[8] K. Fan, *Convex sets and their applications*, Lecture Notes, Appl. Math. Div., Argonne Nat. Lab., Argonne, Ill. 1959.
[9] —, *A generalisation of Tychonoff's fixed point theorem*, Math. Ann. 142 (1961), 305–310.
[10] —, *Sur un théorème minimax*, C. R. Acad. Sci. Paris, Groupe 1, 259 (1964), 3925–3928.
[11] —, *Applications of a theorem concerning sets with convex sections*, Math. Ann. 163 (1966), 189–203.
[12] —, *A minimax inequality and applications*, in: O. Shisha (ed.), Inequalities III, Academic Press, New York and London 1972, 103–113.
[13] —, *Some properties of convex sets related to fixed point theorems*, Math. Ann. 266 (1984), 519–537.
[14] A. Granas, *KKM-maps and their applications to nonlinear problems*, in: The Scottish Book, R. D. Mauldin (ed.), Birkhäuser, Boston 1981, 45–61.
[15] A. Granas and F.-C. Liu, *Remark on a theorem of Ky Fan concerning systems of inequalities*, Bull. Inst. Math. Acad. Sinica 11 (4) (1983), 639–643.
[16] J. Gwinner, *On fixed points and variational inequalities — a circular tour*, Nonlinear Anal. 5 (1981), 565–583.
[17] P. Hartman and G. Stampacchia, *On some non-linear elliptic differential-functional equations*, Acta Math. 115 (1966), 271–310.
[18] C. Horvath, *Points fixes et coïncidences pour les applications multivoques sans convexité*, C. R. Acad. Sci. Paris Sér. I Math. 296 (1983), 403–406.
[19] B. Knaster, C. Kuratowski und S. Mazurkiewicz, *Ein Beweis des Fixpunktsatzes für n-dimensionale Simplexe*, Fund. Math. 14 (1929), 132–137.
[20] M. Lassonde, *On the use of KKM multifunctions in fixed point theory and related topics*, J. Math. Anal. Appl. 97 (1983), 151–201.
[21] F.-C. Liu, *A note on the von Neumann–Sion Minimax Principle*, Bull. Inst. Math. Acad. Sinica 6 (2) (1978), 517–524.
[22] U. Mosco, *Implicit variational problems and quasi variational inequalities*, in: Lecture Notes in Math. 543, Nonlinear Operators and the Calculus of Variations, Bruxelles, 1975, Springer-Verlag, Berlin–Heidelberg–New York 1976, 83–156.
[23] J. Nash, *Non-cooperative games*, Ann. of Math. 54 (1951), 286–295.
[24] J. von Neumann, *Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes*, Ergebnisse eines Mathematischen Kolloquiums 8 (1937), 73–83.
[25] M. Sion, *On general minimax theorems*, Pacific J. Math. 8 (1958), 171–176.
[26] E. Tarafdar and H. B. Thompson, *On Ky Fan's Minimax Principle*, J. Austral. Math. Soc. Ser. A 26 (1978), 220–226.
[27] K. Sakamaki and W. Takahashi, *Systems of convex inequalities and their applications*, J. Math. Anal. Appl. 70 (1979), 445–459.
[28] C. L. Yen, *A minimax inequality and its applications to variational inequalities*, Pacific J. Math. 97 (1981), 477–481.

DÉPARTEMENT DE MATHÉMATIQUES ET DE STATISTIQUE
UNIVERSITÉ DE MONTRÉAL
Montréal, Canada

Received October 12, 1984 (2003)

STUDIA MATHEMATICA, T. LXXXIII. (1986)

The size of sums of sets

by

DANIEL M. OBERLIN* (Tallahassee, Fl.)

Abstract. Lower bounds are obtained for the Haar measure of the set $K+E$ when K and E are suitable subsets of a locally compact abelian group.

Let G be a locally compact abelian group with Haar measure m. Suppose that K and E are measurable subsets of G such that the sum set $K+E = \{k+e\colon k\in K,\ e\in E\}$ is also measurable. What can one say about $m(K+E)$? The papers [1], [2], [3], and a substantial portion of the book [4] are concerned with various aspects of this question. Here we are interested in inequalities which give a lower bound for $m(K+E)$. One example of such an inequality is the following theorem, a corollary of Theorem 2.2 in [4].

THEOREM A. *If G is a torus group $\boldsymbol{R}^n/\boldsymbol{Z}^n$ for some positive integer n, and if $m(K)+m(E) \leqslant m(G)$, then*

$$m(K)+m(E) \leqslant m(K+E).$$

This satisfying inequality provides nontrivial information when both $m(K)$ and $m(E)$ are positive. But what can one say if, for example, $m(K) = 0$? Here the situation has a somewhat different flavor which is typified by the next theorem if $m(K-K) \neq 0$.

THEOREM B. *Suppose K and E are subsets of the locally compact abelian group G with K compact, E and $K+E$ measurable. Then*

$$\sqrt{m(K-K)\,m(E)} \leqslant m(K+E).$$

Theorem B is essentially Proposition 4 below in the case $n = 2$, in which case the constant δ is easily checked to be $m(K_1-K_2)$. Theorem B generalizes and explains the result in [5].

The purpose of this paper, then, is to investigate the existence of inequalities of the form

$$(1)\qquad \varepsilon\cdot m(E)^{\beta} \leqslant m(K+E)$$

holding for some fixed subset $K \subseteq G$, for some $\beta\in(0,1)$ and $\varepsilon > 0$ depending

* Partially supported by the National Science Foundation.

on K, and for every measurable $E \subseteq G$ such that $K+E$ is measurable. Of course such an inequality is nontrivial only when $m(E)$ is small and $m(K) = 0$. Then, at least for $G = \boldsymbol{R}^n$, the smallest allowable value of β seems to be related to the dimension of K in G. For example, we shall see in § 2 that a particular instance of (1) occurs when $G = \boldsymbol{R}^n$ and K is a suitable k-dimensional surface in $\boldsymbol{R}^n$. Then β can be taken to be $1-(k/n)$.

There are three sections to this paper: § 1 contains a theorem on the boundedness of certain multilinear forms. This theorem is a tool which is applied in § 2 to prove inequalities of the form (1) when β is a number $1-(k/n)$ with $k = 1, 2, \ldots, n-1$. In § 3 is a supplementary result concerning the case $k = n-1$.

§ 1. Boundedness of certain multilinear forms. Let G be a locally compact abelian group. Here is some notation: if λ is a measure on G, then $\int \cdot\, d\lambda(x)$ denotes integration over G with respect to λ. When λ is m, Haar measure on G, the integral is written $\int \cdot\, dx$. And for $1 \leqslant p < \infty$, $\|g\|_p$ is the L^p-norm

$$\left(\int |g|^p\, dx\right)^{1/p}$$

of an appropriate function g on G. Letting μ be a Borel measure on the Cartesian product G^n, we wish to study inequalities of the type

$$(2)\qquad \left|\int \int_{G^n} g_1(x-x_1)\, g_2(x-x_2) \ldots g_n(x-x_n)\, d\mu(x_1, \ldots, x_n)\, dx\right| \leqslant C \prod_{j=1}^{n} \|g_j\|_p.$$

The result we prove here is the following theorem.

Theorem 1. *Suppose that $\lambda_1, \ldots, \lambda_n$ are finite positive Borel measures on G and suppose that for some fixed $l = 1, 2, \ldots, n-1$ an inequality of the form*

$$(3)\qquad \left|\int \ldots \int f(x_1+x_2+\ldots+x_l-x_{l+1},\, x_1+\ldots+x_l-x_{l+2}, \ldots, x_1+\ldots+x_l-x_n)\right.$$
$$\left. d\lambda_1(x_1)\, d\lambda_2(x_2) \ldots d\lambda_n(x_n)\right| \leqslant C_1 \|f\|_1$$

holds for some positive C_1 and for all $f \in L^1(G^{n-l})$. Then the inequality

$$\left|\int \int g_1(x-x_1)\, d\lambda_1(x) \int g_2(x-x_2)\, d\lambda_2(x_2) \ldots \int g_n(x-x_n)\, d\lambda_n(x_n)\, dx\right| \leqslant C_2 \prod_{j=1}^{n} \|g_j\|_l$$

holds for some constant C_2 and for all $g_1, \ldots, g_n \in L^l(G)$.

To get a feeling for the hypothesis of Theorem 1 (at least when $l = n-1$) define $\bar{\lambda}_n$ so that $\int g(x)\, d\bar{\lambda}_n(x) = \int g(-x)\, d\lambda_n(x)$. Then when $l = n-1$ the hypothesis is that the convolution $\lambda_1 * \lambda_2 * \ldots * \lambda_{n-1} * \bar{\lambda}_n$ is a bounded function on G.

Proof of Theorem 1. The symbol C denotes a constant which may increase from line to line. Suppose that $g_1, \ldots, g_n$ are nonnegative functions. We start by making the change of variable $x \to x+x_1$ (which requires two

applications of Fubini's theorem) and then applying Hölder's inequality with the exponent pair $(l, l/[l-1])$:

$$(4)\qquad \int \int g_1(x-x_1)\, d\lambda_1(x_1) \ldots \int g_n(x-x_n)\, d\lambda_n(x_n)\, dx$$
$$= \int \ldots \int g_1(x) \left[\int g_2(x+x_1-x_2)\, g_3(x+x_1-x_3) \ldots g_n(x+x_1-x_n)\, d\lambda_1(x_1)\right]$$
$$d\lambda_2(x_2) \ldots d\lambda_n(x_n)\, dx$$
$$\leqslant C \|g_1\|_l \left(\int \ldots \int \left[\int g_2(x+x_1-x_2) \ldots g_n(x+x_1-x_n)\, d\lambda_1(x_1)\right]^{l/(l-1)}\right.$$
$$\left. d\lambda_2(x_2) \ldots d\lambda_n(x_n)\, dx\right)^{(l-1)/l}.$$

In (4) we make the change of variable $x \to x+x_2$ and then apply Hölder's inequality twice—first with the exponent pair $(l, l/[l-1])$ and then with the exponent pair $(l-1, [l-1]/[l-2])$—to obtain

$$(5)\qquad \|g_1\|_l \left(\int \ldots \int \left[\int g_2(x+x_1)\, g_3(x+x_1+x_2-x_3) \ldots\right.\right.$$
$$\left.\left. \ldots g_n(x+x_1+x_2-x_n)\, d\lambda_1(x_1)\right]^{l/(l-1)} d\lambda_2(x_2) \ldots d\lambda_n(x_n)\, dx\right)^{(l-1)/l}$$
$$\leqslant \|g_1\|_l \left(\int \ldots \int \left[\int g_2^l(x+\bar{x}_1)\, d\lambda_1(\bar{x}_1)\right]^{1/(l-1)} \left[\int\int (g_3(x+x_1+x_2-x_3) \ldots\right.\right.$$
$$\left.\left. \ldots g_n(x+x_1+x_2-x_n))^{l/(l-1)}\, d\lambda_1(x_1)\, d\lambda_2(x_2)\right] d\lambda_3(x_3) \ldots d\lambda_n(x_n)\, dx\right)^{(l-1)/l}$$
$$\leqslant C \|g_1\|_l \|g_2\|_l \left(\int \ldots \int \left[\int\int (g_3(x+x_1+x_2-x_3) \ldots g_n(x+x_1+x_2-x_n))^{l/(l-1)}\right.\right.$$
$$\left.\left. d\lambda_1(x_1)\, d\lambda_2(x_2)\right]^{(l-1)/(l-2)} d\lambda_3(x_3) \ldots d\lambda_n(x_n)\, dx\right)^{(l-2)/l}.$$

In (5) we make the change of variable $x \to x+x_3$ and then apply Hölder's inequality twice—first with the exponent pair $(l-1, [l-1]/[l-2])$ and then with the exponent pair $(l-2, [l-2]/[l-3])$ — to obtain as above that (5) is $\leqslant$

$$(6)\qquad C \prod_{j=1}^{3} \|g_j\|_l \left(\int \ldots \int \left[\int\int\int (g_4(x+x_1+x_2+x_3-x_4) \ldots\right.\right.$$
$$\left. \ldots g_n(x+x_1+x_2+x_3-x_4))^{l/(l-2)}\, d\lambda_1(x_1)\, d\lambda_2(x_2)\, d\lambda_3(x_3)\right]^{(l-2)/(l-3)}$$
$$\left. d\lambda_4(x_4) \ldots d\lambda_n(x_n)\, dx\right)^{(l-3)/l}.$$

After $l-4$ more changes of variable and double applications of Hölder's inequality we find that (6) is $\leqslant$

$$(7)\qquad C \prod_{j=1}^{l-1} \|g_j\|_l \left(\int \ldots \int \left[\int \ldots \int (g_l(x_1+\ldots+x_{l-1}-x_l) \ldots\right.\right.$$
$$\left. \ldots g_n(x_1+\ldots+x_{l-1}-x_n))^{l/2}\, d\lambda_1(x_1) \ldots d\lambda_{l-1}(x_{l-1})\right]^2$$
$$\left. d\lambda_l(x_l) \ldots d\lambda_n(x_n)\, dx\right)^{1/l}.$$

If we make the change of variable $x \to x+x_l$ and then apply Hölder's inequality with exponent pair (2,2), we see that (7) is $\leqslant$

(8) $$C\prod_{j=1}^{l-1}\|g_j\|_l\Big(\int\ldots\int\Big[\int\ldots\int\big(g_{l+1}(x+x_1+\ldots+x_l-x_{l+1})\ldots$$
$$\ldots g_n(x+x_1+\ldots+x_l-x_n)\big)^l d\lambda_1(x_1)\ldots d\lambda_n(x_n)\Big]g_l^l(x+\bar{x}_1+\ldots$$
$$\ldots+\bar{x}_{l-1})\,dx\,d\lambda_1(\bar{x}_1)\ldots d\lambda_{l-1}(\bar{x}_{l-1})\Big)^{1/l}.$$

By the hypothesis (3), the quantity in square brackets above is $\leqslant$

$$C_1\prod_{j=l+1}^{n}\|g_j\|_l^l.$$

Thus (8) is $\leqslant$

$$C\prod_{j=1}^{n}\|g_j\|_l.$$

This completes the proof of the theorem.

§ 2. Applications of Theorem 1. Our first result is proved in the same setting as Theorem 1—that of a locally compact abelian group G. The letter m will denote the Haar measure on G.

Proposition 2. *Suppose that $K_1, \ldots, K_n$ are subsets of G and suppose that each K_j carries a nonzero finite positive Borel measure λ_j. If the measures $\lambda_1, \ldots, \lambda_n$ satisfy the hypothesis of Theorem 1 for some $l = 1, \ldots, n-1$, then there is $\delta > 0$ such that the following is true: if $E \subseteq G$ and each K_j+E are m-measurable, then*

(9) $$\delta^l m(E)^l \leqslant \prod_{j=1}^{n} m(K_j+E).$$

Thus if $K = \bigcup_{j=1}^{n} K_j$, then

(10) $$\delta^{l/n} m(E)^{l/n} \leqslant m(K+E).$$

Proof. In Theorem 1 take g_j to be the indicator function of the set $-E-K_j = \{-e-x_j\colon e\in E,\ x_j\in K_j\}$. If $\|\lambda_j\|$ denotes the total mass of λ_j, then for $x\in -E$ we have

$$\int g_j(x-x_j)\,d\lambda_j(x_j) = \|\lambda_j\|.$$

Thus

$$m(-E)\prod_{j=1}^{n}\|\lambda_j\| \leqslant \int\int g_1(x-x_1)\,d\lambda_1(x_1)\ldots\int g_n(x-x_n)\,d\lambda_n(x_n)\,dx$$
$$\leqslant C\prod_{j=1}^{n}\|g_j\|_l = C\prod_{j=1}^{n} m(-E-K_j)^{1/l},$$

where the second inequality is the conclusion of Theorem 1. This gives (9) with $\delta = \big(\prod_{j=1}^{n}\|\lambda_j\|\big)/C$.

We would like to apply Proposition 2 to produce inequalities of the form (1) when K is a k-dimensional surface in $\boldsymbol{R}^n$. A little consideration of low-dimensional cases leads one to believe that the right choice of β here is $1-(k/n)$. But a little more consideration leads to the realization that for some k-dimensional surfaces K no inequality (1) can hold for any $\beta < 1$. The problem here is geometric—a lack of curvature: if K is a bounded subset of some $(n-1)$-dimensional affine hyperplane, then it is easy to choose small sets E such that $m_n(E)$ and $m_n(K+E)$ are comparable. The extra hypothesis on K in the statement of Theorem 3 is present to eliminate this kind of degeneracy. Thus if K is a curve in $\boldsymbol{R}^n$ our hypothesis states that

$$\{x_1+x_2+\ldots+x_{n-1}-x_n\colon x_j\in K\}$$

has positive measure in $\boldsymbol{R}^n$. Our definition of a k-dimensional surface in $\boldsymbol{R}^n$ is the range of a continuously differentiable map $\varphi\colon (0, 1)^k \to \boldsymbol{R}^n$.

Theorem 3. *Suppose that $1 \leqslant k < n$ and that K is a k-dimensional surface in $\boldsymbol{R}^n$. Let $l = n-k$ and suppose that the set*

$$\{(x_1+x_2+\ldots+x_l-x_{l+1},\ x_1+\ldots+x_l-x_{l+2},\ \ldots,\ x_1+\ldots+x_l-x_n)\colon x_j\in K\}$$

has positive nk-dimensional Lebesgue measure in $(\boldsymbol{R}^n)^k$. Then there is $\delta > 0$ such that

(11) $$\delta m_n(E)^{1-(k/n)} \leqslant m_n(K+E)$$

for any measurable $E \subseteq \boldsymbol{R}^n$ such that $K+E$ is measurable.

Proof. By regularity of m_n, (11) is true for any measurable E such that $K+E$ is measurable if it is true for any E which is compact. We will prove (11) for compact E by applying (10) of Proposition 2. And to apply Proposition 2 we will produce nonzero measures $\lambda_1, \ldots, \lambda_n$ satisfying the hypotheses of Theorem 1 (with $l = n-k$) and supported on σ-compact subsets $K_1, \ldots, K_n$ of K.

Suppose $K = \varphi((0, 1)^k)$ where $\varphi\colon (0, 1)^k \to \boldsymbol{R}^n$ is continuously differentiable. Write $I = (0, 1)^k$ and consider the map $\Phi\colon I^n \to (\boldsymbol{R}^n)^k$ defined for $t = (t_1, \ldots, t_n)\in I^n$ by

$$\Phi(t) = \Phi(t_1, \ldots, t_n) = \big(\varphi(t_1)+\varphi(t_2)+\ldots+\varphi(t_l)-$$
$$-\varphi(t_{l+1}),\ \varphi(t_1)+\ldots+\varphi(t_l)-\varphi(t_{l+2}),\ \ldots,\ \varphi(t_1)+\ldots+\varphi(t_l)-\varphi(t_n)\big).$$

The hypothesis is that $m_{nk}(\Phi(I^n)) > 0$. It follows from Sard's theorem, continuity, and the inverse function theorem that there exist $\delta > 0$ and, for

$1 \leqslant j \leqslant n$, nonempty open sets $I_j \subseteq I$ such that

$$|\det \Phi'(t_1, \ldots, t_n)| \geqslant \delta \quad \text{if } t_j \in I_j,\ 1 \leqslant j \leqslant n,$$

and such that Φ is one-to-one on $\prod_{j=1}^{n} I_j \subseteq I^n$. For $1 \leqslant j \leqslant n$ define the measure λ_j on $\boldsymbol{R}^n$ by

$$\int g(x)\, d\lambda_j(x) = \int_{I_j} g(\varphi(t))\, dm_k(t).$$

Then each λ_j is a nonzero finite positive Borel measure supported on a σ-compact set $K_j = \varphi(I_j) \subseteq K$. This theorem will be proved when we show that the measures $\lambda_1, \ldots, \lambda_n$ satisfy the hypothesis (3) of Theorem 1. So suppose $f \in L^1([\boldsymbol{R}^n]^k)$ is nonnegative. Then

$$\int \ldots \int f(x_1 + \ldots + x_l - x_{l+1}, \ldots, x_1 + \ldots + x_l - x_n)\, d\lambda_1(x_1) \ldots d\lambda_n(x_n)$$
$$= \int_{I_1} \ldots \int_{I_n} f(\varphi(t_1) + \ldots + \varphi(t_l) - \varphi(t_{l+1}), \ldots, \varphi(t_1) + \ldots + \varphi(t_l) - \varphi(t_n))$$
$$dm_k(t_1) \ldots dm_k(t_n) = \int_{\prod_{j=1}^{n} I_j} f(\Phi(t))\, dm_{nk}(t)$$
$$\leqslant \delta^{-1} \int_{\prod_{j=1}^{n} I_j} f(\Phi(t))\, |\det \Phi'(t)|\, dm_{nk}(t) = \delta^{-1} \int_{\Phi(\prod_{j=1}^{n} I_j)} f(y)\, dm_{nk}(y).$$

This is enough to establish (3).

§ 3. A generalization of Theorem 3 when $l = 1$. The purpose of this section is to prove the following proposition. Once again G is a locally compact abelian group with Haar measure m.

PROPOSITION 4. *Suppose that $K_1, \ldots, K_n$ are compact subsets of G and suppose that the set*

$$\{(x_1 - x_2, x_1 - x_3, \ldots, x_1 - x_n):\ x_j \in K_j\}$$

has positive Haar measure in G^{n-1}. Then there is $\delta > 0$ such that

$$\delta m(E) \leqslant \prod_{j=1}^{n} m(K_j + E)$$

for any m-measurable $E \subseteq G$ such that each $K_j + E$ is m-measurable. Thus if $K = \bigcup_{j=1}^{n} K_j$, then

$$\delta^{1/n} m(E)^{1/n} \leqslant m(K + E).$$

Proof. The proof is somewhat analogous to the proof of Theorem 3. We will produce a nonzero positive Borel measure μ on $\prod_{j=1}^{n} K_j \subseteq G^n$ such that the inequality

$$(12) \qquad \Big|\int\int_{G^n} g_1(x - x_1) \ldots g_n(x - x_n)\, d\mu(x_1, \ldots, x_n)\, dx\Big| \leqslant \prod_{j=1}^{n} \|g_j\|_1$$

holds for $g_1, \ldots, g_n \in L^1(G)$. (Note that this is (2) for $p = C = 1$.) The proposition will then follow by taking g_j to be the indicator function of $-E - K_j$ as in the proof of Proposition 2.

Making the change of variable $x \to x + x_1$ shows that (12) is equivalent to the inequality

$$\Big|\int g_1(x) \int_{\prod_{j=1}^{n} K_j} g_2(x + x_1 - x_2)\, g_3(x + x_1 - x_3) \ldots$$
$$\ldots g_n(x + x_1 - x_n)\, d\mu(x_1, \ldots, x_n)\, dx\Big| \leqslant \prod_{j=1}^{n} \|g_j\|_1.$$

This inequality will hold if we can choose μ so that

$$\int_{\prod_{j=1}^{n} K_j} f(x_1 - x_2, x_1 - x_3, \ldots, x_1 - x_n)\, d\mu(x_1, \ldots, x_n)$$

is equal to the integral of f over $\{(x_1 - x_2, x_1 - x_3, \ldots, x_1 - x_n):\ x_j \in K_j\}$ with respect to the product Haar measure on G^{n-1} for $f \in L^1(G^{n-1})$. That μ can be so chosen is a consequence of the following simple lemma.

LEMMA. *Suppose that X_1 and X_2 are compact Hausdorff spaces and that $h: X_1 \to X_2$ is a continuous mapping of X_1 onto X_2. If μ_2 is a finite Borel measure on X_2, then there is a finite Borel measure μ_1 on X_1 such that $\mu_2 = \mu_1 \circ h$ in the sense that*

$$\int_{X_2} g\, d\mu_2 = \int_{X_1} g \circ h\, d\mu_1$$

for continuous functions g on X_2.

Proof of the lemma. The lemma is certainly true if μ_2 is a discrete measure. In general, let $\{\mu_2^\alpha\}_{\alpha \in \mathscr{A}}$ be a net of discrete measures each of which has total variation $\leqslant$ that of μ_2 and such that

$$\int g\, d\mu_2^\alpha \xrightarrow{\alpha} \int g\, d\mu_2$$

for continuous g on X_2. For each α let μ_1^α be such that $\mu_2^\alpha = \mu_1^\alpha \circ h$ and such that the total variation of μ_1^α is equal to that of μ_2^α. Then let μ_1 be a weak* limit of some subnet of $\{\mu_1^\alpha\}_{\alpha\in\mathscr{A}}$ in the Banach space of finite Borel measures on X_1 regarded as the dual space of the space of continuous functions on X_1.

References

[1] J. H. B. Kemperman, *On small sumsets in an abelian group*, Acta Math. 103 (1960), 63–88.

[2] M. Kneser, *Summenmengen in lokalkompakten abelschen Gruppen*, Math. Z. 66 (1956), 88–110.

[3] A. M. MacBeath, *On the measure of product sets in a topological group*, J. London Math. Soc. 35 (1960), 403–407.

[4] H. B. Mann, *Addition Theorems: The Addition Theorems of Group Theory and Number Theory*, Wiley-Interscience Tracts in Mathematics 18, New York 1975.

[5] D. Oberlin, *Sums with curves*, J. Math. Anal. Appl. 104 (1984), 477–480.

FLORIDA STATE UNIVERSITY
DEPARTMENT OF MATHEMATICS
Tallahassee, Florida 32306, U.S.A.

Received October 25, 1984 (2006)

STUDIA MATHEMATICA, T. LXXXIII. (1986)

Holomorphic functions of uniformly bounded type on nuclear Fréchet spaces

by

REINHOLD MEISE (Düsseldorf) and DIETMAR VOGT (Wuppertal)

Dedicated to Professor Dr. H.-G. Tillmann on his sixtieth birthday

Abstract. It is studied under what conditions every entire function on a given nuclear Fréchet space E (resp. every holomorphic function on an open polycylindrical set $P \subset E$) is of uniformly bounded type. Necessary as well as sufficient conditions (resp. a characterization) are given in terms of the invariants (LB^∞), $(\tilde{\Omega})$, $(\bar{\Omega})$ known from the theory of linear operators between Fréchet spaces. A holomorphic characterization of nuclear Fréchet spaces with $(\tilde{\Omega})$ is presented and also examples and applications.

For a complex locally convex space E we denote by $H(E)$ the vector space of all entire functions on E, i.e. of all continuous complex functions on E which are Gâteaux-analytic. An entire function on E is called of uniformly bounded type if it is bounded on all multiples of some zero neighbourhood in E. By $H_{\mathrm{ub}}(E)$ we denote the linear space of all entire functions on E which are of uniformly bounded type. Colombeau and Mujica [4] have shown $H(E) = H_{\mathrm{ub}}(E)$ for each (DFM)-space E, while a classical example of Nachbin [16] gives $H_{\mathrm{ub}}(E) \subsetneqq H(E)$ for the nuclear Fréchet space $E = H(\mathbf{C})$. In [14] we have shown that a nuclear locally convex space E satisfies $H(E) = H_{\mathrm{ub}}(E)$ if and only if the entire functions on E are universally extendable in the following sense: Whenever E is a topological linear subspace of a locally convex space F with a fundamental system of continuous semi-norms induced by semi-inner products, then each $f \in H(E)$ has a holomorphic extension to F.

In the present article we investigate necessary as well as sufficient conditions for nuclear Fréchet spaces E to satisfy the relation $H(E) = H_{\mathrm{ub}}(E)$. We prove that this relation defines a subclass which contains all spaces with property $(\tilde{\Omega})$ and which is contained in the subclass of spaces with property (LB^∞). The properties $(\tilde{\Omega})$ and (LB^∞) have been introduced and investigated in Vogt [25], where it has been shown that $(\tilde{\Omega})$ is strictly stronger than (LB^∞). It remains open whether the relation $H(E) = H_{\mathrm{ub}}(E)$ defines a new linear topological invariant which is inherited by quotient spaces or equals one of the invariants (LB^∞) or $(\tilde{\Omega})$.