+\sum_{i=k_0}^{k-1}(n_{2i+2}-n_{2i+1}),\sum_{i=k_0}^{k}(n_{2i+2}-n_{2i+1})\Big).$$

PROPOSITION 4. *Let $n_{2k+1}<n\le n_{2k+3}$. Then*

$$|p(n)|\le(|p(n_1)|+|p(n_0)|)M2^{k-1}A^k|\gamma|^m, \tag{27}$$

where

$$m=\min\Big(n-n_{2k+1}+\sum_{i=0}^{k-1}(n_{2i+2}-n_{2i+1}),\sum_{i=0}^{k}(n_{2i+2}-n_{2i+1})\Big),$$

$$M=\prod_{i=1}^{k+1}(n_{2i+1}-n_{2i}+1) \quad \textit{and} \quad A=\left|\frac{\gamma^2}{\gamma^2-1}\right|.$$

3. Proofs of the main theorems

Proof of Theorem 1. By Lemma 1 we obtain $\operatorname{supp}\mu=[-1,1]\cup[\pi-1,\pi+1]$. Let $x=\pi+\cos\psi$, where $\psi=(q/2^{q_0})\pi$, $\psi\ne k\pi$ and $q,q_0,k\in\mathbb{N}$. Observe that $2^{q_0}\mid(n_{2k+2}-n_{2k+1})$ for $k>q_0/2$. By Proposition 1 there exist a constant $A>0$ and an integer N such that

$$|p(n,x)|>A|\gamma|^m \quad \text{for } N\le n_{2k}<n\le n_{2k+2},$$

where

$$m=\min\Big(n-2^{2k}+\sum_{i=k_0}^{k-1}2^{2i},\sum_{i=k_0}^{k}2^{2i}\Big).$$

Hence $|p(n,x)|^{1/n}>A^{1/n}|\gamma|^{m'}$, where

$$m'\ge\sum_{i=k_0}^{k-1}\frac{2^{2i}}{2^{2k+2}}\xrightarrow{k}\frac{1}{12}.$$

So

$$\liminf_{n\to\infty}|p(n,x)|^{1/n}\ge|\gamma|^{1/12}>1.$$

In the same way for $x=\cos\psi$ (using Proposition 3) we prove

$$\liminf_{n\to\infty}|p(n,x)|^{1/n}\ge|\gamma|^{1/12}>1. \ ■$$

Proof of Theorem 2. Let $x=\cos\psi$, where $\psi=(q/2^{q_0})\pi$, $\psi\ne k\pi$ and $q,q_0,k\in\mathbb{N}$. As above, $2^{q_0}\mid(n_{2k+1}-n_{2k})$ for $k>q_0/2$. By Proposition 3 we get a constant $A>0$ and an integer N such that

$$|p(n,x)|>A|\gamma|^m \quad \text{for } N\le n_{2k+1}<n\le n_{2k+3},$$

where

$$m=\min\Big(n-(k+1)2^k+\sum_{j=k_0+1}^{k-1}(j+1)2^j,\sum_{j=k_0+1}^{k}(j+1)2^j\Big).$$

Hence $|p(n,x)|^{1/n} > A^{1/n}|\gamma|^{m'}$, where

$$m' \geq \sum_{j=k_0+1}^{k-1} \frac{(j+1)2^j}{(k+2)2^{k+1}} \xrightarrow{k} \frac{1}{2}.$$

So

$$\liminf_{n\to\infty} |p(n,x)|^{1/n} \geq |\gamma|^{1/2} > 1.$$

Now let $x \in [\pi-1, \pi+1]$ and ψ, γ be as in (5), (7). By Proposition 2 we get

$$|p(n,x)| \leq M2^{k-1}A^k|\gamma|^m \quad \text{for } n_{2k} < n \leq n_{2k+2}, \tag{28}$$

where

$$m = \min\Big(n - k2^k + \sum_{j=0}^{k-1} 2^j, \sum_{j=0}^{k} 2^j\Big),$$

$$M = \prod_{j=0}^{k}(2^j(j+1)+1) \quad \text{and} \quad A = \left|\frac{\gamma^2}{\gamma^2-1}\right|.$$

Hence $|p(n,x)|^{1/n} \leq M^{1/(k2^k)}(2A)^{1/2^k}|\gamma|^{m'}$, where $m' = \sum_{j=0}^{k} \frac{2^j}{k2^k} \leq \frac{2}{k}$. Moreover, $M \leq \prod_{j=0}^{k} 2^j(j+2) \leq 2^{k^2}(k+2)!$, so $M^{1/(k2^k)} \xrightarrow{k} 1$. Therefore

$$\limsup_{n\to\infty} |p(n,x)|^{1/n} \leq 1.$$

Observe that $|\gamma|$ is bounded away from 1 for all $x \in [\pi-1, \pi+1]$, so A is bounded away from 0. Hence (28) holds with constant A independent of x. So $\limsup_{n\to\infty} |p(n,x)|^{1/n} \leq 1$ uniformly for $x \in [\pi-1, \pi+1]$. ∎

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