

# Some counterexamples to subexponential growth of orthogonal polynomials

by

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**Abstract.** We give examples of polynomials  $p(n)$  orthonormal with respect to a measure  $\mu$  on  $\mathbb{R}$  such that the sequence  $\{p(n, x)\}$  has exponential lower bound for some points  $x$  of  $\text{supp } \mu$ . Moreover, the set of such points is dense in the support of  $\mu$ .

**1. Introduction.** Let  $\mu$  be a probability measure on  $\mathbb{R}$  with all moments finite. Applying the Gram-Schmidt procedure to  $\{x^n\}$  with respect to the inner product  $(f, g) = \int f \bar{g} d\mu$  we get a system of polynomials  $\{p(n, x)\}$  satisfying

$$xp(n, x) = \lambda_{n+1}p(n+1, x) + \beta_n p(n, x) + \lambda_n p(n-1, x),$$

where  $\lambda_n > 0$ ,  $\beta_n \in \mathbb{R}$ .

In [5] J. Zhang has shown that for  $\lambda_n$  and  $\beta_n$  asymptotically periodic and  $\lambda_n$  bounded away from 0 one has

$$\limsup_{n \rightarrow \infty} |p(n, x)|^{1/n} \leq 1$$

uniformly for  $x \in \text{supp } \mu$ . Zhang's proof is a refinement of methods used in [3] where the case of convergent coefficients was considered (see also [1], [2]).

There were suggestions that asymptotic periodicity of  $\lambda_n$  and  $\beta_n$  is essential in the result above. In [4] R. Szwarc has constructed examples where for a point  $x$  in  $\text{supp } \mu$ ,

$$(1) \quad \liminf_{n \rightarrow \infty} |p(n, x)|^{1/n} > 1$$

with  $\lambda_n$  and  $\beta_n$  bounded,  $\lambda_n$  bounded away from 0. One of his examples is  $\lambda_n = 1/2$ ,  $n \geq 0$ , and  $\beta_n = 0$  for  $n \in \mathbb{N} \setminus A$ , where  $A$  is a lacunary subset of  $\mathbb{N}$ .

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In this paper we show that the set of points satisfying (1) can be dense in the support of  $\mu$ . Moreover, we are able to compute  $\text{supp } \mu$  explicitly.

Our results are the following:

**THEOREM 1.** *Let  $\{p(n, x)\}$  be the system of orthogonal polynomials satisfying*

$$xp(n, x) = \frac{1}{2}p(n+1, x) + b_n p(n, x) + \frac{1}{2}p(n-1, x),$$

where

$$b_n = \begin{cases} 0, & n_{2k} \leq n < n_{2k+1}, \\ \pi, & n_{2k+1} \leq n < n_{2k+2}, \end{cases}$$

and  $n_k = 2^k, k > 0, n_0 = 0$ , and  $\mu$  be the corresponding orthogonality measure. Then  $\text{supp } \mu = [-1, 1] \cup [\pi-1, \pi+1]$  and the set of points satisfying (1), i.e.

$$\{x \in \text{supp } \mu \mid \liminf_{n \rightarrow \infty} |p(n, x)|^{1/n} > 1\},$$

is dense in  $\text{supp } \mu$ .

**THEOREM 2.** *Let  $\{p(n, x)\}$  and  $\mu$  be as above, where*

$$n_k = \begin{cases} j2^j, & k = 2j, \\ (j+1)2^j, & k = 2j+1. \end{cases}$$

Then

$$\limsup_{n \rightarrow \infty} |p(n, x)|^{1/n} \leq 1$$

uniformly for  $x \in [\pi-1, \pi+1]$  and the set of points satisfying (1) is dense in  $[-1, 1]$ .

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**2. Some useful lemmas.** Denote by  $p(n, x)$  polynomials satisfying

$$(2) \quad \begin{aligned} xp(n, x) &= \frac{1}{2}p(n+1, x) + b_n p(n, x) + \frac{1}{2}p(n-1, x), \\ p(-1, x) &= 0, \quad p(0, x) = 1, \end{aligned}$$

where

$$b_n = \begin{cases} r_1, & n_{2k} \leq n < n_{2k+1}, \\ r_2, & n_{2k+1} \leq n < n_{2k+2}, \end{cases}$$

$\{n_k\}$  is an increasing sequence of integers,  $n_0 = 0$ , and  $r_1, r_2$  are real numbers. We will write  $p(n) = p(n, x)$  for fixed  $x$ .

With the polynomials  $p(n, x)$  we associate the *Jacobi matrix*  $J$

$$J = \begin{pmatrix} b_0 & \frac{1}{2} & & \\ \frac{1}{2} & b_1 & \frac{1}{2} & \\ & \frac{1}{2} & b_2 & \\ & & & \ddots \end{pmatrix}.$$

If the coefficients  $b_n$  are bounded then the orthogonality measure  $\mu$  coincides with the spectral measure of  $J$ , hence  $\text{supp } \mu$  and the spectrum  $\sigma(J)$  are equal.

Let

$$J_0 = \begin{pmatrix} 0 & \frac{1}{2} & & \\ \frac{1}{2} & 0 & \frac{1}{2} & \\ & \frac{1}{2} & 0 & \\ & & & \ddots \end{pmatrix}.$$

It is well known that  $\sigma(J_0) = [-1, 1]$ .

**LEMMA 1.** *Let  $\{p(n, x)\}$  be the system of orthogonal polynomials satisfying (2), and*

$$(3) \quad \limsup_{k \rightarrow \infty} (n_{2k+1} - n_{2k}) = \limsup_{k \rightarrow \infty} (n_{2k+2} - n_{2k+1}) = \infty.$$

Then  $\text{supp } \mu = [r_1 - 1, r_1 + 1] \cup [r_2 - 1, r_2 + 1]$ , where  $\mu$  is the orthogonality measure.

**Proof.** Without loss of generality we can assume  $r_1 < r_2$ . Hence we have

$$J_0 + r_1 I \leq J \leq J_0 + r_2 I.$$

This gives  $\text{supp } \mu = \sigma(J) \subset [r_1 - 1, r_2 + 1]$ . From (3) we get

$$[r_1 - 1, r_1 + 1] \cup [r_2 - 1, r_2 + 1] \subset \sigma(J).$$

(for example the sequence  $x_k^\psi \in \ell^2(\mathbb{N})$ ,  $x_k^\psi(n) = e^{in\psi} + r_1$  for  $n_{2k} \leq n < n_{2k+1}$  and 0 otherwise, is an approximate eigenvector corresponding to the number  $r_1 + \cos \psi \in \sigma(J)$ ).

Now if  $r_2 - r_1 \leq 2$ , then  $\text{supp } \mu = [r_1 - 1, r_2 + 1]$ . Assume  $r_2 - r_1 > 2$ . Fix  $x \in (r_1 + 1, r_2 - 1)$ . There exists a constant  $c > 1$  such that  $|x - r_i| > c$ . Note that  $p(1) = 2(x - b_0) > 2c > c = cp(0)$ , so  $p(1) - cp(0) > 0$ . From (2) we have

$$|p(n+1)| + |p(n-1)| \geq 2c|p(n)|,$$

hence

$$|p(n+1)| - c|p(n)| \geq c|p(n)| - |p(n-1)| \geq c|p(n)| - c^2|p(n-1)|.$$

By induction we get  $|p(n+1) - c|p(n)| \geq 0$ , hence  $|p(n)| \geq c^n$ . Therefore

$$\forall x \in (r_1 + 1, r_2 - 1) \quad \lim_{n \rightarrow \infty} |p(n, x)| = \infty.$$

Hence  $(r_1 + 1, r_2 - 1) \cap \text{supp } \mu = \emptyset$ . ■

Throughout the rest of this section we use the following notation:

$$(4) \quad r = r_2 - r_1,$$

$$(5) \quad x = r_2 + \cos \psi, \quad \psi \in [0, \pi],$$

$$(6) \quad p(n) = p(n, x).$$

We assume that  $|r| > 2$ . Then there is a unique real number  $\gamma$  satisfying

$$(7) \quad \frac{1}{2}(\gamma + \gamma^{-1}) = r + \cos \psi, \quad |\gamma| > 1.$$

LEMMA 2. Let  $r = r_2 - r_1$  be a transcendental number. For any  $\psi \in \mathbb{Q}\pi$ ,

$$(8) \quad \frac{p(n+1, x)}{p(n, x)} \neq \gamma^{-1}.$$

Proof. By substituting  $x' = x - r_1$  we can reduce ourselves to the case  $r_1 = 0$ ,  $r_2 = r$ . Moreover, without loss of generality we can take  $r > 0$ . Assume now *a contrario* that we have equality in (8), i.e.

$$(9) \quad \frac{p(n+1, r + \cos \psi)}{p(n, r + \cos \psi)} = \gamma^{-1},$$

for some  $n$  and  $\psi$ . From (7) we get

$$\gamma^{-1} = r + \cos \psi - \sqrt{(r + \cos \psi)^2 - 1}.$$

The left hand side of (9) is a rational function in  $r$ , as opposed to the right hand side. Hence (9) cannot be satisfied identically.

We can transform (9) to a polynomial equation in  $r$  with algebraic coefficients because  $\cos \psi$  is algebraic. This implies that  $r$  is algebraic, which contradicts the assumptions. ■

LEMMA 3. We have

$$(10) \quad p(n) = \frac{\gamma p(n_{2k}) - p(n_{2k} - 1)}{\gamma^2 - 1} \gamma^{n - n_{2k} + 1} + \frac{\gamma p(n_{2k} - 1) - p(n_{2k})}{\gamma^2 - 1} \gamma^{-(n - n_{2k})}$$

for  $n_{2k} < n \leq n_{2k+1}$ , and

$$(11) \quad p(n) = \begin{cases} -\frac{\sin(n - n_{2k+1})\psi}{\sin \psi} p(n_{2k+1} - 1) \\ + \frac{\sin(n - n_{2k+1} + 1)\psi}{\sin \psi} p(n_{2k+1}), & \psi \neq 0, \pi, \\ (n - n_{2k+1} + 1)p(n_{2k+1}) - (n - n_{2k+1})p(n_{2k+1} - 1), & \text{otherwise,} \end{cases}$$

for  $n_{2k+1} < n \leq n_{2k+2}$  ( $\psi$  and  $\gamma$  are given by (5), (7)). Moreover, if  $\psi = (q/q_0)\pi$ ,  $\psi \neq 0, \pi$ , and  $q_0 \mid (n_{2k+2} - n_{2k+1})$  for any  $k > k_0$ , then for  $k > k_0$  and  $n_{2k} < n \leq n_{2k+1}$  one has

$$(12) \quad p(n) = \frac{\gamma p(n_{2k_0}) - p(n_{2k_0} - 1)}{\gamma^2 - 1} \gamma^{m_n + 1} + \frac{\gamma p(n_{2k_0} - 1) - p(n_{2k_0})}{\gamma^2 - 1} \gamma^{-m_n},$$

where

$$(13) \quad m_n = n - n_{2k} + \sum_{i=k_0}^{k-1} (n_{2i+1} - n_{2i}).$$

Proof. From (2) we get

$$\frac{1}{2}p(n) + r_1 p(n-1) + \frac{1}{2}p(n-2) = xp(n-1) = \left(\frac{1}{2}(\gamma + \gamma^{-1}) + r_1\right)p(n-1),$$

so

$$(14) \quad p(n) = (\gamma + \gamma^{-1})p(n-1) - p(n-2) \quad \text{for } n_{2k} < n \leq n_{2k+1},$$

and

$$\frac{1}{2}p(n) + r_2 p(n-1) + \frac{1}{2}p(n-2) = xp(n-1) = (r_2 + \cos \psi)p(n-1),$$

so

$$p(n) = 2p(n-1)\cos \psi - p(n-2) \quad \text{for } n_{2k+1} < n \leq n_{2k+2}.$$

Now we can get (10) and (11) by induction.

To prove (12) observe that by (11) we get

$$(15) \quad p(n_{2k}) = p(n_{2k-1}), \quad p(n_{2k} - 1) = p(n_{2k-1} - 1),$$

for  $k > k_0$ . Arrange the numbers  $n$  satisfying

$$n \leq n_{2k_0+1} \quad \text{or} \quad n_{2k} < n \leq n_{2k+1} \quad \text{for } k > k_0$$

into an increasing sequence. A number  $n$  will occupy the position  $\tilde{n}$  given by

$$\tilde{n} = \begin{cases} n & \text{for } n \leq n_{2k_0+1}, \\ n - \sum_{i=k_0}^{k-1} (n_{2i+2} - n_{2i+1}) = m_n + n_{2k_0} & \text{for } k > k_0, n_{2k} < n \leq n_{2k+1}. \end{cases}$$

Define a sequence  $\tilde{p}(n)$  by  $\tilde{p}(\tilde{n}) = p(n)$ . By (14) and (15) we get

$$\tilde{p}(\tilde{n}) = (\gamma + \gamma^{-1})\tilde{p}(\tilde{n} - 1) - \tilde{p}(\tilde{n} - 2) \quad \text{for } \tilde{n} > n_{2k_0}.$$

By induction we obtain

$$\tilde{p}(\tilde{n}) = \frac{\gamma \tilde{p}(\tilde{n}_{2k_0}) - \tilde{p}(\tilde{n}_{2k_0} - 1)}{\gamma^2 - 1} \gamma^{\tilde{n} - \tilde{n}_{2k_0} + 1} + \frac{\gamma \tilde{p}(\tilde{n}_{2k_0} - 1) - \tilde{p}(\tilde{n}_{2k_0})}{\gamma^2 - 1} \gamma^{-(\tilde{n} - \tilde{n}_{2k_0})}.$$

Taking into account that  $\tilde{n} = n$  for  $n \leq n_{2k_0+1}$  and that  $\tilde{p}(\tilde{n}) = p(n)$  gives the conclusion. ■

PROPOSITION 1. Let  $\psi = (q/q_0)\pi$ ,  $\psi \neq 0, \pi$  and  $q, q_0 \in \mathbb{N}$ . Assume  $q_0 \mid (n_{2k+2} - n_{2k+1})$  for all  $k$  starting from some  $k_0$ . Then there are a constant  $A > 0$  and an integer  $N$  such that

$$(16) \quad |p(n)| > A|\gamma|^m \quad \text{for } N \leq n_{2k} < n \leq n_{2k+2},$$

where

$$m = \min \left( n - n_{2k} + \sum_{i=k_0}^{k-1} (n_{2i+1} - n_{2i}), \sum_{i=k_0}^k (n_{2i+1} - n_{2i}) \right).$$

Proof. From (12) (Lemma 3) we get for  $n_{2k} < n \leq n_{2k+1}$ ,  $k > k_0$ ,

$$(17) \quad |p(n)| \geq \left| \frac{\gamma p(n_{2k_0}) - p(n_{2k_0} - 1)}{\gamma^2 - 1} \right| |\gamma|^{m_n+1} - \left| \frac{\gamma p(n_{2k_0} - 1) - p(n_{2k_0})}{\gamma^2 - 1} \right| |\gamma|^{-m_n},$$

where  $m_n$  is given by (13). By Lemma 2,

$$\left| \frac{\gamma p(n_{2k_0+1}) - p(n_{2k_0+1} - 1)}{\gamma^2 - 1} \right| \neq 0.$$

Since  $|\gamma| > 1$  there exist constants  $B > 0$  and  $k_1$  such that

$$(18) \quad |p(n)| > B|\gamma|^{m_n},$$

where  $k > k_1$  and  $n_{2k} < n \leq n_{2k+1}$ .

From (11) we have for  $n_{2k+1} < n \leq n_{2k+2}$ ,

$$(19) \quad |p(n)| = \left| \frac{p(n_{2k+1} - 1)}{\sin \psi} \right| \left| \frac{p(n_{2k+1})}{p(n_{2k+1} - 1)} \sin(n - n_{2k+1} + 1)\psi - \sin(n - n_{2k+1})\psi \right|.$$

Examine now  $p(n_{2k+1})/p(n_{2k+1} - 1)$ . From (12) we have

$$(20) \quad p(n) = \frac{\gamma p(n_{2k_0+1}) - p(n_{2k_0+1} - 1)}{\gamma^2 - 1} \gamma^{m_n+1} + \frac{\gamma p(n_{2k_0+1} - 1) - p(n_{2k_0+1})}{\gamma^2 - 1} \gamma^{-m_n}$$

for  $k > k_0$ ,  $n = n_{2k+1}$ ,  $n_{2k+1} - 1$ . Lemma 2 states that

$$\forall n \in \mathbb{N} \quad \gamma p(n_{2k_0+1}) - p(n_{2k_0+1} - 1) \neq 0,$$

so from (20) we get

$$(21) \quad \frac{p(n_{2k+1})}{p(n_{2k+1} - 1)} \rightarrow \gamma \quad \text{as } k \rightarrow \infty.$$

Now return to (19). We have assumed that  $r$  is transcendental, hence so is  $\gamma$ . Since  $\sin k\psi$  is an algebraic number,

$$\forall k \in \mathbb{N} \quad \gamma \sin k\psi - \sin(k-1)\psi \neq 0.$$

Moreover, there exists a constant  $C > 0$  (depending on  $\psi$ ) such that

$$\forall k \in \mathbb{N} \quad |\gamma \sin k\psi - \sin(k-1)\psi| > C.$$

By (21) there exists  $k_2$  such that  $|p(n_{2k+1})/p(n_{2k+1} - 1) - \gamma| < C/2$  for  $k > k_2$ . Hence for  $n_{2k+1} < n \leq n_{2k+2}$ ,  $k > k_2$ ,

$$\begin{aligned} & \left| \frac{p(n_{2k+1})}{p(n_{2k+1} - 1)} \sin(n - n_{2k+1} + 1)\psi - \sin(n - n_{2k+1})\psi \right| \\ & \geq |\gamma \sin(n - n_{2k+1} + 1)\psi - \sin(n - n_{2k+1})\psi| \\ & - \left| \frac{p(n_{2k+1})}{p(n_{2k+1} - 1)} - \gamma \right| |\sin(n - n_{2k+1} + 1)\psi| \geq \frac{C}{2}. \end{aligned}$$

By (19) we thus get

$$|p(n)| \geq \left| \frac{p(n_{2k+1} - 1)}{\sin \psi} \right| \frac{C}{2},$$

and by (18),

$$(22) \quad |p(n)| > \left| \frac{BC}{2} \right| |\gamma|^m,$$

where  $n_{2k+1} < n \leq n_{2k+2}$ ,  $k > k_1, k_2$ , and

$$m = n_{2k+1} - 1 - n_{2k} + \sum_{i=k_0}^{k-1} (n_{2i+1} - n_{2i}) < \sum_{i=k_0}^k (n_{2i+1} - n_{2i}).$$

Combining (18) and (22) gives the conclusion with

$$A = \min \left( B, \left| \frac{BC}{2 \sin \psi} \right| \right) \quad \text{and} \quad N = \max(n_{2k_0}, n_{2k_1}, n_{2k_2}). \quad \blacksquare$$

PROPOSITION 2. Let  $n_{2k} < n \leq n_{2k+2}$ . Then

$$(23) \quad |p(n)| \leq M 2^{k-1} A^k |\gamma|^m,$$

where

$$m = \min \left( n - n_{2k} + \sum_{i=0}^{k-1} (n_{2i+1} - n_{2i}), \sum_{i=0}^k (n_{2i+1} - n_{2i}) \right),$$

$$M = \prod_{i=0}^k (n_{2i+2} - n_{2i+1} + 1) \quad \text{and} \quad A = \left| \frac{\gamma^2}{\gamma^2 - 1} \right|.$$

Proof. (10) gives

$$p(n) = \frac{\gamma^2 \gamma^m - \gamma^{-m}}{\gamma^2 - 1} p(n_{2k}) + \frac{\gamma \gamma^m - \gamma \gamma^{-m}}{\gamma^2 - 1} p(n_{2k} - 1)$$

for  $n_{2k} < n \leq n_{2k+1}$ , where  $m = n - n_{2k}$ . Observe that  $\gamma^2 \gamma^m$  and  $\gamma^{-m}$  have the same sign, so

$$\left| \frac{\gamma^2 \gamma^m - \gamma^{-m}}{\gamma^2 - 1} \right| \leq \left| \frac{\gamma^2}{\gamma^2 - 1} \right| |\gamma|^m.$$

In the same way we get

$$\left| \frac{\gamma \gamma^m - \gamma \gamma^{-m}}{\gamma^2 - 1} \right| \leq \left| \frac{\gamma^2}{\gamma^2 - 1} \right| |\gamma|^m.$$

Hence

$$(24) \quad |p(n)| \leq \left| \frac{\gamma^2}{\gamma^2 - 1} \right| |\gamma|^{n-n_{2k}} (|p(n_{2k} - 1)| + |p(n_{2k})|).$$

For  $n_{2k+1} < n \leq n_{2k+2}$  we have from (11) (note that  $|\sin n\psi| \leq n|\sin \psi|$ )

$$\begin{aligned} |p(n)| &\leq (n - n_{2k+1})|p(n_{2k+1} - 1)| + (n - n_{2k+1} + 1)|p(n_{2k+1})| \\ &\leq (n_{2k+2} - n_{2k+1} + 1)(|p(n_{2k+1} - 1)| + |p(n_{2k+1})|). \end{aligned}$$

Hence

$$(25) \quad |p(n)| \leq (n_{2k+2} - n_{2k+1} + 1) \left| \frac{\gamma^2}{\gamma^2 - 1} \right| |\gamma|^{n_{2k+1} - n_{2k}} (|p(n_{2k} - 1)| + |p(n_{2k})|).$$

Combining (24) and (25) gives

$$|p(n)| \leq (n_{2k+2} - n_{2k+1} + 1) \left| \frac{\gamma^2}{\gamma^2 - 1} \right| |\gamma|^m (|p(n_{2k} - 1)| + |p(n_{2k})|),$$

for  $n_{2k} < n \leq n_{2k+2}$ , with  $m = \min(n - n_{2k}, n_{2k+1} - n_{2k})$ . Now we get the conclusion by induction. ■

Clearly the above two propositions can be proved for  $x = r_1 + \cos \psi$  using the same arguments. So set now  $r = r_1 - r_2$ . Moreover, define  $x, \psi, \gamma$  as in (5), (7). In this notation we have:

**PROPOSITION 3.** Let  $q_0 \mid (n_{2k+1} - n_{2k})$  for all  $k$  starting from some  $k_0$  and  $\psi = (q/q_0)\pi$ ,  $\sin \psi \neq 0$ ,  $q, q_0 \in \mathbb{N}$ . Then there are a constant  $A > 0$  and an integer  $N$  such that

$$(26) \quad |p(n)| > A|\gamma|^m \quad \text{for } N \leq n_{2k+1} < n \leq n_{2k+3},$$

where

$$m = \min \left( n - n_{2k+1} + \sum_{i=k_0}^{k-1} (n_{2i+2} - n_{2i+1}), \sum_{i=k_0}^k (n_{2i+2} - n_{2i+1}) \right).$$

**PROPOSITION 4.** Let  $n_{2k+1} < n \leq n_{2k+3}$ . Then

$$(27) \quad |p(n)| \leq (|p(n_1)| + |p(n_0)|) M 2^{k-1} A^k |\gamma|^m,$$

where

$$m = \min \left( n - n_{2k+1} + \sum_{i=0}^{k-1} (n_{2i+2} - n_{2i+1}), \sum_{i=0}^k (n_{2i+2} - n_{2i+1}) \right),$$

$$M = \prod_{i=1}^{k+1} (n_{2i+1} - n_{2i} + 1) \quad \text{and} \quad A = \left| \frac{\gamma^2}{\gamma^2 - 1} \right|.$$

### 3. Proofs of the main theorems

**Proof of Theorem 1.** By Lemma 1 we obtain  $\text{supp } \mu = [-1, 1] \cup [\pi - 1, \pi + 1]$ . Let  $x = \pi + \cos \psi$ , where  $\psi = (q/2^{q_0})\pi$ ,  $\psi \neq k\pi$  and  $q, q_0, k \in \mathbb{N}$ . Observe that  $2^{q_0} \mid (n_{2k+2} - n_{2k+1})$  for  $k > q_0/2$ . By Proposition 1 there exist a constant  $A > 0$  and an integer  $N$  such that

$$|p(n, x)| > A|\gamma|^m \quad \text{for } N \leq n_{2k} < n \leq n_{2k+2},$$

where

$$m = \min \left( n - 2^{2k} + \sum_{i=k_0}^{k-1} 2^{2i}, \sum_{i=k_0}^k 2^{2i} \right).$$

Hence  $|p(n, x)|^{1/n} > A^{1/n} |\gamma|^{m'}$ , where

$$m' \geq \sum_{i=k_0}^{k-1} \frac{2^{2i}}{2^{2k+2}} \xrightarrow{k} \frac{1}{12}.$$

So

$$\liminf_{n \rightarrow \infty} |p(n, x)|^{1/n} \geq |\gamma|^{1/12} > 1.$$

In the same way for  $x = \cos \psi$  (using Proposition 3) we prove

$$\liminf_{n \rightarrow \infty} |p(n, x)|^{1/n} \geq |\gamma|^{1/12} > 1. \quad \blacksquare$$

**Proof of Theorem 2.** Let  $x = \cos \psi$ , where  $\psi = (q/2^{q_0})\pi$ ,  $\psi \neq k\pi$  and  $q, q_0, k \in \mathbb{N}$ . As above,  $2^{q_0} \mid (n_{2k+1} - n_{2k})$  for  $k > q_0/2$ . By Proposition 3 we get a constant  $A > 0$  and an integer  $N$  such that

$$|p(n, x)| > A|\gamma|^m \quad \text{for } N \leq n_{2k+1} < n \leq n_{2k+3},$$

where

$$m = \min \left( n - (k+1)2^k + \sum_{j=k_0+1}^{k-1} (j+1)2^j, \sum_{j=k_0+1}^k (j+1)2^j \right).$$

Hence  $|p(n, x)|^{1/n} > A^{1/n} |\gamma|^{m'}$ , where

$$m' \geq \sum_{j=k_0+1}^{k-1} \frac{(j+1)2^j}{(k+2)2^{k+1}} \xrightarrow{k} \frac{1}{2}.$$

So

$$\liminf_{n \rightarrow \infty} |p(n, x)|^{1/n} \geq |\gamma|^{1/2} > 1.$$

Now let  $x \in [\pi - 1, \pi + 1]$  and  $\psi, \gamma$  be as in (5), (7). By Proposition 2 we get

$$(28) \quad |p(n, x)| \leq M 2^{k-1} A^k |\gamma|^m \quad \text{for } n_{2k} < n \leq n_{2k+2},$$

where

$$m = \min \left( n - k 2^k + \sum_{j=0}^{k-1} 2^j, \sum_{j=0}^k 2^j \right),$$

$$M = \prod_{j=0}^k (2^j(j+1) + 1) \quad \text{and} \quad A = \left| \frac{\gamma^2}{\gamma^2 - 1} \right|.$$

Hence  $|p(n, x)|^{1/n} \leq M^{1/(k 2^k)} (2A)^{1/2^k} |\gamma|^{m'}$ , where  $m' = \sum_{j=0}^k \frac{2^j}{k 2^k} \leq \frac{2}{k}$ . Moreover,  $M \leq \prod_{j=0}^k 2^j(j+2) \leq 2^{k^2} (k+2)!$ , so  $M^{1/(k 2^k)} \xrightarrow{k} 1$ . Therefore

$$\limsup_{n \rightarrow \infty} |p(n, x)|^{1/n} \leq 1.$$

Observe that  $|\gamma|$  is bounded away from 1 for all  $x \in [\pi - 1, \pi + 1]$ , so  $A$  is bounded away from 0. Hence (28) holds with constant  $A$  independent of  $x$ . So  $\limsup_{n \rightarrow \infty} |p(n, x)|^{1/n} \leq 1$  uniformly for  $x \in [\pi - 1, \pi + 1]$ . ■

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## Edward Marczewski Collected Mathematical Papers

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Edward Marczewski (Szpilrajn) (1907–1976), one of the most distinguished Polish mathematicians, was a disciple and an active member of the Warsaw School of Mathematics between the two World Wars. His life and work after the Second World War were connected with Wrocław, where he was among the creators of the Polish scientific centre.

Marczewski's main fields of interest were measure theory, descriptive set theory, general topology, probability theory and universal algebra. He also published papers on real and complex analysis, applied mathematics and mathematical logic.

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