[3] Bui Huy Qui, *Weighted Hardy spaces*, Math. Nachr. 103 (1981), 45–65.
[4] —, *Weighted Besov and Triebel spaces: Interpolation by the real method*, Hiroshima Math. J. 12 (1982), 581–605.
[5] R. E. Edwards, *Functional Analysis*, Holt, Rinehart and Winston, New York 1965.
[6] C. Fefferman and E. M. Stein, *Some maximal inequalities*, Amer. J. Math. 93 (1971), 107–115.
[7] B. Jawerth, *Some observations on Besov and Triebel–Lizorkin spaces*, Math. Scand. 40 (1977), 94–104.
[8] J. Marschall, *Pseudo-differential operators with nonregular symbols*, Thesis, Freie Universität Berlin, 1985.
[9] U. Neri, *Fractional integration on the space H^1 and its dual*, Studia Math. 53 (1975), 175–189.
[10] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, 1970.
[11] R. S. Strichartz, *Traces of* BMO-*Sobolev spaces*, Proc. Amer. Math. Soc. 83 (1981), 509–513.
[12] H. Triebel, *Spaces of Besov–Hardy–Sobolev Type*, Teubner-Texte zur Math. 15, Leipzig 1978.
[13] —, *Theory of Function Spaces*, Birkhäuser, Basel 1983.

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Drop property equals reflexivity

by

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Abstract. We prove that in a reflexive Banach space $(X, \|\cdot\|)$ property (H) of Radon–Riesz (if $(x_n)_{n=1}^{\infty}$ is a sequence of elements in X converging weakly to an element x in X such that $\|x_n\| \to \|x\|$, then $(x_n)_{n=1}^{\infty}$ is norm-convergent to x) is equivalent to a geometric condition (the "drop property") introduced by Rolewicz: $\|\cdot\|$ has the drop property if for every closed set S disjoint with B_X (the closed unit ball of X) there exists an element $x \in S$ such that the "drop" defined by x (the convex hull of x and B_X) intersects S only at x. We also prove that a Banach space is reflexive if and only if it has an equivalent norm with drop property.

§ 1. Introduction. Let $(X, \|\cdot\|)$ be a Banach space and B_X its closed unit ball. By the *drop* $D(x, B_X)$ defined by an element $x \in X$, $x \notin B_X$, we shall mean the convex hull of the set $\{x\} \cup B_X$, $\operatorname{conv}(\{x\} \cup B_X)$. In [4], Daneš proved ("Drop Theorem") that, *for any Banach space $(X, \|\cdot\|)$ and every closed set $S \subset X$ at positive distance from B_X, there exists a point $x \in S$ such that $D(x, B_X) \cap S = \{x\}$.*

This result, as its author points out, allows to prove in a simple way certain theorems of Browder [2] and Zabreiko–Krasnosel'skii [17] which are very important in the theory of nonlinear operator equations. In [14], Rolewicz mentions a number of papers where the Daneš' result is used. Recently, Daneš has discussed the relationship between his Drop Theorem and several other results [5].

Motivated by Daneš' theorem, Rolewicz introduced in the aforesaid paper the notion of drop property for the norm in a Banach space: $\|\cdot\|$ in X has the *drop property* if for every closed set S disjoint with B_X there exists an element $x \in S$ such that $D(x, B_X) \cap S = \{x\}$. He proved that *if X is a uniformly convex Banach space then its norm has the drop property*, and also

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that *if the norm in a Banach space has the drop property, then the space is reflexive.* He included an example, suggested by Wojtaszczyk, of a reflexive nonsuperreflexive Banach space such that the norm has the drop property.

The norm $\|\cdot\|$ in a Banach space X is *k-rotund* (k a positive integer greater than 1) if, for every sequence $(x_n)_{n=1}^{\infty}$ in B_X such that

$$\lim_{n_1,\ldots,n_k\to\infty} \|x_{n_1}+\ldots+x_{n_k}\| = k,$$

$(x_n)_{n=1}^{\infty}$ is a convergent sequence. Kutzarova has recently proved [12] that *every 2-rotund norm in a Banach space has the drop property*, extending Rolewicz's result about uniformly convex norms, and she has used a notion also introduced by Rolewicz to give a geometric characterization of reflexivity: Let A be a subset of a Banach space X. The *Kuratowski index of noncompactness* of A, $\alpha(A)$, is the infimum of all positive numbers r such that A can be covered by a finite number of sets of diameter less than r. Given $f\in X^*$ (the topological dual of X) such that $\|f\|=1$ and $0<\varepsilon\leqslant 2$, let $S(f,\varepsilon)=\{x\colon x\in B_X, f(x)>1-\varepsilon\}$. The norm $\|\cdot\|$ in a Banach space X has *property* (α) if, for every $f\in X^*$, $\|f\|=1$,

$$\lim_{\varepsilon\to 0}\alpha\big(S(f,\varepsilon)\big)=0.$$

Then *a Banach space X is reflexive if and only if it admits an equivalent norm with property* (α).

In this paper we shall prove that in a reflexive Banach space the norm has the drop property if and only if it has property (α) and if and only if it has property (H) of Radon–Riesz ($\|\cdot\|$ has *property* (H) if, for any sequence $(x_n)_{n=1}^{\infty}$ in X converging weakly to an element $x\in X$ such that $\|x_n\|\to\|x\|$, $(x_n)_{n=1}^{\infty}$ converges to x in norm). Thanks to Rolewicz's result about the reflexivity of any Banach space with a norm which satisfies the drop property, this theorem reduces the drop property of a norm to the well-known and easy-to-handle property (H). Every k-rotund norm in a Banach space X (k a positive integer greater than 1) (and hence every 2-rotund norm and every uniformly convex norm) has property (H) and the space is reflexive (see, for example, [8], VII, § 2), hence the norm has the drop property. This allows us to give many other examples of norms with the drop property in reflexive nonsuperreflexive Banach spaces.

As a consequence, we shall prove that every reflexive Banach space has an equivalent norm with the drop property. This answers a question of Rolewicz and gives a new characterization of reflexivity.

The last part of this paper deals with the behaviour of the drop property of a norm in subspaces, quotients, substitution and Lebesgue–Bochner spaces.

§ 2. Drop property and reflexivity. We recall some earlier results for further reference:

THEOREM 1 (Drop Theorem, Daneš [4]). *Let X be a Banach space and $S\subset X$ a closed subset of X at positive distance from B_X. Then there exists an element $x\in S$ such that $D(x, B_X)\cap S=\{x\}$.*

A sequence $(x_n)_{n=1}^{\infty}$ in a Banach space X such that $x_{n+1}\in D(x_n, B_X)$, $n=1,2,\ldots$, will be called a *stream*. Notice that $x_n\notin B_X$, $n=1,2,\ldots$, in view of the definition of drop.

The following proposition gives a characterization of a norm with drop property in terms of streams:

PROPOSITION 1 (Rolewicz [14], Proposition 2). *Let $(X,\|\cdot\|)$ be a Banach space. Then $\|\cdot\|$ has the drop property if and only if every stream in X has a convergent subsequence.*

The norm $\|\cdot\|$ in a Banach space X is *uniformly rotund* if given $\varepsilon>0$ there exists $\delta>0$ such that if $x, y\in S_X$ (the unit sphere in X) and $\|x-y\|\geqslant\varepsilon$ then $\|(x+y)/2\|\leqslant 1-\delta$.

THEOREM 2 (Rolewicz [14], Theorems 2, 4 and 5). *Let $(X,\|\cdot\|$ be a Banach space. Consider the following properties*:

(i) *The norm is uniformly rotund.*

(ii) *The norm has the drop property.*

(iii) *The norm has property* (α).

(iv) *X is reflexive.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

It is known that the norm in a Banach space X is $(k+1)$-rotund if it is k-rotund ($k=2,3,\ldots$) and that a Banach space with a k-rotund norm ($k=2,3,\ldots$) is reflexive (cf., for example, [8], VII, § 2). Obviously, there are nonreflexive Banach spaces such that the norm has property (H), as l^1. Kutzarova proved ([12], Theorem 1, Corollary 2) that *for a Banach space X the norm has the drop property if it is 2-rotund* and that *X is reflexive if and only if it admits an equivalent norm with property* (α).

We shall need the following simple lemma:

LEMMA. *Let $x_1,\ldots,x_n$ ($n=2,3,\ldots$) be elements of a normed space $(X,\|\cdot\|)$ such that $x_{i+1}\in D(x_i, B_X)$, $i=1,\ldots,n-1$. Let $z\in\operatorname{conv}(x_1,\ldots,x_{n-1})$. Then $z\notin B_X$ and $x_n\in D(z, B_X)$.*

Proof. For $n=2$ the result is obvious. Let us prove the lemma for $n=3$. We can suppose $x_1\neq x_2$, $z\neq x_i$, $i=1,2$. Thus we can write

(1) $$x_2=\lambda_2 x_1+(1-\lambda_2)b_2,\quad 0<\lambda_2<1,\ b_2\in B_X,$$

(2) $$x_3=\lambda_3 x_2+(1-\lambda_3)b_3,\quad 0\leqslant\lambda_3\leqslant 1,\ b_3\in B_X,$$

(3) $$z=\theta x_1+(1-\theta)x_2,\quad 0<\theta<1,$$

hence $z=\theta x_1+(1-\theta)\big(\lambda_2 x_1+(1-\lambda_2)b_2\big)=kx_1+(1-\theta)(1-\lambda_2)b_2$ where $k=\theta+(1-\theta)\lambda_2\in\,]\lambda_2, 1[$.

Therefore

(4) $$x_1 = \frac{1}{k}[z-(1-\theta)(1-\lambda_2)b_2]$$

and, substituting (4) in (2),

(5) $$x_3 = \frac{1}{k}\lambda_3\lambda_2 z+\frac{1}{k}[k\lambda_3(1-\lambda_2)-\lambda_3\lambda_2(1-\theta)(1-\lambda_2)]b_2+(1-\lambda_3)b_3.$$

A simple computation shows that (5) is a convex combination. Moreover, using (4) in (1) we get

$$x_2 = \frac{1}{k}\lambda_2 z+(1-\lambda_2)\frac{\theta}{k}b_2.$$

Thus z cannot belong to B_X.

Suppose now that the lemma has been proved till $n \geqslant 3$. Let $x_1, \ldots, x_{n+1}$ be elements in X such that $x_{i+1} \in D(x_i, B_X)$, $i = 1, \ldots, n$, and $z \in \operatorname{conv}(x_1, \ldots, x_n)$. Obviously we can find an element $u \in \operatorname{conv}(x_{n-1}, x_n)$ such that $z \in \operatorname{conv}(x_1, \ldots, x_{n-2}, u)$. Since $u \in D(x_{n-2}, B_X)$, the case $n = 3$ gives $u \notin B_X$ and $x_{n+1} \in D(u, B_X)$. The induction hypothesis now yields $z \notin B_X$ and $x_{n+1} \in D(z, B_X)$. ■

THEOREM 3. *Let* $(X, \|\cdot\|$ *be a Banach space. Then the following statements are equivalent*:

(i) *The norm* $\|\cdot\|$ *has the drop property.*

(ii) *The norm* $\|\cdot\|$ *has property* (α).

(iii) *The norm* $\|\cdot\|$ *has property* (H) *and* $(X, \|\cdot\|)$ *is reflexive.*

Proof. (i) ⇒ (ii) is contained in Theorem 2.

(ii) ⇒ (iii). Again by Theorem 2 the space $(X, \|\cdot\|)$ is reflexive. Let $(x_n)_{n=1}^\infty$ be a sequence in S_X (the unit sphere in X) which converges weakly to $x_0 \in S_X$. Let $(y_n)_{n=1}^\infty$ be an arbitrary subsequence of $(x_n)_{n=1}^\infty$. Let $f \in X^*$ be such that $\|f\| = 1$ and $f(x_0) = 1$. Given $\varepsilon > 0$, let $\delta > 0$ be such that $\alpha(S(f, \delta)) < \varepsilon$, where $S(f, \delta) = \{x: x \in B_X, f(x) > 1-\delta\}$. Let n_0 be a positive integer such that, for every $n \geqslant n_0$, $y_n \in S(f, \delta)$. $S(f, \delta)$ can be covered by a finite number of sets of diameter $< \varepsilon$. It is now clear that a diagonal procedure allows us to select a Cauchy subsequence $(z_n)_{n=1}^\infty$ of $(y_n)_{n=1}^\infty$. Then $(z_n)_{n=1}^\infty$ converges to some $z \in X$. But $(z_n)_{n=1}^\infty$ converges weakly to x_0. So $z = x_0$, and because $(y_n)_{n=1}^\infty$ was an arbitrary subsequence of $(x_n)_{n=1}^\infty$ it follows that $(x_n)_{n=1}^\infty$ converges to x_0. Thus the norm $\|\cdot\|$ has property (H).

(iii) ⇒ (i). Let $(x_n)_{n=1}^\infty$ be a non-eventually constant stream in X.

Suppose first that $(\|x_n\|)_{n=1}^\infty$ does not converge to 1. If $(x_n)_{n=1}^\infty$ does not have a convergent subsequence, then $\{x_n: n \in N\}$ is a closed set at positive distance from B_X. By Theorem 1, there exists a point $x \in \{x_n: n \in N\}$ such that $D(x, B_X) \cap \{x_n: n \in N\} = \{x\}$, which is impossible for such a stream. Thus $(x_n)_{n=1}^\infty$ has a convergent subsequence.

On the other hand, if $(\|x_n\|)_{n=1}^\infty$ converges to 1, then by the Eberlein–Shmul'yan Theorem, there exists a point $x_0 \in X$ and a subsequence $(y_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that $(y_n)_{n=1}^\infty$ converges weakly to x_0. By the weak lower semicontinuity of the norm, $\|x_0\| \leqslant 1$. But $x_0 \in \overline{\operatorname{conv}}^\sigma(x_1, x_2, \ldots) = \overline{\operatorname{conv}}(x_1, x_2, \ldots)$ (the weak and norm closure, respectively). By the Lemma, $\|x\| > 1$ for every $x \in \operatorname{conv}(x_1, x_2, \ldots)$, hence $\|x_0\| \geqslant 1$. Therefore $\|x_0\| = 1$. Hence, because the norm $\|\cdot\|$ has property (H), $(x_n)_{n=1}^\infty$ converges in norm to x_0. By Proposition 1, the norm $\|\cdot\|$ has the drop property. ■

The following theorem characterizes reflexive Banach spaces and answers a question of Rolewicz:

THEOREM 4. *Let* X *be a Banach space. Then the following statements are equivalent*:

(i) X *is reflexive.*

(ii) X *has an equivalent norm* $\|\cdot\|$ *with the drop property.*

Proof. (i) ⇒ (ii). A reflexive Banach space is weakly compactly generated. By Troyanski's renorming theorem [18] X has an equivalent norm $\|\cdot\|$ which is locally uniformly rotund (= LUR) (i.e., for any sequence $(x_n)_{n=1}^\infty$ and x in X such that $\|x_n\| \leqslant 1$, $n = 1, 2, \ldots$, $\|x\| = 1$, if

$$\lim_{n\to\infty} \|x+x_n\| = 2,$$

then $(x_n)_{n=1}^\infty$ converges in norm to x). It is well known that a (LUR)-norm has property (H) (cf., for example, [8], VII, § 2). Now (ii) follows from Theorem 3.

(ii) ⇒ (i). This implication is contained in Theorem 2. ■

Kutzarova's characterization of reflexivity (see § 1) follows from Theorems 3 and 4. As we mentioned in § 1, if the norm $\|\cdot\|$ in a Banach space is k-rotund (k a positive integer greater than 1), then $\|\cdot\|$ has property (H) and the space is reflexive. It follows from Theorem 3 that a k-rotund norm in a Banach space has the drop property. By Theorem 4, every reflexive nonsuperreflexive Banach space X furnishes an example of a nonuniformly convex norm with the drop property (in view of Enflo's result [10]). Fan and Glicksberg [11] proved that the reflexive space $(l^2(l^{p_1}, l^{p_2}, \ldots), \|\cdot\|_2)$ (the substitution space or product space; cf., for example, [8], II, § 2) has a norm $(\|\cdot\|_2)$ which is k-rotund for all $k > 1$ and all sequences $(p_n)_{n=1}^\infty$ of integers $1 < p_n < \infty$, $n = 1, 2, \ldots$, and Day proved in [7] that this space is uniformly convex if there exist numbers m and M such that $1 < m \leqslant p_n \leqslant M < +\infty$, $n = 1, 2, \ldots$, and in [6] that otherwise it is not isomorphic to a uniformly convex Banach space. This gives another explicit example of a norm with the drop property in a nonsuperreflexive Banach space.

An alternative proof of Theorem 4 uses, instead, the following Proposition and Corollary 3 in [16]: *A Banach space X is reflexive if and only if it has an equivalent weakly 2-rotund norm* (a norm $\|\cdot\|$ in a Banach space X is *weakly 2-rotund* if any sequence in its unit sphere such that

$$\lim_{n,m\to\infty} \|x_n+x_m\| = 2$$

is weakly Cauchy).

PROPOSITION 2. *Let $(X, \|\cdot\|)$ be a Banach space. Then the norm has the drop property if it is weakly 2-rotund.*

Proof. By Troyanski's result quoted in the previous paragraph, X is reflexive. It is known [3] that, for a reflexive space, the norm $\|\cdot\|$ is weakly 2-rotund if and only if for any sequence $(x_n)_{n=1}^{\infty}$ in the unit sphere the condition

$$\lim_{n,m\to\infty} |f(x_n+x_m)| = 2 \quad \text{for some } f\in X^*,\ \|f\| = 1,$$

implies that $(x_n)_{n=1}^{\infty}$ is convergent.

Let $(x_n)_{n=1}^{\infty}$ be a stream in X. Let $C = \overline{\mathrm{conv}}(x_1, x_2, \ldots)$. By the Lemma, $\|x\| \geqslant 1$ for every $x\in C$. The Separation Theorem gives $f\in X^*$ such that $\|f\| = 1$ and $f(x) \geqslant 1$ for every $x\in C$. In particular, $1 \leqslant f(x_n) \leqslant \|x_n\|$, $n = 1, 2, \ldots$ If $(\|x_n\|)_{n=1}^{\infty}$ does not converge to 1, the argument used in the proof of (iii)$\Rightarrow$(i) in Theorem 3 gives a convergent subsequence of $(x_n)_{n=1}^{\infty}$. If, on the other hand, $(\|x_n\|)_{n=1}^{\infty}$ converges to 1, it follows that $(f(x_n))_{n=1}^{\infty}$ also converges to 1 and so does the double sequence $\big(f(\tfrac{1}{2}(x_n+x_m)\big)$. But the norm is weakly 2-rotund and we conclude that $(x_n)_{n=1}^{\infty}$ converges. In both cases we apply Proposition 1 to deduce that the norm has the drop property. ∎

Note that a weakly 2-rotund norm $\|\cdot\|$ and a norm $|||\cdot|||$ with the drop property in a Banach space X can be different. In fact, every norm in a finite-dimensional Banach space has the drop property (this follows from Proposition 1 or Theorem 3), but not every such norm is weakly 2-rotund.

§ **4. Heredity of the drop property.** Let $(X, \|\cdot\|)$ be a Banach space and Y a closed subspace. The induced norm in Y will be again denoted by $\|\cdot\|$ while the quotient norm in X/Y will be denoted by $\|\cdot\|_Q$. Let $q\colon X\to X/Y$ be the canonical mapping.

The following proposition is easy to prove:

PROPOSITION 3. *Let $(X, \|\cdot\|)$ be a Banach space such that $\|\cdot\|$ has the drop property. Let Y be a closed subspace. Then* (i) *in Y, $\|\cdot\|$ has the drop property,* (ii) *in X/Y, $\|\cdot\|_Q$ also has the drop property.*

Proof. Both Y and X/Y are reflexive Banach spaces. Now, (i) is obvious by Proposition 1. In order to prove (ii), let $(\bar{x}_n)_{n=1}^{\infty}$ be a sequence in $S_{X/Y}$ (the unit sphere in X/Y) and let $\bar{x}_0\in S_{X/Y}$ be such that $(\bar{x}_n)_{n=1}^{\infty}$ converges weakly to $\bar{x}_0$. Let $(x_n)_{n=1}^{\infty}$ be a sequence in X such that $q(x_n) = \bar{x}_n$ and

$$1 = \|\bar{x}_n\|_Q \leqslant \|x_n\| < \|\bar{x}_n\|_Q+1/n = 1+1/n, \qquad n = 1, 2, \ldots$$

The sequence $(x_n)_{n=1}^{\infty}$ is bounded. Let $(y_n)_{n=1}^{\infty}$ be an arbitrary subsequence of $(x_n)_{n=1}^{\infty}$. By the Eberlein–Shmul'yan Theorem we can select a subsequence $(z_n)_{n=1}^{\infty}$ of $(y_n)_{n=1}^{\infty}$ such that $(z_n)_{n=1}^{\infty}$ converges weakly to some element $z_0\in X$. By the weak lower semicontinuity of the norm, $\|z_0\| \leqslant 1$. But $\big(q(z_n)\big)_{n=1}^{\infty}$ converges weakly to $q(z_0)$, hence $q(z_0) = \bar{x}_0$ and thus $\|z_0\| \geqslant \|\bar{x}_0\|_Q = 1$. Hence $\|z_0\| = 1$. By Theorem 3, the norm $\|\cdot\|$ has property (H), so that $(z_n)_{n=1}^{\infty}$ converges in norm to z_0. But $(y_n)_{n=1}^{\infty}$ was an arbitrary subsequence of $(x_n)_{n=1}^{\infty}$ and $\big(q(z_n)\big)_{n=1}^{\infty}$ converges in $\|\cdot\|_Q$ to $q(z_0) = \bar{x}_0$. Hence $(\bar{x}_n)_{n=1}^{\infty}$ converges in $\|\cdot\|_Q$ to $\bar{x}_0$ and $\|\cdot\|_Q$ has property (H). Again by Theorem 3, $\|\cdot\|_Q$ has the drop property. ∎

Let $(X, \|\cdot\|)$ be a Banach space and suppose that the norm $\|\cdot\|$ has property (H). Let $1 \leqslant p < \infty$. Then, in $l^p(X)$, the norm $\|\cdot\|_p$ has property (H) ([1], [13]). The converse is obviously true. Moreover, for $1 < p < \infty$, $l^p(X)$ is reflexive if and only if X is. It is worth noticing that the proof of Theorem 3.1 in [13] for a Banach space X and $l^p(X)$ $(1 \leqslant p < \infty)$ applies word by word to the case of the substitution space $l^p(X_1, X_2, \ldots)$, implying that the norm $\|\cdot\|_p$ in $l^p(X_1, X_2, \ldots)$ has property (H) if and only if the norms in X_n all have property (H) for $n = 1, 2, \ldots$ From this and from Theorem 3 it follows that the norm $\|\cdot\|_p$ in $l^p(X_1, X_2, \ldots)$ has the drop property if and only if the norms in X_n all have the drop property for $n = 1, 2, \ldots$

For the Lebesgue–Bochner spaces $L^p(\mu, X)$, X a Banach space, (T, Σ, μ) a finite measure space, the corresponding result does not hold: Smith and Turett [15] have given an example of a reflexive Banach space $(X, \|\cdot\|)$ such that $\|\cdot\|$ has property (H) and, for every $1 \leqslant p < \infty$, the norm $\|\cdot\|_p$ in $L^p([0, 1], \lambda, X)$ (λ the Lebesgue measure in $[0, 1]$) fails property (H). In view of Theorem 3, $\|\cdot\|$ in X has the drop property, but $\|\cdot\|_p$ in $L^p([0, 1], \lambda, X)$, $1 < p < \infty$, fails this property. This answers in the negative a question of Troyanski (personal communication).

A partial positive answer is given by the following observation: (cf. [15], Question 10): If $(X, \|\cdot\|)$ is a reflexive Banach space such that its norm $\|\cdot\|$ is rotund (= strictly convex, i.e., the unit sphere does not contain any segment) and has property (H), then $\|\cdot\|_p$ in $L^p(\mu, X)$, $1 < p < \infty$, has property (H). Moreover, $L^p(\mu, X)$, $1 < p < \infty$, is reflexive if and only if X is (this fact follows from the well-known result that for (T, Σ, μ) a finite measure space and $1 < p < \infty$, the dual space of $L^p(\mu, X)$ is $L^q(\mu, X)$, where $p^{-1}+q^{-1} = 1$, if and only if X^* has the Radon–Nikodym property with respect to μ ([9], IV.1.1)). Thus if the norm $\|\cdot\|$ in X is rotund and has the drop property, then $\|\cdot\|_p$ in $L^p(\mu, X)$, $1 < p < \infty$, also has the drop property.

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References

[1] K. W. Anderson, *Midpoint local uniform convexity and other geometric properties of Banach spaces*, Dissertation, University of Illinois, 1960.

[2] F. Browder, *Normal solvability and the Fredholm alternative for mappings into infinite dimensional manifolds*, J. Funct. Anal. 8 (1971), 250–274.

[3] D. F. Cudia, *The geometry of Banach spaces. Smoothness*, Trans. Amer. Math. Soc. 110 (1964), 284–314.

[4] J. Daneš, *A geometric theorem useful in non-linear functional analysis*, Boll. Un. Mat. Ital. 6 (1972), 369–372.

[5] —, *Equivalence of some geometric and related results of non-linear functional analysis*, Comment. Math. Univ. Carolinae 26 (1985), 443–454.

[6] M. M. Day, *Reflexive Banach spaces not isomorphic to uniformly convex spaces*, Bull. Amer. Math. Soc. 47 (1941), 313–317.

[7] —, *Some more uniformly convex spaces*, ibid. 47 (1941), 504–507.

[8] —, *Normed Linear Spaces*, 3rd. ed., Springer, 1973.

[9] J. Diestel and J. Uhl, jr., *Vector Measures*, Math. Surv. 15, A.M.S., 1977.

[10] P. Enflo, *Banach spaces which can be given an equivalent uniformly convex norm*, Israel J. Math. 13 (1972), 281–288.

[11] K. Fan and I. Glicksberg, *Some geometric properties of the sphere in a normed linear space*, Duke Math. J. 25 (1958), 553–568.

[12] D. Kutzarova, *A sufficient condition for the drop property*, to appear.

[13] I. E. Leonard, *Banach sequence spaces*, J. Math. Anal. Appl. 54 (1976), 245–265.

[14] S. Rolewicz, *On drop property*, Studia Math. 85 (1987), 27–35.

[15] M. Smith and B. Turett, *Rotundity in Lebesgue–Bochner function spaces*, Trans. Amer. Math. Soc. 257 (1980), 105–118.

[16] S. L. Troyanski, *On locally uniformly convex and differentiable norms in certain non-separable Banach spaces*, Studia Math. 37 (1971), 173–180.

[17] P. P. Zabreiko and M. A. Krasnosel'skii, *On the solvability of non-linear operator equations*, Funktsional. Anal. i Prilozhen. 5 (3) (1971), 42–44 (in Russian).

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