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Infinitely divisible probability measures and the converse Kolmogorov inequality in Banach spaces

by

ALEJANDRO de ACOSTA* (Caracas) and JORGE D. SAMUR** (La Plata)

Abstract. It is proved that every infinitely divisible probability measure on a separable Banach space E has a representation of Khintchine's form if and only if E is a space of cotype 2. This includes in particular results for the L^p spaces, $1 < p < 2$. The converse Kolmogorov inequality is generalized to the case of random vectors.

1. Introduction. Khintchine's form of the Lévy–Khintchine canonical representation of the characteristic function of an infinitely divisible distribution in R^1 associates to every such distribution μ a unique triple (x_0, a, ν), where x_0 is a real number, a is a non-negative real number and ν is a finite non-negative measure satisfying $\nu(\{0\}) = 0$ such that

$$(1)\qquad \hat{\mu}(y) = \exp\Big[ix_0 y - a(y^2/2) + \int K(x, y)\,\nu(dx)\Big] \quad \text{for every } y \in R^1,$$

where

$$K(x, y) = \frac{1+x^2}{x^2}\Big[\exp(ixy) - 1 - \frac{ixy}{1+x^2}\Big] \quad \text{for } x \neq 0,\ y \in R^1$$

(it is irrelevant how $K(x, y)$ is defined for $x = 0$).

In Section 4 of this paper (Theorem 4.2) we prove that if E is a separable Banach space of cotype 2, then every infinitely divisible probability measure on E has a unique representation which is the natural generalization of (1); and, conversely, if a given separable Banach space is such that every infinitely divisible probability measure has a representation of the form (1) (suitably generalized), then the space is of cotype 2.

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* Part of this research was carried out while the first-named author was at the Departamento de Matemáticas, Facultad de Ciencias Exactas y Naturales, Universidad Nacional de Buenos Aires, Argentina.

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In broad outlines our proof is patterned after Khintchine's proof for the one-dimensional case ([8], p. 76); the crucial step is the construction of the measure ν. However, the infinite-dimensional situation presents some serious difficulties, centered around proving the relative weak compactness of certain sets of measures (Theorem 4.1). The basic results which make it possible to handle the weak compactness problems are proved in Sections 2 and 3. Section 2 contains some general results on infinitely divisible probability measures on a separable Banach space. In Section 3 we generalize an important inequality of Kolmogorov (see e.g. [4], p. 117) to the case of Banach space-valued random vectors; this result is of independent interest.

The converse part of the representation theorem (Part (2) of Theorem 4.2) follows easily from results of Maurey and Pisier [15].

We will now make some remarks on the connection of our work with other results in the literature. Theorem 4.2 of the present paper contains Varadhan's representation theorem in the Hilbert space case ([19]; see also [16], Chapter 6, Section 4); our techniques are different from Varadhan's in several important points. Very recently, Dettweiler ([6], Satz 2.5) has obtained a weak form of the Lévy–Khintchine formula valid for infinitely divisible measures on locally convex spaces(1); in this result the information on the "generalized Poisson" part is less specific than in our Theorem 4.2 (as should be expected, in view of part (2) of Theorem 4.2). Satz 2.5 of [6] is based on the decomposition of an infinitely divisible measure as the convolution of a Gaussian measure and a "generalized Poisson" measure (Satz 1.9 of [6]), essentially due to Tortrat ([18] and [18a]). Our work is completely independent of these results; we remark in passing that we do not even use the one-dimensional Lévy–Khintchine formula. More or less simultaneously with the final formulation of the results of the present paper, Araujo and Giné [2] have obtained a representation formula equivalent to Theorem 4.2. Their approach centers around the problem of the integrability of Lévy measures (previously studied by Araujo in [1]). The method of proof of the representation formula in [2] is based on Tortrat's decomposition and is technically different from our work, although it uses in an essential way Theorem 3.1 of the present paper.

We refer to [3] and [16] for definitions and results on weak convergence and compactness, tightness and characteristic functionals of probability measures on Banach spaces. All measures will be defined on the Borel σ-algebra of a separable Banach space E; for $x \in E$, δ_x will denote the probability measure supported by $\{x\}$. If X is an E-valued random vector, its distribution will be denoted by $\mathscr{L}(X)$. The symbol N will denote the set of natural numbers.

(1) A similar result in the case of separable Banach spaces appears in A. de Araujo's doctoral thesis, Department of Statistics, University of California at Berkeley, 1974. (See also J. Multivariate Anal. 8, 4 (1978), pp. 598–613.)

2. Infinitely divisible measures in Banach spaces. Let us recall that a probability measure μ on a separable Banach space E is *infinitely divisible* if for every $n \in N$, there exists a probability measure μ_n on E such that $\mu_n^n = \mu$ (ν^n denotes the nth convolution power of a finite measure ν).

LEMMA 2.1. *Let μ be an infinitely divisible probability measure on a separable Banach space E. Then*

(a) $\hat{\mu}(y) \neq 0$ *for all* $y \in E'$,

(b) *There exists a unique function* $\Phi\colon E' \to C$ *such that*

(b_1) $\Phi(0) = 0$,

(b_2) Φ *is sequentially w^*-continuous,*

(b_3) $\hat{\mu} = \exp \Phi$.

Proof. (a) Let μ_n be such that $\mu_n^n = \mu$, and let $\nu_n = \mu_n * \bar{\mu}_n$, $\nu = \mu * \bar{\mu}$. Then $\hat{\nu}_n \geqslant 0$, $\hat{\nu} \geqslant 0$ and $\hat{\nu}_n = \hat{\nu}^{1/n}$. Using an elementary inequality,

$$[1-\hat{\nu}_n(y_1+y_2)] \leqslant 2[1-\hat{\nu}_n(y_1)]+2[1-\hat{\nu}_n(y_2)]$$

or

$$[1-\hat{\nu}^{1/n}(y_1+y_2)] \leqslant 2[1-\hat{\nu}^{1/n}(y_1)]+2[1-\hat{\nu}^{1/n}(y_2)]$$

for $y_1, y_2 \in E'$. Since $\hat{\nu}(y) > 0$ implies $\lim\limits_n \hat{\nu}^{1/n}(y) = 1$, it follows that $G = \{y\colon \hat{\mu}(y) \neq 0\} = \{y\colon \hat{\nu}(y) > 0\}$ is a norm-open subgroup of E'; since $0 \in G$, it follows that $G = E'$.

(b) Fix $y \in E'$, and define $\varrho_y\colon R^1 \to C$ by $\varrho_y(t) = \hat{\mu}(ty)$ $(t \in R^1)$. By [4], p. 241, there exists a unique function $\lambda_y\colon R^1 \to C$ such that $\lambda_y(0) = 0$, λ_y is continuous and $\varrho_y = \exp(\lambda_y)$. Define $\Phi(y) = \lambda_y(1)$. Then $\Phi(0) = 0$ and $\hat{\mu}(y) = \varrho_y(1) = \exp[\Phi(y)]$ $(y \in E')$.

We show now that Φ is sequentially w^*-continuous. In fact, $y_n \xrightarrow{w^*} y$ implies $\mu \circ y_n^{-1} \xrightarrow{w} \mu \circ y^{-1}$, hence $\varrho_{y_n}(t) = \hat{\mu}(ty_n) = (\mu \circ y_n^{-1})\hat{}\,(t)$ converges to $(\mu \circ y^{-1})\hat{}\,(t) = \hat{\mu}(ty) = \varrho_y(t)$ uniformly on compact intervals of R^1. By [4], p. 242, it follows that $\Phi(y_n) = \lambda_{y_n}(1)$ converges to $\lambda_y(1) = \Phi(y)$.

To prove uniqueness, let Φ_1, Φ_2 be two functions satisfying (b_1), (b_2), (b_3). Then for all $y \in E'$, $\exp[\Phi_1(y)] = \exp[\Phi_2(y)]$, which implies $\Phi_1(y) - \Phi_2(y) = (2\pi i)k(y)$, with $k(y)$ an integer. The function k is norm-continuous and $k(0) = 0$; hence $k(y) = 0$ for all $y \in E'$. ■

Let us remark that in part (b) of Lemma 2.1 we have only used the fact that $\hat{\mu}$ does not vanish (the infinite divisibility of μ is not otherwise relevant).

COROLLARY 2.1. *Let μ be an infinitely divisible probability measure on a separable Banach space. Then for each $n \in N$, μ has a unique n-th convolution root.*

Proof. Assume $\mu_n^n = \mu$; then $\hat{\mu}_n^n = \hat{\mu}$. Since neither $\hat{\mu}$ nor $\hat{\mu}_n$ vanish, there exist unique functions Φ, Φ_n satisfying (b_1) and (b_2) of Lemma 2.1, and such that $\hat{\mu} = \exp\Phi$, $\hat{\mu}_n = \exp\Phi_n$. It follows that $n\Phi_n = \Phi$; hence $\hat{\mu}_n = \exp[\Phi/n]$. This proves the uniqueness of μ_n. ■

We recall the following result, to be used in Lemma 2.2: if $\{\mu_n: n \in N\}$ is a sequence of probability measures on a separable Banach space E such that (1) $\{\mu_n: n \in N\}$ is relatively shift-compact and (2) $\hat{\mu}_n$ converges uniformly on the balls of E' to a function g, then there exists a probability measure μ on E such that $\mu_n \xrightarrow{w} \mu$ and $\hat{\mu} = g$. (This is proved as Theorem 4.5, Chapter 6 of [16].)

LEMMA 2.2. *Let μ be an infinitely divisible probability measure on a separable Banach space, and let μ_n be its n-th convolution root ($n \in N$). If p is a positive integer, then $\{\mu_n^k: n \in N, k \in N, k \leqslant pn\}$ is relatively compact for the weak topology.*

Proof. Let $\mathscr{K} = \{\mu_n^k: n \in N, k \in N, k \leqslant pn\}$. Since $\mu_n^k * \mu_n^{(pn-k)} = \mu_n^{pn} = (\mu_n^n)^p = \mu^p$, it follows from [16] (Theorem 2.2, Chapter 3) that $\mathscr{K}$ is relatively shift-compact. Let $\{\mu_{n_j}^{k_j}: j \in N\}$ be a sequence in $\mathscr{K}$. Since $(k_j/n_j) \leqslant p$, there exists a subsequence $(k_{j'}/n_{j'})$ which converges to a real number $t \in [0, p]$. Since $(\mu_n^k)^\wedge = \exp[(k/n)\Phi]$ (where Φ is defined as in Lemma 2.1) and Φ is bounded over the balls of E', it easily follows that $(\mu_{n_{j'}}^{k_{j'}})^\wedge$ converges to $\exp[t\Phi]$ uniformly on the balls of E'. By the result quoted above, there exists a probability measure ν such that $\hat{\nu} = \exp[t\Phi]$ and $\mu_{n_{j'}}^{k_{j'}}$ converges weakly to ν. This proves that $\mathscr{K}$ is relatively weakly compact. ■

Let us recall that if ν is a finite signed measure on a separable Banach space, then the *exponential of* ν is defined by $\exp\nu = \sum_{n=0}^{\infty} \frac{\nu^n}{n!}$, where $\nu^0 = \delta_0$; the series converges in the total variation norm.

THEOREM 2.1. *Let μ be an infinitely divisible probability measure on a separable Banach space, and let μ_n be its n-th convolution root ($n \in N$). Then*

(a) $\mu_n \xrightarrow{w} \delta_0$ $(n \to \infty)$,

(b) $\exp[n(\mu_n - \delta_0)] \xrightarrow{w} \mu$ $(n \to \infty)$.

Proof. (a) Using the expression $\hat{\mu}_n = \exp[\Phi/n]$ (where Φ is as in Lemma 2.1), it is readily proved that $\hat{\mu}_n \to 1$ $(n \to \infty)$ uniformly over the balls of E'. Since $\mu_n^n = \mu$, it follows from [16] (Theorem 2.2, Chapter 3) that $\{\mu_n: n \in N\}$ is relatively shift-compact. By the result quoted before Lemma 2.2, we conclude that $\mu_n \xrightarrow{w} \delta_0$.

(b) Let $\lambda_n = \exp[n(\mu_n - \delta_0)]$. We will prove

(I) $\hat{\lambda}_n(y) \to \hat{\mu}(y)$ $(n \to \infty)$ for each $y \in E'$,

(II) $\{\lambda_n: n \in N\}$ is tight.

It follows that $\lambda_n \xrightarrow{w} \mu$ $(n \to \infty)$.

To prove (I), observe first that $\hat{\lambda}_n = \exp[n(\hat{\mu}_n - 1)] = \exp[n \times (\exp[\Phi/n] - 1)]$.

We have, for $y \in E'$

$$|n(\exp[\Phi(y)/n] - 1) - \Phi(y)| = \left| n \sum_{k=2}^{\infty} \frac{[\Phi(y)/n]^k}{k!} \right|$$
$$\leqslant n \cdot n^{-2} \sum_{k=2}^{\infty} \frac{|\Phi(y)|^k}{k!} \leqslant n^{-1} \exp(|\Phi(y)|).$$

Therefore, $\hat{\lambda}_n(y) \to \exp[\Phi(y)] = \hat{\mu}(y)$ for each $y \in E'$ (in fact, the convergence is uniform on the balls of E').

For the proof of (II) we shall need the following elementary fact: $\lim_{n\to\infty} \exp(-n) \sum_{k=0}^{2n} \frac{n^k}{k!} = 1$. To prove it, let $\{\xi_j: j \in N\}$ be independent random variables with $\mathscr{L}(\xi_j) =$ Poisson with parameter 1 ($j \in N$), $S_n = \sum_{j=1}^{n} \xi_j$. It is a consequence of the weak law of large numbers that for any $x > 1$, $P[S_n/n \leqslant x] \to 1$ $(n \to \infty)$. But $\mathscr{L}(S_n) =$ Poisson with parameter n. Therefore, as $n \to \infty$

$$\exp(-n) \sum_{k=0}^{2n} \frac{n^k}{k!} = P[S_n \leqslant 2n] \to 1.$$

By Lemma 2.2, for any $\eta < 1$ there exists a compact set $K \subset E$ such that $\mu_n^k(K) > \eta$ for any pair (n, k) of positive integers such that $k \leqslant 2n$. Choose n_0 so that $n \geqslant n_0$ implies $\exp(-n) \sum_{k=0}^{2n} \frac{n^k}{k!} > \eta$. Then for $n \geqslant n_0$

$$\lambda_n(K) = \exp(-n) \sum_{k=0}^{\infty} \frac{n^k}{k!} \mu_n^k(K)$$
$$\geqslant \exp(-n) \sum_{k=0}^{2n} \frac{n^k}{k!} \mu_n^k(K) > \eta^2.$$

The proof that $\{\lambda_n: n \in N\}$ is tight is completed by a standard argument. ■

Remark 2.1. We shall not use in the sequel the full strength of Theorem 2.1 (b). We include this result because of its obvious interest: it provides a canonical way of expressing an arbitrary infinitely divisible probability measure as the weak limit of a sequence of measures of exponential type.

The following result[2] (which will not be used in this paper) may be proved in a similar way.

THEOREM 2.2. *Let $\{\mu_n: n \in N\}$ be a sequence of probability measures on a separable Banach space. Let $\{k_n: n \in N\}$ be a sequence of positive integers such that $k_n \to \infty$. Suppose $\{\mu_n^{k_n}: n \in N\}$ is relatively compact. Then*

(a) $\mu_n \xrightarrow{w} \delta_0$ $(n \to \infty)$,

(b) *If* μ_n *is symmetric* $(n \in N)$, *then* $\{\exp[k_n(\mu_n - \delta_0)]: n \in N\}$ *is relatively compact.*

Remark 2.2. Let $\{\mu_{nj}: n \in N, j = 1, \ldots, k_n\}$ be a triangular array of probability measures on E. It is well known that the relation between weak compactness and weak convergence properties of $\{\prod_j \mu_{nj}: n \in N\}$ and the corresponding properties of $\{\exp[\sum_j (\mu_{nj} - \delta_0)]: n \in N\}$ plays a key role in the general central limit problem for triangular arrays; this fact has been emphasized in [14]. Theorem 2.2 (which deals with the case of identically distributed rows) is of some interest in view of a counterexample of Le Cam ([14], p. 240) for general triangular arrays.

Our next proposition generalizes an inequality in Feller's book ([7], p. 149) and is essential for the proof of some weak compactness results (Theorems 2.3 and 5.1). Let us recall that a *generalized seminorm* q on a real vector space E is a function $q: E \to [0, \infty]$ such that $q(x+y) \leqslant q(x)+q(y)$, $q(\lambda x) = |\lambda| q(x)$ $(x \in E, y \in E, \lambda \in R^1)$.

LEMMA 2.3. *Let E be a separable Banach space, and suppose q is a measurable generalized seminorm on E. Let $\{X_j: j = 1, \ldots, n\}$ be independent symmetric E-valued random vectors, $S_n = \sum_{j=1}^n X_j$. Then for every $t > 0$ such that $P[q(S_n) > t] < (1/2)$,*

$$\sum_{j=1}^n P[q(X_j) > t] \leqslant -\log\big(1 - 2P[q(S_n) > t]\big).$$

Proof. Let $A_k = [q(X_j) \leqslant t,\ j = 1, \ldots, k-1;\ q(X_k) > t]$, $A = [\sup_{1\leqslant k\leqslant n} q(X_k) > t]$. Then $\{A_k: k = 1, \ldots, n\}$ is disjoint and $A = \bigcup_k A_k$. Fix k, and let $Z = 2X_k - S_n$. Then $[q(S_n) \leqslant t] \cap [q(Z) \leqslant t] \subset [q(X_k) \leqslant t]$, and consequently

(1) $$A_k = \big(A_k \cap [q(S_n) > t]\big) \cup \big(A_k \cap [q(Z) > t]\big).$$

Let $Y_j = -X_j$ for $j \neq k$, $= X_k$ for $j = k$. The symmetry and independence of $\{X_j: j = 1, \ldots, n\}$ implies: $\mathscr{L}(Y_1, \ldots, Y_n) = \mathscr{L}(X_1, \ldots, X_n)$. Observe that $A_k = [q(Y_j) \leqslant t,\ j = 1, \ldots, k-1;\ q(Y_k) > t]$ and $Z = \sum_{j=1}^n Y_j$; it follows that $P\big(A_k \cap [q(S_n) > t]\big) = P\big(A_k \cap [q(Z) > t]\big)$, and we obtain from (1)

$$P(A_k) \leqslant 2P\big(A_k \cap [q(S_n) > t]\big).$$

Adding over k,

(2) $$P(A) \leqslant 2P[q(S_n) > t].$$

Now

$$\begin{aligned} P(A) &= 1 - P[\sup_k q(X_k) \leqslant t] \\ &= 1 - \prod_k P[q(X_k) \leqslant t] \\ &\geqslant 1 - \exp\Big(-\sum_k P[q(X_k) > t]\Big) \end{aligned}$$

and therefore

(3) $$\sum_{k=1}^n P[q(X_k) > t] \leqslant -\log\big(1 - P(A)\big).$$

The desired inequality follows from (2) and (3). ∎

THEOREM 2.3. *Let μ be an infinitely divisible probability measure on a separable Banach space, and let μ_n be its n-th convolution root. Then there exists a compact, convex, symmetric set K such that $\{n\mu_n(K^c \cap (\cdot)): n \in N\}$ is relatively compact.*

Proof. Let $\nu = \mu * \bar{\mu}$, $\nu_n = \mu_n * \bar{\mu}_n$ $(n \in N)$. For each $n \in N$, let $\{X_{nj}: j = 1, \ldots, n\}$ and $\{X'_{nj}: j = 1, \ldots, n\}$ be each an independent set of E-valued random vectors with $\mathscr{L}(X_{nj}) = \mathscr{L}(X'_{nj}) = \mu_n$; assume also that the pair of sets $\{X_{nj}: j = 1, \ldots, n\}$ and $\{X'_{nj}: j = 1, \ldots, n\}$ is independent. Let $S_n = \sum_j X_{nj}$, $S'_n = \sum_j X'_{nj}$; then $\mathscr{L}(S_n) = \mathscr{L}(S'_n) = \mu$, $\mathscr{L}(X_{nj} - X'_{nj}) = \nu_n$ $(j = 1, \ldots, n)$ and $\mathscr{L}(S_n - S'_n) = \nu$.

Let C be a compact, convex, symmetric set such that $\mu_n(C) \geqslant (1/2)$ for all $n \in N$ (recall that $\{\mu_n: n \in N\}$ is tight by Theorem 2.1 (a)). Let D be a compact, convex, symmetric set such that $\nu(D^c) < 1/4$, and define $K = C + D$. Define $\lambda_n = n\mu_n\big(K^c \cap (\cdot)\big)$ $(n \in N)$; we will prove that $\{\lambda_n: n \in N\}$ is relatively compact.

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[2] A. de Araujo communicated to us that he was independently aware of this fact.

By the independence of X_{nj} and X'_{nj}

$$\begin{aligned}(1/2)P[X_{nj} \notin K] &\leqslant P[X'_{nj} \in C]P[X_{nj} \notin K] \\ &= P[X'_{nj} \in C, X_{nj} \notin K] \\ &\leqslant P[X_{nj} - X'_{nj} \notin D].\end{aligned}$$

Let q be the Minkowski functional of D. Applying Lemma 2.3, we have for all $n \in N$

$$\begin{aligned}\|\lambda_n\| = n\mu_n(K^c) &= \sum_{j=1}^{n} P[X_{nj} \notin K] \leqslant 2\sum_{j=1}^{n} P[X_{nj} - X'_{nj} \notin D] \\ &= 2\sum_{j=1}^{n} P[q(X_{nj} - X'_{nj}) > 1] \leqslant -2\log\big(1 - 2P[q(S_n - S'_n) > 1]\big) \\ &= -2\log\big(1 - 2\nu(D^c)\big) < 2\log 2.\end{aligned}$$

We have shown that $\sup_n \|\lambda_n\| < \infty$. To prove the tightness of $\{\lambda_n : n \in N\}$, let $\varepsilon > 0$ be given, and let Q be a compact convex symmetric set such that $\nu(Q^c) < (1/2)\big(1 - \exp\big(-(\varepsilon/2)\big)\big)$. Proceeding as above, we obtain, for all $n \in N$

$$\begin{aligned}\lambda_n\big((Q+C)^c\big) &= n\mu_n\big(K^c \cap (Q+C)^c\big) \\ &\leqslant n\mu_n\big((Q+C)^c\big) < \varepsilon.\end{aligned}$$

Remark 2.3. There are other methods of proving Theorem 2.3. One way is to apply results of Le Cam [14], besed on a concentration inequality. A modification of the proof of Theorem 4.3, Chapter 4 of [16], combined with Theorem 2.1 (b) of the present work, also yields an alternative proof. Our approach is a generalization of a method of Feller ([7], p. 309); for the present purposes, it seems to be simpler and more direct than either alternative we have mentioned.

3. The converse Kolmogorov inequality for Banach space valued random vectors. Let E be a separable Banach space, $1 \leqslant p < \infty$. Let $L^p(E)$ be the vector space of (equivalence classes of) E-valued random vectors X such that $E\|X\|^p < \infty$, endowed with the norm $\|X\|_p = (E\|X\|^p)^{1/p}$.

The following inequality (Theorem 2.6 in [9]) is easily proved. It plays a crucial role in Theorem 3.1.

PROPOSITION 3.1. *Let E be a separable Banach space, $1 \leqslant p < \infty$. Let X, Y be independent E-valued random vectors, $X \in L^p(E)$, $Y \in L^p(E)$, $E(X) = 0$. Then*

$$E\|Y\|^p \leqslant E\|X+Y\|^p.$$

THEOREM 3.1. *Let E be a separable Banach space. Let $\{X_j : j = 1, \ldots, n\}$ be independent E-valued random vectors such that, for some $c \in [0, \infty)$, $\|X_j\| \leqslant c$ a.s. and $E(X_j) = 0$ $(j = 1, \ldots, n)$. Let $S_k = \sum_{j=1}^{k} X_j$ $(k = 1, \ldots, n)$, $p \geqslant 1$. Then for every $t > 0$*

$$(*) \qquad P[\sup_{1\leqslant k\leqslant n} \|S_k\| > t] \geqslant 2^{1-p}\left[1 - \frac{(t+c)^p + t^p(1-2^{1-p})}{E\|S_n\|^p}\right].$$

Proof. Let $A = [\sup_{1\leqslant k\leqslant n} \|S_k\| > t]$, $A_k = [\|S_j\| \leqslant t$ for $j = 1, \ldots, k-1;\ \|S_k\| > t]$. Then

$$\begin{aligned}E(\|S_n\|^p I_A) &= \sum_{k=1}^{n} E(\|S_n\|^p I_{A_k}) = \sum_{k=1}^{n} E\big(\|S_k + (S_n - S_k)\|^p I_{A_k}\big) \\ &\leqslant \sum_{k=1}^{n} E\big((\|S_k\| + \|S_n - S_k\|)^p I_{A_k}\big) \\ &\leqslant 2^{p-1}\sum_{k=1}^{n} E(\|S_k\|^p I_{A_k}) + 2^{p-1}\sum_{k=1}^{n} E(\|S_n - S_k\|^p I_{A_k}).\end{aligned}$$

Since $[\|S_{k-1}\| \leqslant t, \|X_k\| \leqslant c] \subset [\|S_k\| \leqslant t+c]$, it follows that

$$\sum_{k=1}^{n} E(\|S_k\|^p I_{A_k}) \leqslant (t+c)^p \sum_{k=1}^{n} P(A_k) = (t+c)^p P(A).$$

On the other hand, by the independence of I_{A_k} and $(S_n - S_k)$ and Proposition 3.1, we have

$$\begin{aligned}E(\|S_n - S_k\|^p I_{A_k}) &= (E\|S_n - S_k\|^p)\big(E(I_{A_k})\big) \\ &\leqslant E\|S_n\|^p \cdot P(A_k)\end{aligned}$$

and therefore

$$\sum_{k=1}^{n} E(\|S_n - S_k\|^p I_{A_k}) \leqslant E\|S_n\|^p P(A).$$

Thus

$$E(\|S_n\|^p I_A) \leqslant 2^{p-1}[(t+c)^p + E\|S_n\|^p]P(A).$$

Since obviously $E(\|S_n\|^p I_{A^c}) \leqslant t^p\big(1 - P(A)\big)$, we have

$$E\|S_n\|^p \leqslant t^p + [2^{p-1}(t+c)^p + 2^{p-1}E\|S_n\|^p - t^p]P(A).$$

Inequality $(*)$ follows now by elementary manipulations. ∎

COROLLARY 3.1. *Let E be a separable Banach space. Let $\{X_j : j = 1, \ldots, n\}$ be independent E-valued symmetric random vectors such that for some $c \in [0, \infty)$, $\|X_j\| \leqslant c$ a.s. $(j = 1, \ldots, n)$. Let $S_n = \sum_{j=1}^{n} X_j$, $p \geqslant 1$. Then for* *every* $t > 0$

$$(**)\qquad P[\|S_n\| > t] \geqslant 2^{-p}\left[1 - \frac{(t+c)^p + t^p(1-2^{1-p})}{E\|S_n\|^p}\right].$$

Proof. According to P. Lévy's inequality for Banach space valued symmetric random vectors ([13], p. 12), $P[\sup_{1\leqslant k\leqslant n} \|S_k\| > t] \leqslant 2P[\|S_n\| > t]$. Inequality $(**)$ follows at once from this fact and Theorem 3.1.

Remark. Let q be a continuous seminorm on a separable Banach space E. Then $(*)$ and $(**)$ hold for q instead of $\|\cdot\|$, with the same proof (using the appropriate versions of Proposition 3.1 and P. Lévy's inequality).

4. Infinitely divisible measures in spaces of cotype 2. Let $\{\varepsilon_j\colon j \in N\}$ be a Bernoulli sequence; that is, $\{\varepsilon_j\}$ is a sequence of independent random variables with $P[\varepsilon_j = 1] = P[\varepsilon_j = -1] = 1/2$. We recall that a Banach space E is of *cotype* 2 if there exists a constant $C > 0$ such that for every finite set $\{x_1, \ldots, x_n\} \subset E$,

$$\sum_{j=1}^{n} \|x_j\|^2 \leqslant CE\Big\|\sum_{j=1}^{n} \varepsilon_j x_j\Big\|^2.$$

(See [15].)

If E is a separable Banach space of cotype 2, there exists a constant $M > 0$ such that if $X_j \in L^2(E)$, $E(X_j) = 0$ $(j = 1, \ldots, n)$ and $\{X_j\colon j = 1, \ldots, n\}$ are independent, then

$$\sum_{j=1}^{n} E\|X_j\|^2 \leqslant ME\Big\|\sum_{j=1}^{n} X_j\Big\|^2.$$

(This is proved as Theorem 2.1 of [10].) It is known that the L^p spaces, $1 \leqslant p \leqslant 2$ are of cotype 2.

Let E be a separable Banach space of cotype 2. If the hypotheses of Theorem 3.1 are fulfilled, then inequality $(*)$ implies (for appropriate t)

$$(\mathrm{I})\qquad \sum_{j=1}^{n} E\|X_j\|^2 \leqslant M[(t+c)^2 + (t^2/2)](1 - 2P[\sup_{1\leqslant k\leqslant n} \|S_k\| > t])^{-1};$$

analogously, if the assumptions of Corollary 3.1 are fulfilled, then inequality $(**)$ implies (for appropriate t)

$$(\mathrm{II})\qquad \sum_{j=1}^{n} E\|X_j\|^2 \leqslant M[(t+c)^2 + (t^2/2)](1 - 4P[\|S_n\| > t])^{-1}.$$

THEOREM 4.1. *Let μ be an infinitely divisible probability measure on a separable Banach space of cotype 2, and let μ_n be its n-th convolution root. Define the measure ν_n by*

$$\nu_n(dx) = \frac{\|x\|^2}{1+\|x\|^2}\, n\mu_n(dx).$$

Then $\{\nu_n\colon n \in N\}$ is relatively compact.

Proof. We will show that $\sup_n \|\nu_n\| < \infty$ and $\{\nu_n\colon n \in N\}$ is tight. First we prove that for any $r \in [0, \infty)$, if $B_r = \{x \in E\colon \|x\| \leqslant r\}$,

$$(1)\qquad \sup_n n\int_{B_r} \|x\|^2 \mu_n(dx) < \infty.$$

The measures $\sigma_n = (1/2)(\mu_n + \bar{\mu}_n)$ are symmetric and the sequence $\{\sigma_n^n\colon n \in N\}$ is relatively compact. In fact, $\{\mu_n^k\colon k \leqslant n, k \in N, n \in N\}$ and $\{\bar{\mu}_n^k\colon k \leqslant n, k \in N, n \in N\}$ are relatively compact by Lemma 2.2. If K is a compact set such that $(\mu_n^k * \bar{\mu}_n^h)(K^c) < \varepsilon$ for all $n \in N$, $k \leqslant n$, $h \leqslant n$, then

$$\sigma_n^n(K^c) = 2^{-n}\sum_{k=0}^{n}\binom{n}{k}(\mu_n^k * \bar{\mu}_n^{n-k})(K^c) < \varepsilon 2^{-n}\sum_{k=0}^{n}\binom{n}{k} = \varepsilon.$$

For each $n \in N$, let Y_{nj} $(j = 1, \ldots, n)$ be independent E-valued random vectors with $\mathscr{L}(Y_{nj}) = \sigma_n$, $T_n = \sum_{j=1}^{n} Y_{nj}$. For $r > 0$, define $X_{nj}^{(r)} = Y_{nj} I_{B_r}(Y_{nj})$, $S_n^{(r)} = \sum_{j=1}^{n} X_{nj}^{(r)}$. Since $\{X_{nj}^{(r)}\colon j = 1, \ldots, n\}$ are independent and symmetric and $\|X_{nj}^{(r)}\| \leqslant r$ a.s., we obtain from inequality (II)

$$(2)\qquad n\int_{B_r} \|x\|^2 \mu_n(dx) = n\int_{B_r} \|x\|^2 \sigma_n(dx) = nE\|X_{n1}^{(r)}\|^2$$
$$= \sum_{j=1}^{n} E\|X_{nj}^{(r)}\|^2 \leqslant M[(t+r)^2 + (t^2/2)](1 - 4P[\|S_n^{(r)}\| > t])^{-1}$$

whenever $t > 0$ satisfies $P[\|S_n^{(r)}\| > t] < 1/4$.

Since $\mathscr{L}(T_n) = \sigma_n^n$, we may choose r arbitrarily large and such that for all $n \in N$, $P[\|T_n\| > r] < 1/24$. Then

$$P[\|S_n^{(r)}\| > r] \leqslant P[\|S_n^{(r)}\| > r \text{ and } \|Y_{nj}\| \leqslant r,\ j = 1, \ldots, n] +$$
$$+ P[\sup_{1\leqslant j\leqslant n} \|Y_{nj}\| > r] \leqslant 3P[\|T_n\| > r] < 1/8$$

by inequality (2) in the proof of Lemma 2.3. By puting $t = r$, we obtain from (2):

$$\sup_n n\int_{B_r} \|x\|^2 \mu_n(dx) \leqslant M[(2r)^2 + (r^2/2)]\cdot 2 = M(9r^2).$$

This proves claim (1).

By Theorem 2.3, there exists a compact (convex, symmetric) set such that $\{n\mu_n(K^c\cap(\cdot)): n\in N\}$ is relatively compact. Choose r so that $K\subset B_r$. Then

$$\|\nu_n\| = \int \frac{\|x\|^2}{1+\|x\|^2}\, n\mu_n(dx) \leqslant n\int_{B_r}\|x\|^2\mu_n(dx)+n\mu_n(B_r^c)$$

and consequently $\sup_n\|\nu_n\|<\infty$.

To prove the tightness of $\{\nu_n: n\in N\}$, let D be a compact set such that $n\mu_n(K^c\cap D^c)<\varepsilon$ for all $n\in N$. If $Q=K\cup D$, then for all $n\in N$

$$\nu_n(Q^c) = \int_{Q^c}\frac{\|x\|^2}{1+\|x\|^2}n\mu_n(dx) \leqslant n\mu_n(Q^c)<\varepsilon . \ \blacksquare$$

Remark. An alternative proof of claim (1) in the proof of Theorem 4.1 may be constructed by using Theorem 2.1 (b) and following the line of proof of Theorem 4.6, Chapter 6 of [16]. The crucial inequality (II) above must be used at the appropriate point.

The following lemma is well-known; it follows from Prohorov's extension theorem for cylinder measures ([3], Exposé No. 7) and the fact that on a separable Banach space every probability measure is a Radon measure.

LEMMA 4.1. *Let μ,ν,λ be cylinder measures on a separable Banach space and assume $\mu=\nu*\lambda$.*

(1) *If ν and λ are symmetric and μ is a measure, then ν and λ are measures.*

(2) *If μ and ν are measures, then λ is a measure.*

Let E be a separable Banach space. Define

$$K(x,y)=\begin{cases}\dfrac{1+\|x\|^2}{\|x\|^2}\left[\exp(i\langle x,y\rangle)-1-\dfrac{i\langle x,y\rangle}{1+\|x\|^2}\right], & x\neq 0,\ y\in E',\\ 0, & x=0,\ y\in E'.\end{cases}$$

For fixed $y\in E'$, $K(\cdot,y)$ is bounded; it is continuous on $E-\{0\}$. If $\dim E=1$ (to simplify, assume $E=R^1$), then for any $y\in E'$ $\lim\limits_{\substack{x\to 0\\ x\neq 0}} K(x,y)$ exists (and is equal to $(-y^2/2)$, which is really the natural definition of $K(0,y)$ in the one-dimensional case); however, the limit does not exist for $y\neq 0$ if $\dim E>1$. This explains a slight technical complication in the proof of Theorem 4.2, as compared to Khintchine's proof in the one-dimensional case (see [8], p. 76).

LEMMA 4.2. *Let ν be a finite non-negative measure on a separable Banach space E, such that $\nu(\{0\})=0$. Let $\Phi(y)=\exp\left[\int K(x,y)\nu(dx)\right]$, $y\in E'$. Then $\Phi(0)=1$ and Φ is positive definite and sequentially w^*-continuous.*

Proof. It is obvious that $\Phi(0)=1$; the fact that Φ is sequentially w^*-continuous on E' follows from Lebesgue's dominated convergence theorem.

To prove that Φ is positive definite, let $\{\lambda_n: n\in N\}$ be a sequence of non-negative measures of finite support disjoint from $\{0\}$ such that $\lambda_n\xrightarrow{w}\nu$. Each λ_n is of the form $\lambda_n=\sum_j a_{nj}\delta_{x_{nj}}$ for certain $a_{nj}\in R^+$, $x_{nj}\in E-\{0\}$. Let

$$z_n = -\sum_j\|x_{nj}\|^{-2}x_{nj},\qquad \beta_{nj}=a_{nj}\left(\frac{1+\|x_{nj}\|^2}{\|x_{nj}\|^2}\right).$$

Then

$$\left(\exp\left[\sum_j\beta_{nj}(\delta_{x_{nj}}-\delta_0)\right]*\delta_{z_n}\right)^{\wedge}(y)$$
$$=\exp\left[\sum_j a_{nj}\left(\exp(i\langle x_{nj},y\rangle)-1-\frac{i\langle x_{nj},y\rangle}{1+\|x_{nj}\|^2}\right)\frac{1+\|x_{nj}\|^2}{\|x_{nj}\|^2}\right]$$
$$=\exp\left[\int K(x,y)\lambda_n(dx)\right]\to\exp\left[\int K(x,y)\nu(dx)\right]$$

for each $y\in E'$, since $\nu(\{x: K(\cdot,y)$ is discontinuous at $x\})=0$. Since positive definiteness is preserved by passage to the limit, the result follows. $\blacksquare$

THEOREM 4.2. *Let E be a separable Banach space.*

(1) *If E is of cotype 2, then for every infinitely divisible probability measure μ on E there exist $x_0\in E$, a centered Gaussian measure γ on E and a finite non-negative measure ν on E satisfying $\nu(\{0\})=0$, such that*

(a) *There exists a probability measure ϱ on E with $\hat\varrho(y)=\exp\left[\int K(x,y)\times\times\nu(dx)\right]$, $y\in E'$.*

(b) $\mu=\delta_{x_0}*\gamma*\varrho$.

The triple (x_0,γ,ν) (subject to the stated conditions) is unique.

(2) *Conversely, suppose that for every infinitely divisible probability measure μ on E there exists a triple (x_0,γ,ν) with the stated properties such that* (a) *and* (b) *hold. Then E is of cotype 2.*

Proof. (1) We will use an auxiliary function $B: E\times E'\to C$, defined by

$B(x,y)=$

$$\begin{cases}\dfrac{1+\|x\|^2}{\|x\|^2}\left[\exp(i\langle x,y\rangle)-1-\dfrac{i\langle x,y\rangle}{1+\|x\|^2}+(1/2)\dfrac{|\langle x,y\rangle|^2}{(1+\|x\|^2)^2}\right], & x\neq 0,\ y\in E',\\ 0, & x=0,\ y\in E'.\end{cases}$$

B has the following properties: for $y \in E'$, $B(\cdot, y)$ is a bounded continuous function on E, $B(x, -y) = B(-x, y)$ $(x \in E, y \in E')$ and

(1) $$B(x, y) = K(x, y) + (1/2)\frac{|\langle x, y\rangle|^2}{\|x\|^2(1+\|x\|^2)}.$$

Let μ_n be the nth convolution root of μ, and let

$$f_n(y) = \int \frac{\langle x, y\rangle}{1+\|x\|^2} n\mu_n(dx), \quad g_n(y) = \int \frac{|\langle x, y\rangle|^2}{(1+\|x\|^2)^2} n\mu_n(dx) \quad (y \in E'),$$

$$\nu_n(dx) = \frac{\|x\|^2}{1+\|x\|^2} n\mu_n(dx).$$

Then one can write, for all $y \in E'$

(2) $$n[\hat{\mu}_n(y)-1] = \int [\exp(i\langle x, y\rangle)-1] n\mu_n(dx) = if_n(y) - (1/2)g_n(y) + \int B(x, y)\nu_n(dx).$$

Therefore

$$n[\hat{\bar{\mu}}_n(y)-1] = n[\hat{\mu}_n(-y)-1] = -if_n(y) - (1/2)g_n(y) + \int B(x, y)\bar{\nu}_n(dx)$$

and

(3) $$n[(\mu_n+\bar{\mu}_n)^\wedge(y)-2] = -g_n(y) + \int B(x, y)(\nu_n+\bar{\nu}_n)(dx).$$

Theorem 2.1 implies that $\exp[n(\hat{\mu}_n(y)-1)] \to \hat{\mu}(y)$ $(y \in E')$, hence $\exp[n((\mu_n+\bar{\mu}_n)^\wedge(y)-2)] \to (\mu*\bar{\mu})^\wedge(y)$. By Theorem 4.1, there exist a subsequence $\{\nu_{n_k}\}$ of $\{\nu_n\}$ and a finite non-negative measure ν' on E such that $\nu_{n_k} \xrightarrow{w} \nu'$. Consequently $\nu_{n_k}+\bar{\nu}_{n_k} \xrightarrow{w} \nu'+\bar{\nu}'$ and from (3) we obtain, letting $k \to \infty$:

(4) $$g(y) = \lim_k g_{n_k}(y) \quad \text{exists}$$

and

(5) $$(\mu*\bar{\mu})^\wedge(y) = \exp\Big[-g(y) + \int B(x, y)(\nu'+\bar{\nu}')(dx)\Big].$$

Let $\nu = \nu' - \nu'(\{0\})\delta_0$. One can write

(6) $$\int B(x, y)\nu'(dx) = \int K(x, y)\nu(dx) + (1/2)h(y),$$

where

$$h(y) = \int \frac{|\langle x, y\rangle|^2}{\|x\|^2(1+\|x\|^2)}\nu(dx).$$

Let $\psi = g - h$. We will prove now that $\psi(y) \geq 0$ for all $y \in E'$. Fix $y \in E'$ and let $u\colon E \to R^1$ be the function

$$u(x) = \begin{cases} \dfrac{|\langle x, y\rangle|^2}{\|x\|^2(1+\|x\|^2)}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then u is lower semicontinuous, bounded and

$$g_n(y) = \int u\,d\nu_n, \quad h(y) = \int u\,d\nu'.$$

Hence

$$h(y) = \int u\,d\nu' \leq \varliminf_k \int u\,d\nu_{n_k} = \lim_k g_{n_k}(y) = g(y)$$

since $\nu_{n_k} \xrightarrow{w} \nu'$.

We obtain now from (5) and (6)

(7) $$(\mu*\bar{\mu})^\wedge(y) = \exp\Big[-\psi(y) + \int K(x, y)(\nu+\bar{\nu})(dx)\Big].$$

Now $\exp[-\psi/2]$ is the characteristic functional of a centered Gaussian cylinder measure γ on E; also, by Lemma 4.2, there exists a cylinder measure ϱ on E such that $\hat{\varrho} = \exp[\int K(x, \cdot)\nu(dx)]$. Equation (7) implies that the cylinder measures $\mu*\bar{\mu}$, $\gamma*\gamma$ and $\varrho*\bar{\varrho}$ satisfy

$$\mu*\bar{\mu} = (\gamma*\gamma)*(\varrho*\bar{\varrho}).$$

Since $\mu*\bar{\mu}$ is a measure on E, it follows from Lemma 4.1 that $\gamma*\gamma$ and $\varrho*\bar{\varrho}$ are measures on E. But $\gamma*\gamma = \gamma(2^{-1/2}(\cdot))$; hence γ is a measure on E.

Let us return to (2). Passing to the limit along the subsequence $\{n_k\}$ we get, by a well-known elementary argument: there exists a linear form $f\colon E' \to R^1$ such that $f_{n_k}(y) \to f(y)$ for all $y \in E'$ and

$$\hat{\mu}(y) = \exp\Big[if(y) - (1/2)g(y) + \int B(x, y)\nu(dx)\Big]$$

or

(8) $$\hat{\mu}(y) = \exp\Big[if(y) - (1/2)\psi(y) + \int K(x, y)\nu(dx)\Big].$$

Now $\hat{\mu}$, $\exp[-\psi/2]$ and $\hat{\varrho}$ are sequentially w^*-continuous (the first two, because they are characteristic functionals; the third, by Lemma 4.2). It follows that f is a sequentially w^*-continuous linear form on E'; therefore there exists $x_0 \in E$ such that $f(y) = \langle x_0, y\rangle$, $y \in E'$ ([17], p. 150).

Equation (8) implies that the cylinder measures μ, δ_{x_0}, γ and ϱ satisfy

$$\mu = (\delta_{x_0}*\gamma)*\varrho.$$

Since μ and $(\delta_{x_0} * \gamma)$ are measures on E, so is ϱ by Lemma 4.1. This proves the existence statement of the theorem. We omit the proof of uniqueness; it is carried out as in ([16], p. 110). Let us remark that the proof of uniqueness is valid in any separable Banach space; the cotype 2 condition is not used.

(2) Let σ be the Poisson distribution with parameter 1, and let $\{\varrho_j : j \in N\}$ be a sequence of independent random variables with $\mathscr{L}(\varrho_j) = \sigma * \tilde{\sigma}$ $(j \in N)$. We first prove: $\sum_j \varrho_j x_j$ converges in $L^2(E)$ implies $\sum_j \|x_j\|^2 < \infty$. In fact, suppose $X = L^2(E) - \lim_n \sum_{j=1}^{n} \varrho_j x_j$, $\mu = \mathscr{L}(X)$, and let $\lambda = \sum_j (\delta_{x_j} + \delta_{(-x_j)})$. Then

$$\hat{\mu}(y) = \prod_{j=1}^{\infty} \mathscr{L}(\varrho_j x_j)^{\wedge}(y) = \exp\Big[\int [\exp(i\langle x, y\rangle) - 1]\lambda(dx)\Big]$$
$$= \exp\Big[\int K(x, y)\eta(dx)\Big],$$

where

$$\eta(dx) = \frac{\|x\|^2}{1+\|x\|^2}\lambda(dx).$$

Since μ is infinitely divisible, by hypothesis there exist a point $x_0 \in E$, a centered Gaussian measure γ on E and a finite non-negative measure ν with $\nu(\{0\}) = 0$ on E, such that

$$\hat{\mu}(y) = \exp\Big[i\langle x_0, y\rangle - (1/2)\psi(y) + \int K(x, y)\nu(dx)\Big].$$

The uniqueness of the representation implies $x_0 = 0$, $\psi = 0$, $\nu = \eta$. Therefore

$$2\sum_j \frac{\|x_j\|^2}{1+\|x_j\|^2} = \int \frac{\|x\|^2}{1+\|x\|^2}\lambda(dx) = \eta(E) = \nu(E) < \infty$$

and consequently $\sum_j \|x_j\|^2 < \infty$.

By an argument based on the closed graph theorem and analogous to that given in [10], p. 588, it follows that there exists a constant $C > 0$ such that for every finite set $\{x_1, \ldots, x_n\} \subset E$,

$$\sum_{j=1}^{n} \|x_j\|^2 \leqslant CE\Big\|\sum_{j=1}^{n} \varrho_j x_j\Big\|^2. \tag{9}$$

In order to complete the proof that is of cotype 2, it is enough to prove that c_0 is not finitely representable in E ([15], p. 49). For the proof that (a) $\Rightarrow$ (b) in Corollary 1.2, p. 67, of [15] then shows that there exists a constant $M > 0$ such that for every finite set $\{x_1, \ldots, x_n\} \subset E$,

$$E\Big\|\sum_{j=1}^{n} \varrho_j x_j\Big\|^2 \leqslant ME\Big\|\sum_{j=1}^{n} \varepsilon_j x_j\Big\|^2. \tag{10}$$

Inequalities (9) and (10) show that E is of cotype 2.

Suppose now that c_0 is finitely representable in E. By ([15], p. 50, Remarque 0.2), for every $n \in N$ there exists an n-uple $(x_1^{(n)}, \ldots, x_n^{(n)})$ of vectors in E with $\|x_j^{(n)}\| \geqslant 1/2$ such that for every sequence $(a_j)_{j \in N}$ in R^1,

$$\Big\|\sum_{j=1}^{n} a_j x_j^{(n)}\Big\| \leqslant \sup_{1 \leqslant j \leqslant n} |a_j|. \tag{11}$$

Putting $a_j = a_j \varrho_j$ with $a_j \in R^1$, taking expectations and combining (9) and (11), we obtain

$$(1/4)\sum_{j=1}^{n} |a_j|^2 \leqslant \sum_{j=1}^{n} \|a_j x_j^{(n)}\|^2 \leqslant CE\Big\|\sum_{j=1}^{n} a_j \varrho_j x_j^{(n)}\Big\|^2 \leqslant CE(\sup_{1 \leqslant j \leqslant n} |a_j \varrho_j|)^2.$$

Since $E(\sup_{1 \leqslant j \leqslant n} |a_j \varrho_j|)^2 \leqslant (E|\varrho_1|^p \sum_{j=1}^{n} |a_j|^p)^{2/p}$ (p any fixed real number > 2), a contradiction is obtained by appropriate choice of $(a_j)_{j \in N}$. ■

Note. A complete characterization of the Gaussian covariance ψ by continuity properties has been obtained by E. Giné and the first-named author. This will appear elsewhere.

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INSTITUTO VENEZOLANO DE INVESTIGACIONES CIENTIFICAS
CARACAS, VENEZUELA
and
UNIVERSIDAD NACIONAL DE LA PLATA,
LA PLATA, ARGENTINA

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Multiply self-decomposable probability measures on Banach spaces

by

NGUYEN VAN THU (Wrocław)

Abstract. In the present paper we define multiply self-decomposable probability measures on a Banach space and give a general form of their characteristic functionals.

1. Introduction. This paper is concerned with probability measures defined on Borel subsets of a real separable Banach space X. For a probability measure μ on X, the characteristic functional $\hat{\mu}$ is defined on the dual space X^* by the formula

$$\hat{\mu}(y) = \int_X e^{i\langle y,x\rangle}\,\mu(dx) \qquad (y \in X^*),$$

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing between X and X^*.

Recall that a probability measure μ on X is self-decomposable if for every number c in $(0,1)$ there exists a probability measure μ_c on X such that

$$\hat{\mu}(y) = \hat{\mu}(cy)\hat{\mu}_c(y) \qquad (y \in X^*). \tag{1.1}$$

The problem of describing the class of characteristic functionals of self-decomposable probability measures has been completely solved by Urbanik [8]. In the same paper the author has obtained a general form of characteristic functionals even for a larger class of probability measures, namely, for Levy's measures on X.

We now introduce a concept of multiply self-decomposable probability measures on Banach spaces. Let $L_1(X)$ denote the class of all self-decomposable probability measures on X. For every integer $n > 1$, let $L_n(X)$ denote the class of all measures μ in $L_1(X)$ such that for every number c in $(0,1)$ the component μ_c in (1.1) belongs to $L_{n-1}(X)$. Every measure in $L_n(X)$ will be called *n-times self-decomposable*. Further, every measure in $L_\infty(X) := \bigcap_{n=1}^{\infty} L_n(X)$ will be called *completely self-decomposable*. Since every stable measure on X is completely self-decomposable (Proposition 1.9, [3]), the set $L_\infty(X)$ is non-empty.