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MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
Sendai 980, Japan

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A cheaper Swiss cheese

by

T. W. KÖRNER (Cambridge)

Abstract. We simplify some of the computations required for McKissick's example of a normal uniform algebra.

§ 1. Introduction. McKissick [1] has constructed an elegant "Swiss cheese" K with the following property.

THEOREM 1.1. *There is a compact subset K of $\mathbf{C}$ such that $R(K)$ is normal but is not equal to $C(K)$.*

His proof depends on the following preliminary lemma.

LEMMA 1.2. *Given any $\varepsilon > 0$ we can find a sequence of open discs $\{\Delta_k\}$ and a sequence of rational functions $\{f_n\}$ such that:*

(a) *If r_k is the radius of Δ_k then $\sum_{k=1}^{\infty} r_k < \varepsilon$.*

(b) *The poles of the f_n lie in $\bigcup_{k=1}^{\infty} \Delta_k$.*

(c) *The sequence f_n tends uniformly to zero on $\{z\colon |z| \geqslant 1\} \setminus \bigcup_{k=1}^{\infty} \Delta_k$ and uniformly to some nowhere zero function on $\{z\colon |z| < 1\} \setminus \bigcup_{k=1}^{\infty} \Delta_k$.*

The standard proof of McKissick's lemma relies on a construction of Beurling and in the textbook detail of [3] fills 7 pages. The object of this note is to give a computationally simpler derivation of the lemma. Our method will have the further minor advantage of fulfilling two further conditions.

LEMMA 1.2′. *In addition to conditions* (a), (b), *and* (c) *of Lemma* 1.2 *we can demand:*

(d) $\bigcup_{k=1}^{\infty} \Delta_k \subseteq \{z\colon 1-\varepsilon < |z| < 1\}$.

(e) *$\Delta_1, \Delta_2, \ldots$ are disjoint.*

(Condition (d) is required for [2] but can also be obtained by modifying the original construction.)

§ 2. **The construction.** Our basic observation is the following.

LEMMA 2.1. *If* $N \geqslant 2$ *is an integer and* $h_N(z) = 1/(1-z^N)$ *then*
(i) $|h_N(z)| \leqslant 2|z|^{-N}$ *for* $|z|^N \geqslant 2$,
(ii) $|1-h_N(z)| \leqslant 2|z|^N$ *for* $|z|^N \leqslant 2^{-1}$,
(iii) $h_N(z) \neq 0$ *for all* z.
Further if $(8\log N)^{-1} > \delta > 0$ *then*
(iv) $|h_N(z)| \leqslant 2\delta^{-1}$ *provided only that* $|z-w| \geqslant \delta N^{-1}$ *whenever* $w^N = 1$.

Proof. Parts (i), (ii) and (iii) are obvious. To see (iv) set $\omega = \exp(2\pi i/N)$ and observe that

$$|h_N(z)| = \left|\frac{1}{N}\sum_{r=0}^{N-1}\frac{\omega^r}{\omega^r - z}\right| \leqslant \frac{1}{N}\sum_{r=0}^{N-1}\frac{1}{|\omega^r - z|}.$$

By symmetry we may suppose that $0 \leqslant \arg z \leqslant \pi/N$. But if $1 \leqslant k \leqslant N/4$, we have

$$|\omega^k - z| \geqslant |\sin(\arg(\omega^k z^{-1}))| \geqslant \sin((2k-1)\pi/2N) \geqslant (2|k|-1)/N$$

and, similarly, if $-N/4 \leqslant k \leqslant -1$, $|\omega^k - z| \geqslant (2|k|+1)/N$, whilst if $N/2 \geqslant |k| \geqslant N/4$, $|\omega^k - z| \geqslant 1 \geqslant 2|k|/N$. Thus

$$|h_N(z)| \leqslant \frac{1}{N}\left(\frac{1}{|1-z|} + \sum_{r\neq 0}\frac{1}{|\omega^r - z|}\right) \leqslant \frac{1}{N}\left(N\delta^{-1} + 2\sum_{r=1}^{N}\frac{N}{r}\right)$$
$$\leqslant \frac{1}{N}(N\delta^{-1} + 8N\log N) = \delta^{-1} + 8\log N \leqslant 2\delta^{-1}. \quad \blacksquare$$

LEMMA 2.2. *If in Lemma* 2.1 *we set* $N = n2^{2n}$ *with* n *sufficiently large (e.g.* $n \geqslant 10^8$*) then*
(i) $|h_N(z)| \leqslant (n+1)^{-4}$ *for* $|z| \geqslant 1+2^{-(2n+1)}$,
(ii) $|1-h_N(z)| \leqslant (n+1)^{-4}$ *for* $|z| \leqslant 1-2^{-(2n+1)}$,
(iii) $h_N(z) \neq 0$ *for all* z,
(iv) $|h_N(z)| \leqslant 2n^2$ *provided only that* $|z-w| \geqslant n^{-3}2^{-2n}$ *whenever* $w^N = 1$.

Proof. Since $(1+m^{-1})^m \geqslant e/2$ for m sufficiently large it follows that $|z|^N \geqslant (1+2^{-(2n+1)})^N \geqslant (e/2)^{n/2} \geqslant (n+1)^4$ whenever $|z| \geqslant 1+2^{-(2n+1)}$ for n sufficiently large. Thus Lemma 2.2 (i) follows from Lemma 2.1 (i). A similar argument applies for (ii). Part (iv) follows on putting $\delta = n^{-2}$. $\blacksquare$

We now apply a dilatation to the result.

LEMMA 2.3. *Provided only that* n *is sufficiently large we can find a finite collection* $A(n)$ *of disjoint open discs and a rational function* g_n *such that*
(i) $\sum_{\Delta\in A(n)} \text{radius}\,\Delta = n^{-2}$,
(ii) *the poles of* g_n *lie in* $\bigcup_{\Delta\in A(n)} \Delta$,

(iii) $|g_n(z)| \leqslant (n+1)^{-4}$ *for* $|z| \geqslant 1-2^{-(2n+1)}$,
(iv) $|1-g_n(z)| \leqslant (n+1)^{-4}$ *for* $|z| \leqslant 1-2^{-(2n-1)}$,
(v) $|g_n(z)| \leqslant 2n^2$ *for* $z \notin \bigcup_{\Delta\in A(n)} \Delta$,
(vi) $g_n(z) \neq 0$ *for all* z,
(vii) $\bigcup_{\Delta\in A(n)} \Delta \subseteq \{z\colon 1-2^{-(2n-1)} \leqslant |z| \leqslant 1-2^{-(2n+1)}\}$.

Proof. Let $N = n2^n$, $\omega = \exp(2\pi i/N)$ and $g_n(z) = h_N((1-2^{-2n})^{-1}z)$. If we take $A(n)$ to be the collection of discs with radii $n^{-3}2^{-2n}$ and centres $(1-2^{-2n})\,\omega^r$ $[0 \leqslant r \leqslant N-1]$ the required results are either trivial like (i), (ii) and (vii) or follow directly from Lemma 2.2 on scaling by a factor $1-2^{-2n}$. $\blacksquare$

Multiplying the g_n together produces the required result.

THEOREM 2.4. *Given any* $\varepsilon > 0$ *there exists an* $m(\varepsilon)$ *such that if we adopt the notation of Lemma* 2.3, *write* $f_n = 4^{-1}(m+1)^{-4}\prod_{r=m}^{n} g_r$ *and let* $\{\Delta_k\}$ *be a sequence enumerating the discs of* $\bigcup_{r=m}^{\infty} A(r)$ *then the conclusions of Lemmas* 1.2 *and* 1.2′ *hold.*

Proof. Observe that

$$\sum_{k=1}^{\infty} r_k = \sum_{r=m}^{\infty}\sum_{\Delta\in A(r)} \text{radius}\,\Delta = \sum_{r=m}^{\infty} r^{-2} < \varepsilon$$

provided only that $m(\varepsilon) > 2\varepsilon^{-1}+1$. Thus, conclusions (a), (b), (d) and (e) are easy to verify. To prove (c) set $K = \prod_{r=1}^{\infty}(1+(r+1)^{-4})$ and observe that, if $z \notin \bigcup_{k=1}^{\infty}\Delta_k$, a simple induction gives

$$|f_n(z)| \leqslant K \qquad \text{for } |z| \leqslant 1-2^{-2n-1},$$
$$|f_n(z)| \leqslant n^{-2} \leqslant K \qquad \text{for } 1-2^{-2n-1} < |z| < 1-2^{-2n+1},$$
$$|f_n(z)| \leqslant 4^{-1}(n+1)^{-4} \qquad \text{for } 1-2^{-2n+1} < |z|.$$

Using the trivial equality

$$|f_{n+1}(z) - f_n(z)| = |f_n(z)|\,|1-g_{n+1}(z)|$$

we see that, provided $z \notin \bigcup_{k=1}^{\infty}\Delta_k$,

$$|f_{n+1}(z) - f_n(z)| \leqslant K(n+1)^{-4} \qquad \text{for } |z| \leqslant 1-2^{-2n+1}$$
$$|f_{n+1}(z) - f_n(z)| \leqslant 4^{-1}(n+1)^{-4}\cdot 2(1+(n+1)^2) \leqslant (n+1)^{-2}$$
$$\text{for } 1-2^{-2n+1} < |z|.$$

Thus $|f_{n+1}(z)-f_n(z)| \leqslant K(n+1)^{-2}$ for all $z \notin \bigcup_{k=1}^{\infty} \Delta_k$ and, by for example the Weierstrass M test, f_n converges uniformly to f say.

To see that $f(z) \neq 0$ for $|z|<1$, $z \notin \bigcup_{k=1} \Delta_k$, note that if $|z| \leqslant 1-2^{-2n-1}$ then $f_n(z) \neq 0$, and

$$\sum_{r=n+1}^{\infty} |1-g_r(z)| \leqslant \sum_{r=n+1}^{\infty} (r+1)^{-4} < \infty$$

so by a basic result on infinite products

$$f(z) = f_n(z) \prod_{r=n+1}^{\infty} g_r(z) \neq 0. \blacksquare$$

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TRINITY HALL
Cambridge CB2 1SB, England

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Some results on intersection properties of balls in complex Banach spaces

by

Å. LIMA (Aas) and A. K. ROY (Calcutta)

Abstract. Predual real L^1-spaces are characterized by the 4.2. intersection property. The structure of real spaces with the 3.2. intersection property and of real and complex spaces with the 4.3. intersection property is fairly well understood. In this paper we study complex spaces with the *n.k.* intersection property when $n > k \geqslant 4$. We show that the 5.4. intersection property characterizes complex L^1-preduals, and that the $(2n+1).2n$. intersection property implies the almost $(2n+1).(2n-1)$. intersection property in the complex case.

1. Introduction. Let A be a Banach space over the complex scalars C. $B(a, r)$ denotes the closed ball in A with centre a and radius r. Let n,k be integers with $n > k \geqslant 2$. We say that A has the *almost n.k.I.P.* (to be read as the almost *n.k.* intersection property) if for every family $\{B(a_j, r_j)\}_{j=1}^{n}$ of n balls in A such that for any k of them,

$$\bigcap_{m=1}^{k} B(a_{j_m}, r_{j_m}) \neq \emptyset,$$

we have

$$\bigcap_{j=1}^{n} B(a_j, r_j+\varepsilon) \neq \emptyset \quad \text{for all } \varepsilon > 0.$$

(If we can take $\varepsilon = 0$, we say that A has the *n.k.*I.P.) Introducing the space

$$H^n(A^*) = \{(x_1, \ldots, x_n) \in (A^*)^n \colon \sum_{k=1}^{n} x_k = 0\}$$

with the norm $\|(x_1, \ldots, x_n)\| = \sum_{k=1}^{n} \|x_k\|$, it was proved in [7] that A has the almost *n.k.*I.P. if and only if each extreme point $(x_1, \ldots, x_n)$ in the unit ball of $H^n(A^*)$ has at most k nonzero components. Thus examination of the extreme point structure of the unit ball of $H^n(A^*)$ furnishes a useful analytic device for the study of the intersection properties of balls in Banach spaces, and this has been effectively used in obtaining various characterizations of