*derivation with $D(z)$ a function of bounded variation on $[a, 1]$ for all $0 < a < 1$. Suppose that the continuity ideal is $I(D) = M_{n,k-1}$ for some $1 \leqslant k \leqslant n$. Then there exist discontinuous linear functionals $\alpha_1, \ldots, \alpha_k$ on $C^n[0, 1]$ such that*

$$D(f) = T(f) + \frac{1}{(n-1)!} \sum_{j=n-k}^{n-1} \binom{n-1}{j} (-1)^j \alpha_{n-j}(f) z^j, \quad f \in M_{n,n},$$

*where $T$ is a continuous linear map from $C^n[0, 1]$ to $L_p(0, 1)$ which is completely determined by $D(z)$ and $\varrho_n$.*

Proof. Since $L_p(0, 1) \subseteq L_1(0, 1)$ for $p \geqslant 1$, we can consider $D$ as a derivation from $C^n[0, 1]$ into $L_1(0, 1)$. By Theorem 3.5 we can write

$$D(f) = T(f) + \frac{1}{(n-1)!} \sum_{j=n-k}^{n-1} \binom{n-1}{j} (-1)^j \alpha_{n-j}(f) z^j, \quad f \in M_{n,n},$$

so that

$$T(f) = D(f) - \frac{1}{(n-1)!} \sum_{j=n-k}^{n-1} \binom{n-1}{j} (-1)^j \alpha_{n-j}(f) z^j, \quad f \in M_{n,n}.$$

Since all the terms on the left-hand side are in $L_p(0, 1)$, $T(f) \in L_p(0, 1)$ for all $f \in M_{n,n}$. Let $y \in L_p(0, 1)$ be in the separating space $S(T)$ of $T$. There exists $f_m \to 0$ in $C^n[0, 1]$ and $T(f_m) \to y$ in $L_p(0, 1)$. By Theorem 3.5, $T$ is a continuous linear map from $C^n[0, 1]$ into $L_1(0, 1)$, so that $T(f_m) \to 0$ in $L_1(0, 1)$. Thus $y = 0$, and we conclude that $T$ is continuous. This completes the proof.

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DEPARTMENT OF MATHEMATICS
CALIFORNIA STATE UNIVERSITY
Long Beach, California 90840, U.S.A.

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# On the integrability and $L^1$-convergence of double trigonometric series

by

FERENC MÓRICZ (Szeged)

**Abstract.** We study double cosine and sine series whose coefficients form a null sequence of bounded variation. In particular, we consider the special cases where the double sequence of coefficients is monotone decreasing, or convex, or quasiconvex. We are mainly concerned with the following problems: (i) the series in question converges pointwise, (ii) the sum of the series is integrable, (iii) the series is the Fourier series of its sum, (iv) the series converges in $L^1$-norm.

Among other things, we extend the classical theorems of Kolmogorov and Young from one-dimensional cosine and sine series to two-dimensional ones in an essentially more general setting. Our basic tools are Sidon type inequalities.

**0. Introduction.** The following theorems are well known for one-dimensional cosine and sine series.

THEOREM A (Kolmogorov [6] and see also [11, Vol. 1, pp. 183–184]). *If $\{a_j: j \geqslant 0\}$ is a quasiconvex null sequence, then the cosine series*

$$\tfrac{1}{2} a_0 + \sum_{j=1}^{\infty} a_j \cos jx \tag{0.1}$$

*converges, except possibly at $x = 0$, to an integrable function $f(x)$, is the Fourier series of $f$, and the partial sums converge in $L^1(0, \pi)$-norm to $f$ if and only if $a_j \ln j \to 0$ as $j \to \infty$.*

THEOREM B (W. H. Young [10] and see also [11, Vol. 1, pp. 185–186]). *If $\{a_j: j \geqslant 1\}$ is a monotone decreasing null sequence, then the sine series*

$$\sum_{j=1}^{\infty} a_j \sin jx \tag{0.2}$$

*converges to a function $g(x)$ at every $x$, and $g$ is integrable if and only if $\sum (a_j - a_{j+1}) \ln j < \infty$. If this condition is satisfied, then (0.2) is the Fourier series of $g$, and the partial sums converge in $L^1(0, \pi)$-norm to $g$.*

In this paper we will extend these results to two-dimensional trigonometric series (see Corollary 3 in Section 2 and Theorem 5 in Section 6) in an essentially

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more general setting. Our basic tools are Sidon type inequalities. To reveal the essence, here we formulate two special cases (as to the more powerful inequalities, see Lemmas 3 and 5 in Section 7): There exist positive constants $C$ and $\tilde{C}$ such that for all double sequences $\{a_{jk}\}$ of real numbers, and for all nonnegative integers $m, n$ we have

$$\int_0^{\pi}\int_0^{\pi}\Big|\sum_{j=0}^{m}\sum_{k=0}^{n} a_{jk}D_j(x)D_k(y)\Big|\,dx\,dy \leqslant C(m+1)(n+1)\max_{0\leqslant j\leqslant m}\ \max_{0\leqslant k\leqslant n}|a_{jk}|, \tag{0.3}$$

$$\int_0^{\pi}\int_0^{\pi}\Big|\sum_{j=1}^{m}\sum_{k=1}^{n} a_{jk}D_j'(x)D_k'(y)\Big|\,dx\,dy \leqslant \tilde{C}mn\max_{1\leqslant j\leqslant m}\ \max_{1\leqslant k\leqslant n} jk|a_{jk}|, \tag{0.4}$$

where $D_m$ is the Dirichlet kernel (see (1.5) below) and prime means derivative. The first of these is the extension of the celebrated Sidon inequality from one-dimensional to two-dimensional trigonometric polynomials, obtained earlier by Telyakovskii [9] using a different method.

In the sequel, we will study double cosine and sine series whose coefficients form a null sequence of bounded variation (see (1.2) and (1.3) below). In particular, we will consider the special cases where the double sequence of coefficients is monotone decreasing, or convex, or quasiconvex. We will mainly be concerned with the following problems:

(i) the series converges pointwise,
(ii) the sum of the series is integrable,
(iii) the series is the Fourier series of its sum,
(iv) the series converges in $L^1$-norm.

**1. Cosine series.** We consider double cosine series

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\lambda_j\lambda_k a_{jk}\cos jx\cos ky \tag{1.1}$$

on the positive quadrant $T^2 = [0, \pi]\times[0, \pi]$ of the two-dimensional torus, where $\lambda_0 = \frac{1}{2}$ and $\lambda_j = 1$ if $j \geqslant 1$; and the real coefficients $a_{jk}$ form a null sequence of bounded variation, that is,

$$a_{jk}\to 0 \quad \text{as } \max(j, k)\to\infty, \tag{1.2}$$

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\Delta_{11}a_{jk}| < \infty. \tag{1.3}$$

We remind the reader that the differences $\Delta_{pq}a_{jk}$ are defined for all pairs of nonnegative integers $p$ and $q$ as follows:

$$\Delta_{pq}a_{jk} = \sum_{i=0}^{p}\sum_{l=0}^{q}(-1)^{i+l}\binom{p}{i}\binom{q}{l}a_{j+i,k+l} \qquad (j, k \geqslant 0).$$

Then the following recurrence relations hold: $\Delta_{00}a_{jk} = a_{jk}$,



$$\Delta_{pq}a_{jk} = \Delta_{p-1,q}a_{jk} - \Delta_{p-1,q}a_{j+1,k} \qquad (p \geqslant 1),$$
$$\Delta_{pq}a_{jk} = \Delta_{p,q-1}a_{jk} - \Delta_{p,q-1}a_{j,k+1} \qquad (q \geqslant 1).$$

The next simple observation will be useful on many occasions: If (1.2) is satisfied and for some $p, q \geqslant 0$,

$$\Delta_{pq}a_{jk} \geqslant 0 \qquad (j, k \geqslant 0),$$

then for all $0 \leqslant p_1 \leqslant p$ and $0 \leqslant q_1 \leqslant q$,

$$\Delta_{p_1q_1}a_{jk} \geqslant 0 \qquad (j, k \geqslant 0).$$

Consequently, the sequence $\{\Delta_{p_1q_1}a_{jk}\}$ is monotone decreasing in both $j$ and $k$ provided either $p_1 < p$ and $q_1 \leqslant q$ or $p_1 \leqslant p$ and $q_1 < q$.

We denote by

$$s_{mn}(x, y) = \sum_{j=0}^{m}\sum_{k=0}^{n}\lambda_j\lambda_k a_{jk}\cos jx\cos ky \qquad (m, n \geqslant 0)$$

the rectangular partial sums of series (1.1). Performing a double summation by parts yields

$$\begin{aligned} s_{mn}(x, y) = {} & \sum_{j=0}^{m}\sum_{k=0}^{n} D_j(x)D_k(y)\Delta_{11}a_{jk} + \sum_{j=0}^{m} D_j(x)D_n(y)\Delta_{10}a_{j,n+1} \\ & + \sum_{k=0}^{n} D_m(x)D_k(y)\Delta_{01}a_{m+1,k} + a_{m+1,n+1}D_m(x)D_n(y), \end{aligned} \tag{1.4}$$

where $D_m$ is the Dirichlet kernel:

$$D_m(x) = \frac{1}{2} + \sum_{j=1}^{m}\cos jx = \frac{\sin(m+\frac{1}{2})x}{2\sin\frac{1}{2}x} \qquad (m \geqslant 0). \tag{1.5}$$

Since $|D_m(x)| \leqslant \pi/(2x)$ $(m \geqslant 0;\ 0 < x \leqslant \pi)$, by (1.3), for all $0 < x, y \leqslant \pi$,

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|D_j(x)D_k(y)\Delta_{11}a_{jk}| < \infty.$$

By (1.2),

$$\Delta_{10}a_{j,n+1} = \sum_{k=n+1}^{\infty}\Delta_{11}a_{jk}, \tag{1.6}$$

whence, by (1.3),

$$\sum_{j=0}^{\infty}|\Delta_{10}a_{j,n+1}| \leqslant \sum_{j=0}^{\infty}\sum_{k=n+1}^{\infty}|\Delta_{11}a_{jk}| \to 0 \qquad \text{as } n\to\infty. \tag{1.7}$$

This implies that, for all $0 < x, y \leqslant \pi$,

$$\sum_{j=0}^{m} D_j(x)D_n(y)\Delta_{10}a_{j,n+1} \to 0 \qquad \text{as } n\to\infty,$$

uniformly in $m$. Similarly, for all $0 < x, y \leqslant \pi$,

$$\sum_{k=0}^{n} D_m(x) D_k(y) \Delta_{01} a_{m+1,k} \to 0 \quad \text{as } m \to \infty,$$

uniformly in $n$, and $a_{m+1,n+1} D_m(x) D_n(y) \to 0$ as $\max(m, n) \to \infty$. Consequently, series (1.1) converges to the function $f$ defined by

$$f(x, y) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} D_j(x) D_k(y) \Delta_{11} a_{jk} \tag{1.8}$$

for all $0 < x, y \leqslant \pi$ in the sense that $s_{mn}(x, y) \to f(x, y)$ as $\min(m, n) \to \infty$.

Motivated by (1.4), we introduce the modified rectangular partial sums $u_{mn}$ of series (1.1) defined by

$$\begin{aligned} u_{mn}(x, y) = s_{mn}(x, y) &- \sum_{j=0}^{m} D_j(x) D_n(y) \Delta_{10} a_{j,n+1} \\ &- \sum_{k=0}^{n} D_m(x) D_k(y) \Delta_{01} a_{m+1,k} - a_{m+1,n+1} D_m(x) D_n(y). \end{aligned} \tag{1.9}$$

It will turn out that the $u_{mn}$ approximate $f$ better than $s_{mn}$ since they converge to $f$ in $L^1$-norm when $s_{mn}$ may not.

According to (1.4) and (1.9),

$$u_{mn}(x, y) = \sum_{j=0}^{m} \sum_{k=0}^{n} D_j(x) D_k(y) \Delta_{11} a_{jk}. \tag{1.10}$$

Another representation for $u_{mn}$ can be obtained by performing two single summations by parts:

$$u_{mn}(x, y) = \sum_{j=0}^{m} \sum_{k=0}^{n} \Big(\sum_{i=j}^{m} \sum_{l=k}^{n} \Delta_{11} a_{il}\Big) \lambda_j \lambda_k \cos jx \cos ky.$$

We note that for one-dimensional cosine series analogous modified partial sums were introduced by Garrett and Stanojević [3].

Our first main result is the following.

THEOREM 1. *If a double sequence $\{a_{jk}\}$ satisfies* (1.2) *and for some $p > 1$,*

$$\begin{aligned} \mathscr{A}_p = |\Delta_{11} a_{00}| &+ \sum_{m=0}^{\infty} 2^m \Big[2^{-m} \sum_{j=2^m}^{2^{m+1}-1} |\Delta_{11} a_{j0}|^p\Big]^{1/p} \\ &+ \sum_{n=0}^{\infty} 2^n \Big[2^{-n} \sum_{k=2^n}^{2^{n+1}-1} |\Delta_{11} a_{0k}|^p\Big]^{1/p} \\ &+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 2^{m+n} \Big[2^{-m-n} \sum_{j=2^m}^{2^{m+1}-1} \sum_{k=2^n}^{2^{n+1}-1} |\Delta_{11} a_{jk}|^p\Big]^{1/p} < \infty, \end{aligned} \tag{1.11}$$



*then*

$$\|u_{mn} - f\| \to 0 \quad \textit{as } \min(m, n) \to \infty. \tag{1.12}$$

Here $\|\cdot\|$ denotes the two-dimensional $L^1(T^2)$-norm. Later on, $\|\cdot\|$ will stand for this norm or for the one-dimensional $L^1(T)$-norm, $T = [0, \pi]$. It will be clear from the context what the case is.

As a by-product of the proof, we can get

$$\Big\|\sum_{j=0}^{m} \sum_{k=0}^{n} D_j(x) D_k(y) \Delta_{11} a_{jk}\Big\| \leqslant C_p \mathscr{A}_p \qquad (m, n \geqslant 0;\ p > 1).$$

Here and in the sequel, $C_p$, $\tilde{C}_p$, $C_p^*$, etc. denote positive constants depending only on $p$ and not necessarily the same at different occurrences.

We draw three corollaries of Theorem 1.

COROLLARY 1. *Under the conditions of Theorem* 1, *the sum $f$ of series* (1.1) *is integrable and* (1.1) *is the Fourier series of $f$.*

COROLLARY 2. *If a double sequence $\{a_{jk}\}$ satisfies* (1.2), (1.11), *and*

$$\ln(n+2)\Big\{|\Delta_{10} a_{0,n+1}| + \sum_{m=0}^{\infty} 2^m \Big[2^{-m} \sum_{j=2^m}^{2^{m+1}-1} |\Delta_{10} a_{j,n+1}|^p\Big]^{1/p}\Big\} \to 0 \quad \textit{as } n \to \infty, \tag{1.13}$$

$$\ln(m+2)\Big\{|\Delta_{01} a_{m+1,0}| + \sum_{n=0}^{\infty} 2^n \Big[2^{-n} \sum_{k=2^n}^{2^{n+1}-1} |\Delta_{01} a_{m+1,k}|^p\Big]^{1/p}\Big\} \to 0 \quad \textit{as } m \to \infty \tag{1.14}$$

*for some $p > 1$, then*

$$\|s_{mn} - f\| \to 0 \quad \textit{as } \min(m, n) \to \infty \tag{1.15}$$

*if and only if*

$$a_{mn} \ln(m+2) \ln(n+2) \to 0 \quad \textit{as } \min(m, n) \to \infty. \tag{1.16}$$

We note that Theorem 1 and Corollaries 1 and 2 can be considered the extensions of the corresponding results of Fomin [2] from one-dimensional to two-dimensional cosine series.

**2. Special cases of condition (1.11).** By Hölder's inequality,

$$\mathscr{A}_{p_1} \leqslant \mathscr{A}_{p_2} \qquad (0 < p_1 < p_2). \tag{2.1}$$

Putting $p_1 = 1$, it follows that if (1.11) is satisfied for some $p > 1$, then $\{a_{jk}\}$ is of bounded variation, i.e. (1.3) is also satisfied. On the other hand, we can easily see that for all $p > 0$, (2.2) $$\mathscr{A}_p \leqslant \mathscr{A}_\infty = |\Delta_{11} a_{00}| + \sum_{m=0}^{\infty} 2^m \max_{2^m \leqslant j < 2^{m+1}} |\Delta_{11} a_{j0}| + \sum_{n=0}^{\infty} 2^n \max_{2^n \leqslant k < 2^{n+1}} |\Delta_{11} a_{0k}| + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 2^{m+n} \max_{2^m \leqslant j < 2^{m+1}} \max_{2^n \leqslant k < 2^{n+1}} |\Delta_{11} a_{jk}|.$$

Following Nosenko [8] and Telyakovskii [9], a null sequence $\{a_{jk}\}$ is said to belong to the class $\mathscr{S}$ if there exists a double sequence $\{\varepsilon_{jk}\}$ of nonnegative numbers such that

(2.3) $$|\Delta_{11} a_{jk}| \leqslant \varepsilon_{jk}, \quad \Delta_{11} a_{jk} \geqslant 0, \quad \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \varepsilon_{jk} < \infty.$$

We note that the last condition here is equivalent to the condition

$$\sum_{m=0}^{\infty} 2^m \varepsilon_{2^m,0} + \sum_{n=0}^{\infty} 2^n \varepsilon_{0,2^n} + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 2^{m+n} \varepsilon_{2^m,2^n} < \infty.$$

It follows from this and (2.2) that if $\{a_{jk}\} \in \mathscr{S}$, then $\mathscr{A}_\infty < \infty$, and consequently, (1.11) is satisfied for all $p > 0$. Thus, the conclusions of Theorem 1 and of Corollaries 1 and 2 hold true in case $\{a_{jk}\} \in \mathscr{S}$; some of them were proved in [8] and [9].

EXAMPLE 1. Define a sequence $\{a_j: j > 0\}$ by

$$\Delta a_{2^m} = [2^{m/2}(m+1)\ln^2(m+2)]^{-1} \quad \text{if } m = 0, 1, \ldots;$$
$$\Delta a_j = 0 \quad \text{if } 2^m < j < 2^{m+1} \text{ for some } m > 0,$$

where $\Delta a_j = a_j - a_{j+1}$; and let $a_{jk} = a_j a_k$. Then $\mathscr{A}_2 < \infty$ but $\{a_{jk}\} \notin \mathscr{S}$. This example shows that $\mathscr{S}$ is a proper subclass of the double sequences $\{a_{jk}\}$ for which (1.11) is satisfied with $p = 2$.

We remind the reader that a double sequence $\{a_{jk}\}$ is said to be *quasiconvex* if

(2.4) $$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (j+1)(k+1)|\Delta_{22} a_{jk}| < \infty.$$

By setting $\varepsilon_{jk} = \sum_{m=j}^{\infty} \sum_{n=k}^{\infty} |\Delta_{22} a_{mn}|$, it is not hard to verify that $\mathscr{S}$ contains all quasiconvex null sequences.

We note that it follows from (1.2) and (2.4) that the single sequence $\{a_{jk}: j \geqslant 0\}$ is then quasiconvex:

$$\sum_{j=0}^{\infty} (j+1)|\Delta_{20} a_{jk}| < \infty \quad (k \geqslant 0).$$

Similarly, $\{a_{jk}: k \geqslant 0\}$ is also quasiconvex for each fixed $j \geqslant 0$.



As is known, $\{a_{jk}\}$ is said to be *convex* if

(2.5) $$\Delta_{22} a_{jk} \geqslant 0 \quad (j, k \geqslant 0).$$

It is a routine matter to prove that every bounded convex sequence is also quasiconvex.

Before formulating Corollary 3 we make some preparations. Clearly, under conditions (1.2) and (1.3), $a_{j0} \to 0$ as $j \to \infty$, and $\sum_{j=0}^{\infty} |\Delta_{10} a_{j0}| < \infty$. So, the first row in (1.1) (i.e. when $k = 0$) converges, except possibly at $x = 0$:

(2.6) $$\sum_{j=0}^{\infty} \lambda_j a_{j0} \cos jx = f_1(x), \quad \text{say}.$$

Actually, we have dropped here the factor $\lambda_0 = \frac{1}{2}$ corresponding to $k = 0$. Analogously, the first column in (1.1) (i.e. when $j = 0$) converges, except possibly at $y = 0$:

(2.7) $$\sum_{k=0}^{\infty} \lambda_k a_{0k} \cos ky = f_2(y), \quad \text{say}.$$

We denote by $s_m^{(1)}(x)$ and $s_n^{(2)}(y)$ the partial sums of series (2.6) and (2.7), respectively.

COROLLARY 3. *If a double sequence $\{a_{jk}\}$ satisfies* (1.2) *and* (2.4), *then the sum $f$ of series* (1.1) *is integrable and* (1.1) *is the Fourier series of $f$. If, in addition,*

(2.8) $$\Delta_{20} a_{jk} \geqslant 0, \quad \Delta_{02} a_{jk} \geqslant 0 \quad (j, k \geqslant 0),$$

*then*

(2.9) $$\begin{cases} \|s_{mn} - f\| \to 0 & \text{as } \min(m, n) \to \infty, \\ \|s_m^{(1)} - f_1\| \to 0 & \text{as } m \to \infty, \\ \|s_n^{(2)} - f_2\| \to 0 & \text{as } n \to \infty, \end{cases}$$

*if and only if*

(2.10) $$a_{mn} \ln(m+2) \ln(n+2) \to 0 \quad \text{as } \max(m, n) \to \infty.$$

However, it is an open problem whether the equivalence (2.9)$\Leftrightarrow$(2.10) holds under conditions (1.2) and (2.4) (without (2.8)).

We note that if (1.2), (2.4), and (2.10) are satisfied, then each row and each column in series (1.1) converges both pointwise (except possibly at $x = 0$ or $y = 0$, respectively) as well as in $L^1$-norm. Furthermore, the equivalence (2.9)$\Leftrightarrow$(2.10) can be reformulated as the equivalence of the regular convergence of (1.1) in $L^1(T^2)$-norm with the fulfillment of (2.10). (Concerning the notion of regular convergence, we refer to [4].)

Corollary 3 is an extension of the famous Kolmogorov result (see [6] and also [11, Vol. 1, pp. 183–184] for the convex case) from one-dimensional to two-dimensional cosine series.

It is of some interest to observe that (1.2) and

$$\Delta_{21} a_{jk} \geqslant 0, \quad \Delta_{12} a_{jk} \geqslant 0 \quad (j, k \geqslant 0) \tag{2.11}$$

imply that $\Delta_{11} a_{jk}$ is nonnegative and nonincreasing in both $j$ and $k$. Thus, we can conclude that $\mathscr{A}_p < \infty$ in (1.11), independently of $p$, and Theorem 1 applies. This means that Corollary 3 remains true if (2.4) and (2.8) are replaced by (2.11).

**3. Another sufficient condition for cosine series.** The condition in question is

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\Delta_{11} a_{jk}| \ln(j+2) \ln(k+2) < \infty, \tag{3.1}$$

which is clearly stronger that (1.3). It is routine to show that if (1.2) is satisfied and

$$\Delta_{11} a_{jk} \geqslant 0 \quad (j, k \geqslant 0), \tag{3.2}$$

then (3.1) is equivalent to the condition

$$\sum_{j=1}^{\infty} \frac{a_{j0}}{j} + \sum_{k=1}^{\infty} \frac{a_{0k}}{k} + \sum_{j=1}^{k} \sum_{k=1}^{\infty} \frac{a_{jk}}{jk} < \infty. \tag{3.3}$$

THEOREM 2. *If a double sequence $\{a_{jk}\}$ satisfies (1.2) and (3.1), then the sum $f$ of series (1.1) is integrable, (1.1) is the Fourier series of $f$, and (1.15) holds.*

We note that if $\{a_{jk}\}$ is a convex null sequence, then (3.1) is equivalent to the condition

$$\sum_{m=0}^{\infty} (m+1) 2^m \Delta_{11} a_{2^m,0} + \sum_{n=0}^{\infty} (n+1) 2^n \Delta_{11} a_{0,2^n} + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (m+1)(n+1) 2^{m+n} \Delta_{11} a_{2^m,2^n} < \infty, \tag{3.4}$$

while (1.11), for any $p > 0$, is equivalent to the condition

$$\sum_{m=0}^{\infty} 2^m \Delta_{11} a_{2^m,0} + \sum_{n=0}^{\infty} 2^n \Delta_{11} a_{0,2^n} + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 2^{m+n} \Delta_{11} a_{2^m,2^n} < \infty. \tag{3.5}$$

It is striking that (3.5) is more general that (3.4). Thus, condition (1.11) is stronger than (3.1) for convex null sequences.

On the other hand, for nonconvex sequences (3.1) may be satisfied, while (1.11) is not.

EXAMPLE 2. Define $\{a_j : j \geqslant 0\}$ by $\Delta a_{2^m} = m^{-3}$ if $m \geqslant 0$ and $\Delta a_j = 0$ otherwise (cf. Example 1), and set $a_{jk} = a_j a_k$. Then (3.1) is satisfied, but (1.11) is not satisfied for any $p > 1$.



**4. Sine series.** We consider double sine series

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk} \sin jx \sin ky \tag{4.1}$$

on $T^2 = [0, \pi] \times [0, \pi]$, where the real coefficients $a_{jk}$ satisfy conditions (1.2) and (4.3) below.

Following an idea of Kano [5], we represent the rectangular partial sums

$$s_{mn}(x, y) = \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk} \sin jx \sin ky \quad (m, n \geqslant 1)$$

of series (4.1) in the form

$$s_{mn}(x, y) = \sum_{j=1}^{m} \sum_{k=1}^{n} b_{jk} [\cos jx]' [\cos ky]'$$

where prime means derivative and

$$b_{jk} = a_{jk}/(jk) \quad (j, k \geqslant 1). \tag{4.2}$$

We will assume that

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} jk |\Delta_{11} b_{jk}| < \infty. \tag{4.3}$$

This condition expresses a modified version of bounded variation, which seems to be more appropriate for sine series than bounded variation in the ordinary sense.

Performing a double summation by parts, we get another representation of $s_{mn}$ as follows:

$$s_{mn}(x, y) = \sum_{j=1}^{m} \sum_{k=1}^{n} D_j'(x) D_k'(y) \Delta_{11} b_{jk} + \sum_{j=1}^{m} D_j'(x) D_n'(y) \Delta_{10} b_{j,n+1} + \sum_{k=1}^{n} D_m'(x) D_k'(y) \Delta_{01} b_{m+1,k} + b_{m+1,n+1} D_m'(x) D_n'(y). \tag{4.4}$$

It is not hard to check that $|D_m'(x)| \leqslant Cm/x^2$ $(m \geqslant 1;\ 0 < x \leqslant \pi)$. Now, a routine argument gives that if (1.2) and (4.3) are satisfied, then for all $x, y$,

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |D_j'(x) D_k'(y) \Delta_{11} b_{jk}| < \infty.$$

By (1.2), $\Delta_{10} b_{j,n+1} = \sum_{k=n+1}^{\infty} \Delta_{11} b_{jk}$ (cf. (1.6)), whence by (4.3),

$$\sum_{j=1}^{\infty} jn |\Delta_{10} b_{j,n+1}| \leqslant \sum_{j=1}^{\infty} \sum_{k=n+1}^{\infty} jk |\Delta_{11} b_{jk}| \to 0 \quad \text{as } n \to \infty.$$

This implies that for all $x$, $y$

$$\sum_{j=1}^{m} D_j'(x)D_n'(y)\Delta_{10}b_{j,n+1}\to 0 \quad \text{as } n\to\infty,$$

uniformly in $m$. Similarly, for all $x$, $y$

$$\sum_{k=1}^{n} D_m'(x)D_k'(y)\Delta_{01}b_{m+1,k}\to 0 \quad \text{as } m\to\infty,$$

uniformly in $n$, and $b_{m+1,n+1}D_m'(x)D_n'(y)\to 0$ as $\max(m,n)\to\infty$. To sum up, we can write for all $x$, $y$,

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk}\sin jx\sin ky = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} D_j'(x)D_k'(y)\Delta_{11}b_{jk} = g(x,y), \quad \text{say.} \tag{4.5}$$

Motivated by (4.4), we introduce the modified rectangular partial sums $v_{mn}$ of series (4.1) defined by

$$v_{mn}(x,y) = s_{mn}(x,y) - \sum_{j=1}^{m} D_j'(x)D_n'(y)\Delta_{10}b_{j,n+1} - \sum_{k=1}^{n} D_m'(x)D_k'(y)\Delta_{01}b_{m+1,k} - b_{m+1,n+1}D_m'(x)D_n'(y). \tag{4.6}$$

According to (4.4) and (4.6),

$$v_{mn}(x,y) = \sum_{j=1}^{m}\sum_{k=1}^{n} D_j'(x)D_k'(y)\Delta_{11}b_{jk}. \tag{4.7}$$

Another representation for $v_{mn}$ can be obtained by performing two single summations by parts which result in

$$v_{mn}(x,y) = \sum_{j=1}^{m}\sum_{k=1}^{n}\Big(\sum_{i=j}^{m}\sum_{l=k}^{n}\Delta_{11}b_{il}\Big)\sin jx\sin ky + \sum_{j=1}^{m}\Big(\sum_{i=j}^{m}\Delta_{10}a_{i,n+1}\Big)\tilde{K}_n(y)\sin jx + \sum_{k=1}^{n}\Big(\sum_{l=k}^{n}\Delta_{01}a_{m+1,l}\Big)\tilde{K}_m(x)\sin ky + a_{m+1,n+1}\tilde{K}_m(x)\tilde{K}_n(y),$$

where $\tilde{K}_m$ is the conjugate Fejér kernel:

$$\tilde{K}_m(x) = \frac{1}{m+1}\sum_{j=1}^{m}\tilde{D}_j(x) = \sum_{j=1}^{m}\Big(1-\frac{j}{m+1}\Big)\sin jx,$$

while $\tilde{D}_m$ is the conjugate Dirichlet kernel:

$$\tilde{D}_m(x) = \sum_{j=1}^{m}\sin jx = \tfrac{1}{2}\cot\tfrac{1}{2}x - \frac{\cos(m+\frac{1}{2})x}{2\sin\frac{1}{2}x} \quad (m\geqslant 1). \tag{4.8}$$

We note that for one-dimensional sine series analogous modified partial sums were defined in [7].



Our second main result reads as follows.

THEOREM 3. *If a double sequence $\{a_{jk}\}$ satisfies* (1.2) *and for some $p>1$,*

$$\mathscr{B}_p = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} 2^{m+n}\Big[2^{-m-n}\sum_{j=2^m}^{2^{m+1}-1}\sum_{k=2^n}^{2^{n+1}-1} j^p k^p|\Delta_{11}b_{jk}|^p\Big]^{1/p} < \infty, \tag{4.9}$$

*then*

$$\|v_{mn}-g\|\to 0 \quad \textit{as } \min(m,n)\to\infty. \tag{4.10}$$

As a by-product of the proof, we can see that

$$\Big\|\sum_{j=1}^{m}\sum_{k=1}^{n} D_j'(x)D_k'(y)\Delta_{11}b_{jk}\Big\| \leqslant \tilde{C}_p\mathscr{B}_p \quad (m,n\geqslant 1;\ p>1).$$

We draw two corollaries of Theorem 3.

COROLLARY 4. *Under the conditions of Theorem* 3, *the sum $g$ of series* (4.1) *is integrable and* (4.1) *is the Fourier series of $g$.*

COROLLARY 5. *If a double sequence $\{a_{jk}\}$ satisfies* (1.2), (4.9), *and*

$$n\ln(n+2)\sum_{m=0}^{\infty}2^m\Big[2^{-m}\sum_{j=2^m}^{2^{m+1}-1} j^p|\Delta_{10}b_{j,n+1}|^p\Big]^{1/p}\to 0 \quad \textit{as } n\to\infty, \tag{4.11}$$

$$m\ln(m+2)\sum_{n=0}^{\infty}2^n\Big[2^{-n}\sum_{k=2^n}^{2^{n+1}-1} k^p|\Delta_{01}b_{m+1,k}|^p\Big]^{1/p}\to 0 \quad \textit{as } m\to\infty \tag{4.12}$$

*for some $p>1$, then*

$$\|s_{mn}-g\|\to 0 \quad \textit{as } \min(m,n)\to\infty \tag{4.13}$$

*if and only if condition* (1.16) *is satisfied.*

We note that Theorem 3 and Corollaries 4 and 5 can be considered the extensions of the corresponding results in [7] from one-dimensional to two-dimensional sine series.

Finally, we compare conditions (1.11) and (4.9). To this end assume that $a_{jk}=0$ if $\min(j,k)=0$. By (4.2),

$$jk\Delta_{11}b_{jk} = \Delta_{11}a_{jk} + \frac{\Delta_{10}a_{j,k+1}}{k+1} + \frac{\Delta_{01}a_{j+1,k}}{j+1} + \frac{a_{j+1,k+1}}{(j+1)(k+1)} \quad (j,k\geqslant 1). \tag{4.14}$$

Hence it follows that if (3.2) is satisfied, then $0\leqslant\Delta_{11}a_{jk}\leqslant jk\Delta_{11}b_{jk}$ and (4.9) implies (1.11).

The converse implication is not true in general.

EXAMPLE 3. Let $a_j=[\ln(j+2)\ln\ln(j+4)]^{-1}$ $(j\geqslant 0)$ and let $a_{jk}=a_ja_k$. Then (1.11) does hold, while (4.9) does not hold for any $p>1$.

To be more precise, by (4.14), conditions (1.11) and (4.9) are equivalent for some $p>1$ provided

$$(4.15)\quad \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Big\{2^m\Big[2^{-m-n}\sum_{j=2^m}^{2^{m+1}-1}\sum_{k=2^n}^{2^{n+1}-1}|\Delta_{10}a_{j,k+1}|^p\Big]^{1/p} +2^n\Big[2^{-m-n}\sum_{j=2^m}^{2^{m+1}-1}\sum_{k=2^n}^{2^{n+1}-1}|\Delta_{01}a_{j+1,k}|^p\Big]^{1/p} +\Big[2^{-m-n}\sum_{j=2^m}^{2^{m+1}-1}\sum_{k=2^n}^{2^{n+1}-1}|a_{j+1,k+1}|^p\Big]^{1/p}\Big\}<\infty.$$

Now, assume that (1.2) and (3.2) are satisfied. Then (4.15), for all $p>0$, is equivalent to each of the following conditions:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\{2^m\Delta_{10}a_{2^m,2^n}+2^n\Delta_{01}a_{2^m,2^n}+a_{2^m,2^n}\}<\infty,$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\Big\{\frac{\Delta_{10}a_{jk}}{k}+\frac{\Delta_{01}a_{jk}}{j}+\frac{a_{jk}}{jk}\Big\}<\infty,$$

and

$$(4.16)\quad \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}a_{jk}/(jk)<\infty.$$

To sum up, under (1.2), (3.2), and (4.16), conditions (1.11) and (4.9) are equivalent.

**5. Another sufficient condition for sine series.** The condition in question is

$$(5.1)\quad \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\Delta_{11}a_{jk}|\ln(j+1)\ln(k+1)<\infty,$$

which is essentially (3.1). Again, if (1.2) and (3.2) (for $j,k\geqslant 1$) are satisfied, then (5.1) is equivalent to (4.16) (cf. (3.3)).

THEOREM 4. *If a double sequence $\{a_{jk}\}$ satisfies* (1.2) *and* (5.1), *then the sum $g$ of series* (4.1) *is integrable,* (4.1) *is the Fourier series of $g$, and* (4.13) *holds.*

In the sequel, we compare conditions (4.9) and (5.1).

(a) If (1.2) is satisfied and $\{b_{jk}\}$ is convex in the sense of (2.5), then (4.9) and (5.1) are equivalent.

In fact, in this case (4.9), for any $p>0$, is equivalent to each of the conditions

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}2^{2(m+n)}\Delta_{11}b_{2^m,2^n}<\infty$$



and

$$(5.2)\quad \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}jk\Delta_{11}b_{jk}<\infty.$$

On the other hand, (5.1) is equivalent to (4.16), which can be rewritten as $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}b_{jk}<\infty$. This is also equivalent to (5.2), due to (1.2) and the monotone decreasing property of $b_{jk}$ in $j$ and $k$.

(b) Now assume only (3.2). By (4.14) and Hölder's inequality,

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{a_{jk}}{jk}=a_{11}+\sum_{m=0}^{\infty}\sum_{j=2^m}^{2^{m+1}-1}\frac{a_{j+1,1}}{j+1}+\sum_{n=0}^{\infty}\sum_{k=2^n}^{2^{n+1}-1}\frac{a_{1,k+1}}{k+1} +\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{j=2^m}^{2^{m+1}-1}\sum_{k=2^n}^{2^{n+1}-1}\frac{a_{j+1,k+1}}{(j+1)(k+1)}$$
$$\leqslant a_{11}+\sum_{m=0}^{\infty}2^{m(1-1/p)}\Big[\sum_{j=2^m}^{2^{m+1}-1}\Big(\frac{a_{j+1,1}}{j+1}\Big)^p\Big]^{1/p} +\sum_{n=0}^{\infty}2^{n(1-1/p)}\Big[\sum_{k=2^n}^{2^{n+1}-1}\Big(\frac{a_{1,k+1}}{k+1}\Big)^p\Big]^{1/p} +\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}2^{(m+n)(1-1/p)}\Big[\sum_{j=2^m}^{2^{m+1}-1}\sum_{k=2^n}^{2^{n+1}-1}\Big(\frac{a_{j+1,k+1}}{(j+1)(k+1)}\Big)^p\Big]^{1/p}$$
$$\leqslant a_{11}+3\mathscr{B}_p.$$

This shows that (4.16) (or equivalently, (5.1)) is stronger than (4.9), at least in the case of (3.2).

(c) However, (4.9) and (5.1) are not comparable in general. First, in the case of Example 2 in Section 3, condition (5.1) holds, while (4.9) does not hold for any $p>1$. Second, in the next example (4.9) holds for all $p>0$, while (5.1) does not.

EXAMPLE 4. Let

$$a_j=\begin{cases}\dfrac{j}{2^{2^m+m}} & \text{if } 2^{2^m}\leqslant j<2^{2^m+1},\\[2mm] \dfrac{2^{2^m+m}-j}{2^{m-1}(2^{2^m+m}-2^{2^m+1})} & \text{if } 2^{2^m+1}\leqslant j<2^{2^m+m},\\[2mm] 0 & \text{if } 2^{2^m+m}\leqslant j<2^{2^{m+1}-1},\\[2mm] \dfrac{j}{2^{2^{m+1}+m}}-\dfrac{1}{2^{m+1}} & \text{if } 2^{2^{m+1}-1}\leqslant j<2^{2^{m+1}},\end{cases}$$

for $m=1,2,\ldots$; let $a_j=1$ for $j\leqslant 3$; and let $a_{jk}=a_ja_k$.

It is not hard to check that (4.11) and (4.12) are also satisfied for all $p > 0$, while (1.16) is not. According to Theorem 3 and Corollary 5, in this case (4.1) is a Fourier sine series, but its rectangular partial sums fail to converge in $L^1(T^2)$-norm.

**6. Special cases of conditions (4.9) and (5.1).** By Hölder's inequality, $\mathscr{B}_{p_1} \leqslant \mathscr{B}_{p_2}$ $(0 < p_1 < p_2)$. Putting $p_1 = 1$, we can see that the fulfillment of (4.9) for any $p > 1$ implies that of (4.3). On the other hand, we can conclude that

$$\mathscr{B}_p \leqslant \mathscr{B}_\infty = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 2^{m+n} \max_{2^m \leqslant j < 2^{m+1}} \max_{2^n \leqslant k < 2^{n+1}} jk|\Delta_{11} b_{jk}|.$$

Following Telyakovskii [9], a null sequence $\{a_{jk}\}$ is said to belong to the class $\tilde{S}$ if there exists a double sequence $\{\varepsilon_{jk}\}$ of nonnegative numbers such that

$$|\Delta_{11} b_{jk}| \leqslant \varepsilon_{jk}, \quad \Delta_{11} \varepsilon_{jk} \geqslant 0, \quad \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} jk\varepsilon_{jk} < \infty$$

(cf. (2.3)). Since the last condition here is equivalent to the condition

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 2^{2(m+n)} \varepsilon_{2^m, 2^n} < \infty,$$

it follows that if $\{a_{jk}\} \in \tilde{S}$, then $\mathscr{B}_\infty < \infty$, and consequently, condition (4.9) is satisfied for all $p > 1$. Thus, the corresponding results of Telyakovskii [9] are particular cases of Theorem 3 and Corollary 4.

Example 1 in Section 2 shows that $\tilde{S}$ is a proper subclass of those double sequences $\{a_{jk}\}$ for which (4.9) is satisfied with $p = 2$.

Finally, we consider the special case where (1.2) and (3.2) are satisfied. Clearly, the first row of series (4.1), apart from $\sin y$, is

$$\sum_{j=1}^{\infty} a_{j1} \sin jx = g_1(x), \quad \text{say;} \tag{6.1}$$

and the first column, apart from $\sin x$, is

$$\sum_{k=1}^{\infty} a_{1k} \sin ky = g_2(y), \quad \text{say.} \tag{6.2}$$

The pointwise convergence of these series follows from the fact that their coefficients form monotone decreasing null sequences.

THEOREM 5. *If a double sequence $\{a_{jk}\}$ satisfies* (1.2) *and* (3.2), *then the sums $g, g_1, g_2$ of series* (4.1), (6.1), (6.2), *respectively, are integrable if and only if* (5.1) *is satisfied. If* (5.1) *is satisfied, then* (4.1) *is the Fourier series of $g$ and* (4.13) *holds.*

We note that under the conditions of Theorem 5, we also have $\|s_m^{(1)} - g_1\| \to 0$ as $m \to \infty$, and $\|s_n^{(2)} - g_2\| \to 0$ as $n \to \infty$, where $s_m^{(1)}(x)$ and $s_n^{(2)}(y)$ are the partial sums of (6.1) and (6.2), respectively. Even, each row and each column in (4.1) converges both pointwise and in $L^1$-norm.



Theorem 5 is an extension of the familiar result of W. H. Young (see [10] and also [11, Vol. 1, pp. 185–186]) from one-dimensional to two-dimensional sine series.

**7. Auxiliary results.** We will need five lemmas whose proofs or references are given in this section.

LEMMA 1 (Bojanic and Stanojević [1]). *If $1 < p \leqslant 2$, then for all single sequences $\{a_j\colon j \geqslant 0\}$ and integers $m \geqslant 1$,*

$$\Big\|\sum_{j=m}^{2m-1} a_j D_j\Big\| \leqslant C_p^* m\Big[m^{-1} \sum_{j=m}^{2m-1} |a_j|^p\Big]^{1/p}. \tag{7.1}$$

The following more general inequality can be derived easily from (7.1) (see e.g. [7, Lemma 6]).

LEMMA 2. *If $1 < p \leqslant 2$, then for all single sequences $\{a_j\}$ and integers $m \geqslant 0$,*

$$\Big\|\sum_{j=0}^{m} a_j D_j\Big\| \leqslant C_p^*\Big\{|a_0| + \sum_{i=0}^{\infty} 2^i\Big[2^{-i} \sum_{j=2^i}^{2^{i+1}-1} |a_j|^p\Big]^{1/p}\Big\}.$$

LEMMA 3. *If $1 < p \leqslant 2$, then for all double sequences $\{a_{jk}\colon j, k \geqslant 0\}$ and integers $m, n \geqslant 0$,*

$$\begin{aligned} \Sigma_{mn} &= \Big\|\sum_{j=0}^{m} \sum_{k=0}^{n} a_{jk} D_j(x) D_k(y)\Big\| \\ &\leqslant C_p(m+1)(n+1)\Big[\frac{1}{(m+1)(n+1)} \sum_{j=0}^{m} \sum_{k=0}^{n} |a_{jk}|^p\Big]^{1/p}. \end{aligned} \tag{7.2}$$

Proof. Let $I_m = [0, 1/(m+1)]$, $J_n = [0, 1/(n+1)]$, $CI_m = [0, \pi]\backslash I_m$ and $CJ_n = [0, \pi]\backslash J_n$ $(m, n \geqslant 0)$. We split the double integral in (7.2) into four parts as follows:

$$\begin{aligned} \Sigma_{mn} &= \Big\{\int_{I_m}\int_{J_n} + \int_{I_m}\int_{CJ_n} + \int_{CI_m}\int_{J_n} + \int_{CI_m}\int_{CJ_n}\Big\}\Big|\sum_{j=0}^{m} \sum_{k=0}^{n} a_{jk} D_j(x) D_k(y)\Big|\,dx\,dy \\ &= \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4, \quad \text{say.} \end{aligned} \tag{7.3}$$

First, using the elementary inequality

$$|D_j(x)| < j+1 \quad (j \geqslant 0;\ 0 \leqslant x \leqslant \pi), \tag{7.4}$$

we get

$$\Sigma_1 \leqslant \frac{1}{(m+1)(n+1)} \sum_{j=0}^{m} \sum_{k=0}^{n} (j+1)(k+1)|a_{jk}|.$$

Hence, by Hölder's inequality with exponents $p$ and $q = p/(p-1)$, we find that

$$\Sigma_1 \leqslant C_{p1}[(m+1)(n+1)]^{1/q}\Big[\sum_{j=0}^{m}\sum_{k=0}^{n}|a_{jk}|^p\Big]^{1/p}. \tag{7.5}$$

Second, applying Fubini's theorem, (7.4), and Lemma 1, we obtain

$$\Sigma_2 \leqslant \sum_{j=0}^{m}\int_{I_m}|D_j(x)|\,dx\int_{CJ_n}\Big|\sum_{k=0}^{n}a_{jk}D_k(y)\Big|\,dy$$

$$\leqslant \sum_{j=0}^{m}C_p^*(n+1)^{1/q}\Big[\sum_{k=0}^{n}|a_{jk}|^p\Big]^{1/p}\int_{I_m}(j+1)\,dx$$

$$= \frac{C_p^*(n+1)^{1/q}}{m+1}\sum_{j=0}^{m}(j+1)\Big[\sum_{k=0}^{n}|a_{jk}|^p\Big]^{1/p}.$$

Hence, by Hölder's inequality,

$$\Sigma_2 \leqslant C_{p2}[(m+1)(n+1)]^{1/q}\Big[\sum_{j=0}^{m}\sum_{k=0}^{n}|a_{jk}|^p\Big]^{1/p}. \tag{7.6}$$

By symmetry, we can deduce the same upper bound for $\Sigma_3$, too.

Third, using representation (1.5), we have

$$\Sigma_4 = \int_{CI_m}\int_{CJ_n}\Big|\sum_{j=0}^{m}\sum_{k=0}^{n}a_{jk}\frac{\sin(j+\frac12)x\sin(k+\frac12)y}{4\sin\frac12 x\sin\frac12 y}\Big|\,dx\,dy$$

$$\leqslant \int_{CI_m}\int_{CJ_n}\Big|\sum_{j=0}^{m}\sum_{k=0}^{n}\frac{a_{jk}e^{i(jx+ky)}}{4\sin\frac12 x\sin\frac12 y}\Big|\,dx\,dy.$$

By Hölder's inequality,

$$\Sigma_4 \leqslant \Big[\int_{CI_m}\int_{CJ_n}\frac{dx\,dy}{(4\sin\frac12 x\sin\frac12 y)^p}\Big]^{1/p}$$

$$\times\Big[\int_{CI_m}\int_{CJ_n}\Big|\sum_{j=0}^{m}\sum_{k=0}^{n}a_{jk}e^{i(jx+ky)}\Big|^q dx\,dy\Big]^{1/q}.$$

Hence, evaluating the first integral and applying the Hausdorff–Young inequality [11, Vol. 2, p. 101] extended to two-dimensional Fourier series (observe that $q \geqslant 2$ due to the assumption $1 < p \leqslant 2$) gives

$$\Sigma_4 \leqslant C_{p3}[(m+1)(n+1)]^{1/q}\Big[\sum_{j=0}^{m}\sum_{k=0}^{n}|a_{jk}|^p\Big]^{1/p}. \tag{7.7}$$

Combining (7.3), (7.5), (7.6) and its symmetric counterpart for $\Sigma_3$, and (7.7) results in (7.2).

LEMMA 4 (Móricz [7]). *If $1 < p \leqslant 2$, then for all single sequences $\{a_j\colon j \geqslant 1\}$ and integers $m \geqslant 1$,*



$$\Big\|\sum_{j=1}^{m}a_jD_j'\Big\| \leqslant \bar{C}_p^*\sum_{l=0}^{\infty}2^l\Big[2^{-l}\sum_{j=2^l}^{2^{l+1}-1}j^p|a_j|^p\Big]^{1/p}.$$

LEMMA 5. *If $1 < p \leqslant 2$, then for all double sequences $\{a_{jk}\colon j, k \geqslant 1\}$ and integers $m, n \geqslant 1$,*

$$\Big\|\sum_{j=1}^{m}\sum_{k=1}^{n}a_{jk}D_j'(x)D_k'(y)\Big\| \leqslant \bar{C}_p mn\Big[(mn)^{-1}\sum_{j=1}^{m}\sum_{k=1}^{n}j^pk^p|a_{jk}|^p\Big]^{1/p}. \tag{7.8}$$

Proof. Without loss of generality, we may assume that $m$ and $n$ are of the form $m = 2^u - 1$ and $n = 2^v - 1$ with some integers $u, v \geqslant 1$.

Let $i, l \geqslant 1$ and $q = p/(p-1)$. By applying Bernstein's inequality (see e.g. [11, Vol. 2, p. 11]), it follows from (7.2) that

$$\Big\|\sum_{j=2^{i-1}}^{2^i-1}\sum_{k=2^{l-1}}^{2^l-1}a_{jk}D_j'(x)D_k'(y)\Big\| \leqslant 2^{i+l}\Big\|\sum_{j=2^{i-1}}^{2^i-1}\sum_{k=2^{l-1}}^{2^l-1}a_{jk}D_j(x)D_k(y)\Big\|$$

$$\leqslant C_p2^{(i+l)(1+1/q)}\Big[\sum_{j=2^{i-1}}^{2^i-1}\sum_{k=2^{l-1}}^{2^l-1}|a_{jk}|^p\Big]^{1/p}$$

$$\leqslant 4C_p2^{(i+l)/q}\Big[\sum_{j=2^{i-1}}^{2^i-1}\sum_{k=2^{l-1}}^{2^l-1}j^pk^p|a_{jk}|^p\Big]^{1/p}.$$

Making use of the triangle inequality, then Hölder's inequality gives

$$\Big\|\sum_{j=1}^{2^u-1}\sum_{k=1}^{2^v-1}a_{jk}D_j'(x)D_k'(y)\Big\| \leqslant \sum_{i=1}^{u}\sum_{l=1}^{v}\Big\|\sum_{j=2^{i-1}}^{2^i-1}\sum_{k=2^{l-1}}^{2^l-1}a_{jk}D_j'(x)D_k'(y)\Big\|$$

$$\leqslant 4C_p\sum_{i=1}^{u}\sum_{l=1}^{v}2^{(i+l)/q}\Big[\sum_{j=2^{i-1}}^{2^i-1}\sum_{k=2^{l-1}}^{2^l-1}j^pk^p|a_{jk}|^p\Big]^{1/p}$$

$$\leqslant 4C_p\Big[\sum_{i=1}^{u}\sum_{l=1}^{v}2^{i+l}\Big]^{1/q}\Big[\sum_{i=1}^{u}\sum_{l=1}^{v}\sum_{j=2^{i-1}}^{2^i-1}\sum_{k=2^{l-1}}^{2^l-1}j^pk^p|a_{jk}|^p\Big]^{1/p}$$

$$\leqslant \bar{C}_p2^{(u+v)/q}\Big[\sum_{j=1}^{2^u-1}\sum_{k=1}^{2^v-1}j^pk^p|a_{jk}|^p\Big]^{1/p}.$$

This is (7.8) for $m = 2^u - 1$ and $n = 2^v - 1$.

On closing, we mention that (7.2) and (7.8) in the particular case $p = 2$ provide inequalities (0.3) and (0.4) presented in the Introduction.

## 8. Proofs of the theorems and corollaries

Proof of Theorem 1. By (2.1), condition (1.11) is less restrictive if $p$ is closer to 1. Therefore, we may assume that $1 < p \leqslant 2$.

By (1.8) and (1.10),

$$f(x, y) - u_{mn}(x, y) = \sum\sum_{(j,k)\in R_{mn}}D_j(x)D_k(y)\Delta_{11}a_{jk},$$

where $R_{mn}$ denotes the set of all lattice points $(j, k)$ with nonnegative integer coordinates $j$ and $k$ such that $j \geqslant m$ or/and $k \geqslant n$. We may assume that $m, n \geqslant 1$ and define the integers $M, N \geqslant 0$ such that

(8.1) $$2^M \leqslant m < 2^{M+1}, \quad 2^N \leqslant n < 2^{N+1}.$$

We represent $R_{mn}$ as an infinite disjoint union of (partly dyadic) rectangles and estimate accordingly as follows:

(8.2) $$\begin{aligned}\|f-u_{mn}\| \leqslant & \Big\|\sum_{j=m}^{2^{M+1}-1}\sum_{k=n}^{2^{N+1}-1} D_j(x)D_k(y)\Delta_{11}a_{jk}\Big\| + \Big\|\sum_{j=m}^{2^{M+1}-1} D_j(x)D_0(y)\Delta_{11}a_{j0}\Big\| \\ & + \sum_{v=0}^{N-1}\Big\|\sum_{j=m}^{2^{M+1}-1}\sum_{k=2^v}^{2^{v+1}-1} D_j(x)D_k(y)\Delta_{11}a_{jk}\Big\| + \Big\|\sum_{k=n}^{2^{N+1}-1} D_0(x)D_k(y)\Delta_{11}a_{0k}\Big\| \\ & + \sum_{u=0}^{M-1}\Big\|\sum_{j=2^u}^{2^{u+1}-1}\sum_{k=n}^{2^{N+1}-1} D_j(x)D_k(y)\Delta_{11}a_{jk}\Big\| + \sum_{u=M+1}^{\infty}\Big\|\sum_{j=2^u}^{2^{u+1}-1} D_j(x)D_0(y)\Delta_{11}a_{j0}\Big\| \\ & + \sum_{v=N+1}^{\infty}\Big\|\sum_{k=2^v}^{2^{v+1}-1} D_0(x)D_k(y)\Delta_{11}a_{0k}\Big\| \\ & + \sum\sum_{(u,v)\in R_{M+1,N+1}}\Big\|\sum_{j=2^u}^{2^{u+1}-1}\sum_{j=2^v}^{2^{v+1}-1} D_j(x)D_k(y)\Delta_{11}a_{jk}\Big\|.\end{aligned}$$

We apply Lemma 3 to obtain

$$\begin{aligned}\|f-u_{mn}\| \leqslant C_p\Big\{ & \sum_{u=M}^{\infty} 2^{(u+1)(1-1/p)}\Big[\sum_{j=2^u}^{2^{u+1}-1}|\Delta_{11}a_{j0}|^p\Big]^{1/p} \\ & + \sum_{v=N}^{\infty} 2^{(v+1)(1-1/p)}\Big[\sum_{k=2^v}^{2^{v+1}-1}|\Delta_{11}a_{0k}|^p\Big]^{1/p} \\ & + \sum\sum_{(u,v)\in R_{MN}} 2^{(u+v+2)(1-1/p)}\Big[\sum_{j=2^u}^{2^{u+1}-1}\sum_{k=2^v}^{2^{v+1}-1}|\Delta_{11}a_{jk}|^p\Big]^{1/p}\Big\}.\end{aligned}$$

Now, (1.12) is an immediate consequence of (1.11).

Proof of Corollary 1. It follows from (1.12) that $f \in L^1(T^2)$. Furthermore, it is a commonplace that convergence in $L^1$-norm (the so-called strong



convergence) implies weak convergence. Thus, by (1.9) and (1.12), for fixed $i, l \geqslant 1$,

(8.3) $$\begin{aligned}\frac{4}{\pi^2}\int_0^\pi\int_0^\pi f(x, y)\cos ix \cos ly\, dx\, dy & = \lim_{m,n\to\infty}\frac{4}{\pi^2}\int_0^\pi\int_0^\pi u_{mn}(x, y)\cos ix\cos ly\, dx\, dy \\ & = a_{il} - \lim_{m,n\to\infty}\Big\{\sum_{j=0}^{m}\Delta_{10}a_{j,n+1} + \sum_{k=0}^{n}\Delta_{01}a_{m+1,k} + a_{m+1,n+1}\Big\} = a_{il}.\end{aligned}$$

Here we have taken into account that the limit of each term in braces is zero, thanks to (1.2), (1.7) and its symmetric counterpart for $\Delta_{01}a_{m+1,k}$. An analogous argument works when $i=0$ or/and $l=0$. These prove that (1.1) is the Fourier series of $f$.

Proof of Corollary 2. *Sufficiency.* As is well known, there exist positive constants $C_1$ and $C_2$ such that

(8.4) $$C_1\ln(m+2) \leqslant \|D_m\| \leqslant C_2\ln(m+2) \quad (m \geqslant 0)$$

(see e.g. [11, Vol. 1, p. 67]). By this and (1.9),

(8.5) $$\begin{aligned}\|f-s_{mn}\| \leqslant \|f-u_{mn}\| & + C_2\ln(n+2)\Big\|\sum_{j=0}^{m} D_j\Delta_{10}a_{j,n+1}\Big\| \\ & + C_2\ln(m+2)\Big\|\sum_{k=0}^{n} D_k\Delta_{01}a_{m+1,k}\Big\| \\ & + C_2^2|a_{m+1,n+1}|\ln(m+2)\ln(n+2).\end{aligned}$$

Conditions (1.12), (1.13), (1.14) (via Lemma 2), and (1.16) imply (1.15).

*Necessity.* By (1.9) and (8.4),

(8.6) $$\begin{aligned}\|f-s_{mn}\| \geqslant & C_1^2|a_{m+1,n+1}|\ln(m+2)\ln(n+2) \\ & -\|f-u_{mn}\| - C_2\ln(n+2)\Big\|\sum_{j=0}^{m} D_j\Delta_{10}a_{j,n+1}\Big\| \\ & - C_2\ln(m+2)\Big\|\sum_{k=0}^{n} D_k\Delta_{01}a_{m+1,k}\Big\|\end{aligned}$$

and (1.16) follows from (1.12), (1.13), (1.14) (via Lemma 2) and (1.15).

Proof of Corollary 3. As we have seen in Section 2, it follows from (1.2) and (2.4) that $\{a_{jk}\} \in \mathscr{S}$. Consequently, $\mathscr{A}_\infty < \infty$ and (1.11) is satisfied for all $p > 0$. The first part of Corollary 3 is a special case of Corollary 1.

Now we are going to prove the equivalence (2.9)$\Leftrightarrow$(2.10) under condition (2.8).

*Sufficiency.* Under conditions (1.2) and (2.8), (1.13) is equivalent to the condition

$$\ln(n+2)\{\Delta_{10}a_{0,n+1}+\sum_{m=0}^{\infty}2^m\Delta_{10}a_{2^m,n+1}\}\to 0 \quad \text{as } n\to\infty.$$

This can be rewritten as

$$\ln(n+2)\sum_{j=0}^{\infty}\Delta_{10}a_{j,n+1}=a_{0,n+1}\ln(n+2)\to 0 \quad \text{as } n\to\infty,$$

which is a special case of (2.10). Analogously, (2.10) also implies (1.14). Thus, the first relation in (2.9) follows from (2.10) via Corollary 2. Since each of the single sequences $\{a_{jk}\colon j\geqslant 0\}$ and $\{a_{jk}\colon k\geqslant 0\}$ is convex by (2.8), the second and third relations in (2.9) follow via the corresponding one-dimensional result [11, Vol. 1, p. 184].

*Necessity.* Applying the one-dimensional result just referred to, we conclude from the second and third relations in (2.9) that

$$a_{m0}\ln(m+2)\to 0 \quad \text{and} \quad a_{0n}\ln(n+2)\to 0 \tag{8.7}$$

as $m\to\infty$ and $n\to\infty$, respectively. As we have seen in the proof of sufficiency, these conditions ensure the fulfillment of (1.13) and (1.14). We can apply Corollary 2, according to which (1.16) follows from (1.15).

Clearly, the couple of conditions (1.16) and (8.7) is equivalent, in the monotone decreasing case, to (2.10).

Proof of Theorem 2. By (1.8), (3.1), and (8.4) we have $f\in L^1(T^2)$. By (1.4) and (1.8),

$$\begin{aligned}f(x,y)-s_{mn}(x,y)=&\sum\sum_{(j,k)\in R_{mn}}D_j(x)D_k(y)\Delta_{11}a_{jk}-\sum_{j=0}^{m}D_j(x)D_n(y)\Delta_{10}a_{j,n+1}\\&-\sum_{k=0}^{n}D_m(x)D_k(y)\Delta_{01}a_{m+1,k}-a_{m+1,n+1}D_m(x)D_n(y),\end{aligned}$$

where $R_{mn}$ was defined in the proof of Theorem 1. Hence, by (8.4),

$$\begin{aligned}\|f-s_{mn}\|\leqslant C_2^2\{&\sum\sum_{(j,k)\in R_{mn}}|\Delta_{11}a_{jk}|\ln(j+2)\ln(k+2)\\&+\sum_{j=0}^{m}|\Delta_{10}a_{j,n+1}|\ln(j+2)\ln(n+2)\\&+\sum_{k=0}^{n}|\Delta_{01}a_{m+1,k}|\ln(m+2)\ln(k+2)\\&+|a_{m+1,n+1}|\ln(m+2)\ln(n+2)\}=C_2^2(S_1+S_2+S_3+S_4), \quad \text{say.}\end{aligned}$$

By (3.1), $S_1\to 0$ as $\min(m,n)\to\infty$. Using (1.6), we get



$$\begin{aligned}S_2&\leqslant\ln(n+2)\sum_{j=0}^{m}\sum_{k=n+1}^{\infty}|\Delta_{11}a_{jk}|\ln(j+2)\\&\leqslant\sum_{j=0}^{m}\sum_{k=n+1}^{\infty}|\Delta_{11}a_{jk}|\ln(j+2)\ln(k+2)\to 0 \quad \text{as } n\to\infty,\end{aligned}$$

independently of $m$. Similarly, $S_3\to 0$ as $m\to\infty$, idependently of $n$. Finally,

$$\begin{aligned}S_4&\leqslant\ln(m+2)\ln(n+2)\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}|\Delta_{11}a_{jk}|\\&\leqslant\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}|\Delta_{11}a_{jk}|\ln(j+2)\ln(k+2)\to 0\end{aligned}$$

as $\max(m,n)\to\infty$. Combining these estimates yields (1.15). Obviously, (1.15) implies that (1.1) is the Fourier series of $f$.

Proof of Theorem 3. We may assume again that $p\leqslant 2$. By (4.5) and (4.7),

$$g(x,y)-v_{mn}(x,y)=\sum\sum_{(j,k)\in R_{mn}}D_j'(x)D_k'(y)\Delta_{11}a_{jk}$$

where this time $R_{mn}$ is the set of lattice points $(j,k)$ with positive integer coordinates $j$ and $k$ such that $j\geqslant m$ or/and $k\geqslant n$. The rest is similar to the proof of Theorem 1, with the only exception that we apply Lemma 5 instead of Lemma 3.

Proof of Corollary 4. It is modelled after the proof of Corollary 1. This time (cf. (8.3)),

$$\begin{aligned}&\frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi}v_{mn}(x,y)\sin ix\sin ly\,dx\,dy\\&\qquad=a_{il}+il\{\sum_{j=1}^{m}\Delta_{10}b_{j,n+1}+\sum_{k=1}^{n}\Delta_{01}b_{m+1,k}+b_{m+1,n+1}\}.\end{aligned}$$

Proof of Corollary 5. It runs along the same lines as that of Corollary 2 with two modifications. First, instead of (8.4) we use the estimate

$$C_1 m\ln(m+1)\leqslant\|D_m'\|\leqslant C_2 m\ln(m+1) \quad (m\geqslant 1)$$

(see [7]). Second, we apply Lemma 4 instead of Lemma 2.

Proof of Theorem 4. It is similar to that of Theorem 2. But we have to replace (4.4) by the following more common representation (as a result of another double summation by parts):

$$(8.8)\qquad s_{mn}(x, y) = \sum_{j=1}^{m}\sum_{k=1}^{n} \tilde{D}_j(x)\tilde{D}_k(y)\Delta_{11}a_{jk} - \sum_{j=1}^{m} \tilde{D}_j(x)\tilde{D}_n(y)\Delta_{10}a_{j,n+1}$$
$$- \sum_{k=1}^{n} \tilde{D}_m(x)\tilde{D}_k(y)\Delta_{01}a_{m+1,k} - a_{m+1,n+1}\tilde{D}_m(x)\tilde{D}_n(y),$$

where the conjugate Dirichlet kernel $\tilde{D}_m$ is defined in (4.8). The same procedure that resulted in (4.5) now leads to

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk}\sin jx \sin ky = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \tilde{D}_j(x)\tilde{D}_k(y)\Delta_{11}a_{jk} = g(x, y).$$

Finally, instead of (8.4) we use the estimate (see [11, Vol. 1, p. 67])

$$(8.9)\qquad C_1\ln(m+1) \leqslant \|\tilde{D}_m\| \leqslant C_2\ln(m+1) \quad (m \geqslant 1).$$

Proof of Theorem 5. *Sufficiency.* This part is essentially a special case of Theorem 4. It remains to prove that $g_1$ and $g_2$ are also integrable. For example, by (1.6) and (5.1),

$$\sum_{j=1}^{\infty} |\Delta_{10}a_{j1}|\ln(j+1) \leqslant \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} |\Delta_{11}a_{jk}|\ln(j+1) < \infty$$

and the corresponding one-dimensional theorem [11, Vol. 1, p. 185] yields that $g_1 \in L^1(T)$.

*Necessity.* Assume that $g, g_1$, and $g_2$ are integrable. We rather use the modified conjugate Dirichlet kernel defined by

$$(8.10)\qquad \tilde{D}_m^*(x) = \tilde{D}_m(x) - \tfrac{1}{2}\sin mx = \frac{1-\cos mx}{2\tan\frac{1}{2}x} \quad (m \geqslant 1)$$

(cf. (4.8)). Clearly, $\tilde{D}_m^*$ is nonnegative, and if we substitute $\tilde{D}_m^*$ for $\tilde{D}_m$, the inequalities in (8.9) remain true. (See [11, Vol. 1, pp. 50 and 67].) By (8.10),

$$(8.11)\quad g(x, y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \tilde{D}_j^*(x)\tilde{D}_k^*(y)\Delta_{11}a_{jk} + \tfrac{1}{2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \tilde{D}_j^*(x)(\sin ky)\Delta_{11}a_{jk}$$
$$+ \tfrac{1}{2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} (\sin jx)\tilde{D}_k^*(y)\Delta_{11}a_{jk}$$
$$+ \tfrac{1}{4}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} (\sin jx \sin ky)\Delta_{11}a_{jk}$$
$$= g^*(x, y) + h_1(x, y) + h_2(x, y) + h_3(x, y), \quad \text{say.}$$

By (5.1), $h_3$ is continuous everywhere. By (1.2) and (3.2),

$$(8.12)\quad |h_1(x, y)| \leqslant \tfrac{1}{2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \tilde{D}_j^*(x)\Delta_{11}a_{jk} = \tfrac{1}{2}\sum_{j=1}^{\infty} \tilde{D}_j^*(x)\Delta_{10}a_{j1} = g_1^*(x), \quad \text{say}$$

(cf. (1.6)). Furthermore, by (1.2), (6.2), and (8.10),



$$(8.13)\qquad g_1(x) = \sum_{j=1}^{\infty} \tilde{D}_j(x)\Delta_{10}a_{j1}$$
$$= \sum_{j=1}^{\infty} \tilde{D}_j^*(x)\Delta_{10}a_{j1} + \tfrac{1}{2}\sum_{j=1}^{\infty} (\sin jx)\Delta_{10}a_{j1} = g_1^*(x) + h_4(x), \quad \text{say.}$$

Since $h_4$ is continuous and $g_1$ is integrable by assumption, it follows that $g_1^*$ is also integrable. Combining (8.12) and (8.13) yields that $h_1 \in L^1(T^2)$. In an analogous way, we can deduce that $h_2 \in L^1(T^2)$.

To sum up, it follows from (8.11) that $g^* \in L^1(T^2)$. The series defining $g^*$ has nonnegative terms, and since the integral of $\tilde{D}_m^*$ over $T$ is exactly of order $\ln(m+1)$ (see the remark made after (8.10)), we conclude that $g^* \in L^1(T^2)$ if and only if (5.1) is satisfied.

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BOLYAI INSTITUTE
UNIVERSITY OF SZEGED
Aradi vértanúk tere 1, 6720 Szeged, Hungary

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