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Rank is not a spectral invariant

by

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Abstract. We construct two spectrally isomorphic automorphisms, the first of which is of rank one while the second does not satisfy the weak closure theorem, hence, it is not of rank one.

DEFINITION 1. A system $(\Omega, \mathscr{A}, T, \mu)$ is of *rank at most r* if for every partition P of Ω and for every $\varepsilon > 0$, there exist subsets $F_1, \ldots, F_r$ of Ω, integers $h_1, \ldots, h_r$ and a partition P' such that:

– the $T^i F_j$, $1 \leqslant j \leqslant r$, $0 \leqslant i \leqslant h_j - 1$, are disjoint,

– $|P - P'| < \varepsilon$ for the usual distance between partitions,

– P' is refined by the partition consisting of the $T^i F_j$, $1 \leqslant j \leqslant r$, $0 \leqslant i \leqslant h_j - 1$, and of the set $\Omega \setminus \bigcup_{j=1}^{r} \bigcup_{i=0}^{h_j-1} T^i F_j$.

A system is of *rank r* if it is of rank at most r and not of rank at most $r-1$.

The rank is of course an invariant of metrical isomorphism; it is also known ([1]) that the rank of a system is not smaller than its spectral multiplicity; it was not yet known whether the rank is an invariant of spectral isomorphism, i.e. of isomorphism between the associated unitary operators on $L^2(\Omega)$. We answer this question by constructing two spectrally isomorphic systems: the first will be immediately recognized as a rank one system, while we shall show that the second cannot be of rank one since it does not satisfy the weak closure theorem (see [5]).

This example will make use of the notion of Gaussian–Kronecker systems:

DEFINITION 2. Given a symmetrical probability σ on the circle we define a *(real) Gaussian system* of spectral measure σ to be the dynamical system $(\Omega, \mathscr{A}, T, \mu)$ where:

– Ω is $\mathbf{R}^{\mathbf{Z}}$,

– $\mathscr{A}$ is the borelian σ-algebra,

– T is the shift $(Tx)_n = x_{n+1}$,

– μ is defined on cylinders by letting $\mu(x_{j_1} \in A_1, \ldots, x_{j_n} \in A_n)$ be the probability of visiting the set $A_1 \times \ldots \times A_n$ for a gaussian vector $(X_{j_1}, \ldots, X_{j_n})$ of zero mean and covariances

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$$\mathrm{Cov}(X_{j_s}, X_{j_t}) = \int_{[-\pi,\pi]} e^{i(j_s-j_t)u}\,d\sigma(u).$$

Such a system is then generated by a real strongly stationary gaussian process, namely $X_n = X_0 \circ T^n$, $n \in \mathbf{Z}$, where $X_0(x) = x_0$.

DEFINITION 3. A subset K' of the circle is called a *Kronecker set* if for every continuous complex function f of modulus one on K' and for each $\varepsilon > 0$, there exists an integer n such that $\sup_{t\in K'}|f(t)-e^{int}| < \varepsilon$.

We say that the Gaussian system of spectral measure σ is a *Gaussian–Kronecker system* if σ is a continuous measure concentrated on $K' \cup -K'$, where K' is a Kronecker set contained in $[0, \pi]$ (for a construction of Kronecker sets see [2]). In this case, following [6], we call the set $K' \cup -K'$ a *semi-Kronecker set*.

DEFINITION 4. A symmetrical measure σ on the circle (identified with the interval $[-\pi, \pi]$) is called a *semi-Kronecker measure* if for every even complex function $f \in L^2(S^1, \sigma)$ of modulus 1 and for each $\varepsilon > 0$, there exists an integer n such that $\|f-\chi_n\|_2 < \varepsilon$, where $\chi_n(t) = e^{int}$ (this is a slightly weaker notion than requiring σ to be concentrated on a semi-Kronecker set).

Construction of the rank one system. By [3], there exists a symmetrical measure σ, concentrated on $\Lambda = K' \cup -K'$, such that the Gaussian–Kronecker system of spectral measure σ is of rank one. Let (X, T, μ) denote that system. Since σ is concentrated on a semi-Kronecker set (in particular, K' is a set without rational relations),

$$L^2(X) = \bigoplus_{n\geqslant 0} K_n,$$

where K_0 is the space of constants, K_n are T-invariant subspaces and moreover $T\colon K_n \to K_n$ is (spectrally) isomorphic to the multiplication $V\colon L^2(S^1, \sigma^{*n}) \to L^2(S^1, \sigma^{*n})$, where $Vf(z) = zf(z)$ and σ^{*n} denotes the n-fold convolution of σ. In addition, the measures σ^{*n}, σ^{*m} are pairwise singular and T has simple spectrum (see [2], 14. §4).

In (X, T, μ), following Newton and Parry [7], we define an invariant sub-σ-algebra $\mathscr{B}$ by the following equivalence relation: for $x, x' \in X$, $x \equiv x'$ iff $x_n = x'_n$ for all n or $x_n = -x'_n$ for all n. Let us denote by (Y, T, μ) the factor system defined by σ. Being a factor of a rank one system, (Y, T, μ) must be of rank one ([4]). The spectral analysis of (Y, T, μ) was made by Newton and Parry in [7]. In particular, they proved that

$$L^2(Y) = \bigoplus_{n\geqslant 0} K_{2n}.$$

Construction of the second system. Let (W, S) be the real Gaussian system of spectral measure $\sigma * \sigma$. The measure $\sigma * \sigma$ is concentrated on the set $\Lambda_1 = \Lambda + \Lambda$, which satisfies

$$\underbrace{\Lambda_1 + \ldots + \Lambda_1}_{n} \cap \underbrace{\Lambda_1 + \ldots + \Lambda_1}_{m} \text{ is empty, } m \neq n.$$

Therefore (by [2], 14. §4) $L^2(W) = \bigoplus_{n\geqslant 0} L_n$, where L_0 is the space of constants and $S\colon L_n \to L_n$ is isomorphic to $V\colon L^2(S^1, (\sigma*\sigma)^{*n}) \to L^2(S^1, (\sigma*\sigma)^{*n})$, where $Vf(z) = zf(z)$. This latter operator is obviously isomorphic to $V\colon L^2(S^1, \sigma^{*2n}) \to L^2(S^1, \sigma^{*2n})$. Hence, (W, S) *is spectrally isomorphic to* (Y, T).

DEFINITION 5. By the *centralizer* $C(\theta)$ of a system (Z, θ, ν) we mean the set of all ν-preserving transformations commuting with θ. We say that (Z, θ, ν) satisfies the *weak closure theorem* (WCT) if for each $\theta' \in C(\theta)$ there exists a sequence of positive integers (n_k) such that $\|\theta^{n_k}(h) - \theta'(h)\|_2 \to 0$ for each $h \in L^2(Z)$.

In [5], King proved that every rank one system satisfies WCT. In [8], Thouvenot showed that every Gaussian–Kronecker system satisfies WCT. We prove the following.

PROPOSITION 1. *Let* (Ω, T, μ) *be a Gaussian system with spectral measure* σ *and having simple spectrum. Then* T *satisfies WCT iff* σ *is a semi-Kronecker measure.*

Proof. It follows from [2], 14. §1, that $L^2(\Omega) = \bigoplus_{n\geqslant 0} K_n$, K_0 is the space of constants, K_1 is the T-invariant subspace generated by $\{T^n X_0\}_{n=-\infty}^{+\infty}$ and K_n, $n \geqslant 2$, are invariant subspaces generated by Hermite–Itô polynomials. Moreover,

(1) $$T\colon K_1 \to K_1 \text{ is isomorphic to } V\colon L^2(S^1, \sigma) \to L^2(S^1, \sigma),$$

where $Vf(z) = zf(z)$.

Assume that $S \in C(T)$. Since T has simple spectrum, there exists a sequence of polynomials $p_n(T)$ of T such that

$$\|p_n(T)(h) - Sh\|_2 \to 0 \quad \text{for each } h \in L^2(\Omega).$$

Consequently, $p_n(T)(X_0) \to SX_0$ in $L^2(\Omega)$, which means that $SX_0 \in K_1$. Thus, S restricted to K_1 determines an isometry of K_1. Conversely, if S is an isometry of K_1 commuting with T on K_1 then by the special structure of the Gaussian space $L^2(\Omega)$, we can extend S to a multiplication preserving map of $L^2(\Omega)$ commuting with T on $L^2(\Omega)$, hence to an element of $C(T)$. Therefore, $C(T)$ can be identified with the set of all isometries of K_1 commuting with T. By (1), this latter set is the same as the set of all isometries P of $L^2(S^1, \sigma)$ commuting with V. Since $P(z^m) = P(V^m(1)) = V^m(P(1)) = z^m P(1)$, such an isometry is the multiplication by $h = P(1)$. As P is an isometry, the modulus of h must be equal to 1.

Now, T satisfies WCT iff V satisfies WCT in the sense that each isometry commuting with V is the strong limit of powers of V. This latter condition is equivalent to σ being a semi-Kronecker measure.

PROPOSITION 2. *A measure of the form $\sigma * \sigma$ can never be a semi-Kronecker measure.*

Proof. This measure has positive Fourier coefficients; hence it is impossible to approximate the constant -1 by characters.

COROLLARY 1. (W, S), *which is spectrally isomorphic to a rank one system, is not of rank one.*

COROLLARY 2. *Rank is not a spectral invariant. Also, WCT is not a spectral property, and it is still not a spectral property when we restrict ourselves to the class of systems which are spectrally isomorphic to rank one systems.*

Remarks. The rank of (W, S) is not known.

For any measure σ on the circle, we can define the systems (Y, T) and (W, S) in the same way. These give spectrally isomorphic systems; in several other cases we can prove that they are not metrically isomorphic, for example when σ is singular and $\sigma * \sigma$ is absolutely continuous ([7]) or when σ is concentrated on a semi-Kronecker set (using Thouvenot's [8] theory of Gaussian–Kronecker factors). Is this true for every singular σ?

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On the Fourier transform of $e^{-\psi(x)}$

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Abstract. We prove that the Fourier transform of $e^{-\psi(x)}$ where $\psi(x)$ is a convex polynomial with positive coefficients can be estimated by $e^{-\tilde{\psi}(x)}$ where $\tilde{\psi}(x)$ is the Legendre transform of $\psi(x)$.

1. Introduction. In this paper we investigate the behavior of the Fourier transform of the function $e^{-\psi(x)}$ where ψ is a convex polynomial on $\mathbf{R}$. Since $e^{-\psi(x)}$ belongs to the Schwartz class, we know that the Fourier transform of $e^{-\psi(x)}$ decays faster than the reciprocal of any polynomial. But, since the decay of $e^{-\psi(x)}$ is exponential, we should be able to say more about its Fourier transform. In fact, we prove that the behavior of the Fourier transform of $e^{-\psi(x)}$ is controlled by $e^{-\tilde{\psi}(x)}$ where $\tilde{\psi}$ is the Legendre transform of ψ when ψ belongs to a certain class of functions. The Legendre transform of a convex function $\psi(x)$ such that $\psi(0) = \psi'(0) = 0$ is defined by

$$\tilde{\psi}(x) = \sup_{p \in \mathbf{R}} (xp - \psi(p)). \tag{1}$$

For a geometric meaning of the Legendre transform, see [1]. A precise statement of our result is as follows.

THEOREM. *Let $\psi(x) = \sum_{j=1}^{m} a_j x^{2j}$ be an even convex polynomial. Assume that $a_j \geqslant 0$ for all j. Then there are positive constants C and ε depending only on m such that*

$$\Big|\int_{-\infty}^{\infty} e^{ixt - \psi(x)}\, dx\Big| \leqslant C\psi^{-1}(1) e^{-\varepsilon\tilde{\psi}(t)} \tag{2}$$

where $\psi^{-1}(1)$ is the positive number u such that $\psi(u) = 1$.

If $\psi(x) = x^2$, then $\tilde{\psi}(x) = x^2$, and hence the theorem holds for e^{-x^2} since the Fourier transform of e^{-x^2} is $\sqrt{\pi} e^{-x^2/(16\pi^2)}$ [2].

2. Proofs. Let Γ be the class of all nonzero even convex polynomials $\psi(x) = \sum_{j=1}^{m} a_j x^{2j}$ where $a_j \geqslant 0$ (not all zero). We prove the theorem by induction on the number of terms in the polynomial in Γ. We begin with some preliminary observations on convex functions and their Legendre transforms.

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