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# Weighted norm inequalities for Riesz potentials and fractional maximal functions in mixed norm Lebesgue spaces

by

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**Abstract.** We prove a norm inequality between the Riesz potential $R_\alpha f$ and the fractional maximal function $M_\alpha f$ in $\mathbf{R}^d$, $0<\alpha<d$. The norm is a weighted mixed Lebesgue norm $L^{p,q}_{w_0,w_1}(\mathbf{R}^m\times\mathbf{R}^n)$, where $0<p,q<\infty$ and $d=m+n$, with weights in $A_\infty$. Our proof makes extensive use of the concept of independence of weights in $A_p$. It is shown that many of the well known properties of Muckenhoupt weights are true in this more general form, among them the P. W. Jones Factorization Theorem for $A_p$-weights.

**0. Introduction.** Let $\mathbf{R}^d$ be the $d$-dimensional Euclidean space. The Riesz potential of order $\alpha$, $0<\alpha<d$, of a function $f$ is defined by

$$R_\alpha f(\xi)=\int|\xi-\eta|^{\alpha-d}f(\eta)\,d\eta.$$

For $0\leqslant\alpha<d$ we also define the fractional maximal operator $M_\alpha f(\xi)$ by

$$M_\alpha f(\xi)=\sup|Q|^{\alpha/d-1}\int_Q|f(\eta)|\,d\eta,$$

where the supremum is over all cubes $Q$ with sides parallel to the axes and containing $\xi$. When $\alpha=0$ we get the usual Hardy–Littlewood maximal operator.

Muckenhoupt and Wheeden [MW, Theorem 1] proved that if $0<p<\infty$ and $0<\alpha<d$ then

$$\int|R_\alpha f(\xi)|^p w(\xi)\,d\xi\leqslant C\int M_\alpha f(\xi)^p w(\xi)\,d\xi, \tag{0.1}$$

where $w$ is a weight in the Muckenhoupt class $A_\infty$ and the constant $C$ is independent of $f$. The purpose of this paper is to extend (0.1) to certain weighted Lebesgue spaces $L^{p,q}_{w_0,w_1}(\mathbf{R}^d)$ with mixed norm (see Definition 1.2). More precisely, we prove that

$$\|R_\alpha f\|_{p,q,w_0,w_1}\leqslant C\,\|M_\alpha f\|_{p,q,w_0,w_1}, \tag{0.2}$$

where $0<p,q<\infty$, $0<\alpha<d$ and $w_0$, $w_1$ are weights in the Muckenhoupt

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class $A_\infty$ (Theorem 2.1). The constant $C$ is here independent of $f$ and, moreover, independent of $w_0$ and $w_1$ in $A_\infty$.

Our proof of (0.2) makes extensive use of the notion of independence of weights in $A_p$, introduced by A. Torchinsky in [T, Ch. IX]. We observe that several well known properties of $A_p$-weights hold in this setting (Section 3). In particular, this is true for the Rubio de Francia factorization lemma for $A_p$-weights (Lemma 3.5). We prove (0.2) following an idea used by E. Hernández [H] for the Hardy–Littlewood maximal function.

Section 1 of this paper contains our notation and basic definitions. Our main result (Theorem 2.1) is stated in Section 2 and proved in Section 4. The lemmas necessary for the proof are given in Section 3.

**1. Notation and definitions.** We consider the $d$-dimensional Euclidean space $\mathbf{R}^d = \mathbf{R}^m \times \mathbf{R}^n$, with $d = m+n$. Points in $\mathbf{R}^d$ are denoted by $\xi = (x,s)$, $\eta = (y,t)$ and $\zeta = (z,u)$, where $x,y,z \in \mathbf{R}^m$ and $s,t,u \in \mathbf{R}^n$. We use $Q$ to denote cubes in $\mathbf{R}^d$ having its sides parallel to the coordinate axes. Next we define the Muckenhoupt classes $A_p$. A *weight* $w$ in $\mathbf{R}^d$ is a nonnegative and locally integrable function which is not identically zero.

Definition 1.1. Let $w$ be a weight in $\mathbf{R}^d$. Then

(i) $w$ belongs to $A_1$ if

$$|Q|^{-1} \int_Q w(\xi)\,d\xi \leqslant C \operatorname*{ess\,inf}_{\eta \in Q} w(\eta),$$

for every cube $Q$ and $C$ is independent of $Q$.

(ii) $w$ belongs to $A_p$, $1 < p < \infty$, if

$$\Big(|Q|^{-1} \int_Q w(\xi)\,d\xi\Big)\cdot\Big(|Q|^{-1} \int_Q w(\xi)^{-1/(p-1)}\,d\xi\Big)^{p-1} \leqslant C;$$

for every cube $Q$ and $C$ is independent of $Q$.

(iii) $w$ belongs to $A_\infty$ if for every $0 < \varepsilon < 1$ there is $0 < \delta < 1$ such that

$$|E| \leqslant \delta |Q| \quad \text{implies} \quad \int_E w(\xi)\,d\xi \leqslant \varepsilon \int_Q w(\xi)\,d\xi,$$

for every cube $Q$ and every measurable set $E \subset Q$.

The best constant $C$ in (i) and (ii) in Definition 1.1 is called the *$A_p$-constant* of $w$, $1 \leqslant p < \infty$. A statement which does not depend on the individual weight $w$ but only on the $A_p$-constant of $w$ is said to be *independent of $w$ in $A_p$*, $1 \leqslant p < \infty$. Analogously a statement is said to be *independent of $w$ in $A_\infty$* if it only depends on the function $\delta = \delta(\varepsilon)$ in the definition of $A_\infty$ and not on the particular weight $w$. For general properties of $A_p$-weights, see [T, Ch. IX].

In the following we also consider weights and Muckenhoupt classes in $\mathbf{R}^m$ and $\mathbf{R}^n$. These concepts are defined in complete analogy with Definition 1.1. We let $w(\xi)$, $w(x)$ and $w(s)$ denote weights in $\mathbf{R}^d$, $\mathbf{R}^m$ and $\mathbf{R}^n$ respectively.



Definition 1.2. Let $0 < p,q < \infty$ and let $w_0(x)$ and $w_1(s)$ be weights in $\mathbf{R}^m$ and $\mathbf{R}^n$ respectively. Define

$$\|f\|_{p,q,w_0,w_1} = \Big(\int\Big(\int |f(x,s)|^p w_0(x)\,dx\Big)^{q/p} w_1(s)\,ds\Big)^{1/q}.$$

We denote by $L^{p,q}_{w_0,w_1}(\mathbf{R}^d)$ the linear space of measurable functions $f$ in $\mathbf{R}^d$ with $\|f\|_{p,q,w_0,w_1} < \infty$.

When $p = q$ we write $\|f\|_{p,w_0,w_1}$ and $L^p_{w_0,w_1}(\mathbf{R}^d)$, and $w_0$, $w_1$ are dropped from the notation if $w_0 \equiv w_1 \equiv 1$. The standard reference on mixed norm Lebesgue spaces is [BP].

Various constants depending on parameters $\alpha$, $\beta$, ... are denoted by $c(\alpha, \beta, \ldots)$ and may have different values at different occurrences.

**2. The main result.** Our main result is the following weighted norm inequality for the Riesz potential and the fractional maximal function.

Theorem 2.1. *Let $0 < p,q < \infty$ and $0 < \alpha < d$ and let $w_0(x)$ and $w_1(s)$ be $A_\infty$-weights in $\mathbf{R}^m$ and $\mathbf{R}^n$ respectively. Then*

$$\|R_\alpha f\|_{p,q,w_0,w_1} \leqslant C\,\|M_\alpha f\|_{p,q,w_0,w_1}, \tag{2.1}$$

*with a constant $C$ independent of $f$ and independent of $w_0$ and $w_1$ in $A_\infty$.*

Theorem 2.1 was proved for $p = q$ and $A_\infty$-weights in $\mathbf{R}^d$ by Muckenhoupt and Wheeden [MW, Theorem 1], except that $C$ may be chosen independently of $w$ in $A_\infty$. However, this follows easily from their proof. In fact, it is the concept of independence of weights in the Muckenhoupt classes that plays a key role in the proof of Theorem 2.1 (Section 4). This will also be evident from the series of lemmas in Section 3.

**3. Some lemmas.** In this section we give the lemmas needed for the proof of Theorem 2.1. They are all known, except for the statement of independence in the respective Muckenhoupt classes, see [T, Ch. IX]. For the reader's convenience we state them in a form suitable for our purposes. We begin with a version of the P. W. Jones factorization theorem.

Lemma 3.1. *Let $1 < p < \infty$ and let $w \in A_p$. Then there are $w_0$ and $w_1$ in $A_1$ such that $w = w_0 w_1^{1-p}$, and the $A_1$-norms of $w_0$ and $w_1$ are independent of $w$ in $A_p$.*

Lemma 3.1 was proved in [J, Theorem, p. 511], except for the independence of $w$ in $A_p$. A careful examination of the more general result in [RdF, Theorem 2] gives the full statement of the lemma. We omit the details.

Lemma 3.2. *Let $1 \leqslant p \leqslant q \leqslant \infty$ and let $w \in A_p$. Then $w \in A_q$ independently of $w$ in $A_p$.*

Proof. When $q < \infty$ this follows from the definition of the classes $A_p$. For

$q = \infty$ we use Lemma 3.1 and the proof of the Reverse Hölder Inequality in [T, Ch. IX, Theorem 3.5 and Proposition 4.5].

Our next lemma is a partial converse of Lemma 3.2.

LEMMA 3.3. *Let $w$ be a weight in $A_\infty$. Then there exist $1 < \sigma < \infty$ and $0 < C < \infty$, independent of $w$ in $A_\infty$, such that $w \in A_\sigma$ and the $A_\sigma$-constant of $w$ is at most $C$.*

This result is mainly due to Muckenhoupt [M, p. 104]. It follows from his proof that the numbers $\sigma$ and $C$ can be chosen independently of $w$ in $A_\infty$.

LEMMA 3.4. *Let $1 \leqslant p \leqslant \infty$ and let $w_0(x)$ and $w_1(s)$ belong to $A_p$ in $\mathbf{R}^m$ and $\mathbf{R}^n$ respectively. Then $w(x,s) = w_0(x)w_1(s)$ belongs to $A_p$ in $\mathbf{R}^d$ independently of $w_0(x)$ and $w_1(s)$ in $A_p$.*

Lemma 3.4 follows immediately from the definition of the classes $A_p$ if $1 \leqslant p < \infty$. When $p = \infty$ it follows from Lemmas 3.2 and 3.3. We have now come to the key lemma for the proof of Theorem 2.1. It is [RdF, Lemma, p. 539], except for the statement about independence of $w$ in $A_p$, and can be proved along the same lines. The proof is therefore omitted.

LEMMA 3.5. *Let $w \in A_p$, $1 < p < \infty$. If $1 \leqslant r < \infty$ and $1/s = |1 - r/p|$ then for every nonnegative function $u \in L^s_w(\mathbf{R}^d)$ there exists $v \in L^s_w(\mathbf{R}^d)$ such that $u(\xi) \leqslant v(\xi)$ and*

$$vw \in A_r \quad \text{if } r \leqslant p,$$
$$v^{-1}w \in A_r \quad \text{if } p < r.$$

*Moreover, $\|v\|_{s,w} \leqslant c(d,r,p)\|u\|_{s,w}$ and the $A_r$-constants of $vw$ and $v^{-1}w$ are independent of $w$ in $A_p$.*

**4. Proof of Theorem 2.1.** Let $f$, $w_0$ and $w_1$ be as in the theorem and assume that $f \geqslant 0$. We begin with the case $0 < p < q < \infty$. Then $r = q/p > 1$ and we define $r'$ by $1/r + 1/r' = 1$. Duality now gives

$$\text{(4.1)} \qquad \|R_\alpha f\|^p_{p,q,w_0,w_1} = \Big(\int \Big(\int R_\alpha f(x,s)^p w_0(x)\,dx\Big)^{q/p} w_1(s)\,ds\Big)^{p/q} = \int \Big(\int R_\alpha f(x,s)^p w_0(x)\,dx\Big) w_1(s) g(s)\,ds,$$

for some nonnegative $g \in L^{r'}_{w_1}(\mathbf{R}^n)$ with norm one. By Lemma 3.3 there are $1 < \sigma < \infty$ and $0 < M < \infty$, independent of $w_1$ in $A_\infty$, such that $w_1(s) \in A_\sigma$ in $\mathbf{R}^n$ with $A_\sigma$-constant at most $M$. Define $p_1 = \lambda p$ and $q_1 = \lambda q$, where $\lambda > 1$ is such that $p_1 > \sigma$. Now we apply Lemma 3.5 to $w_1(s) \in A_{q_1}$ and

$$\frac{1}{r'} = 1 - \frac{p}{q} = 1 - \frac{p_1}{q_1}.$$

Hence there is $v(s) \in L^{r'}_{w_1}(\mathbf{R}^n)$ such that $g(s) \leqslant v(s)$, $\|v\|_{r',w_1} \leqslant c(n,p,q,\lambda)\|g\|_{r',w_1}$ and $w_1(s)v(s)$ belongs to $A_{p_1}$ independently of $w_1$ in $A_{q_1}$. Then (4.1) yields the estimate



$$\|R_\alpha f\|^p_{p,q,w_0,w_1} \leqslant \iint R_\alpha f(x,s)^p w_0(x) w_1(s) v(s)\,dx\,ds \leqslant C_1 \iint M_\alpha f(x,s)^p w_0(x) w_1(s) v(s)\,dx\,ds,$$

by [MW, Theorem 1] and Lemma 3.2 with $C_1$ independent of $w_0(x)$ and $w_1(s)$ in $A_\infty$. Hölder's inequality gives

$$\|R_\alpha f\|^p_{p,q,w_0,w_1} \leqslant C_1^{1/p}\|v\|^{1/p}_{r',w_1}\|M_\alpha f\|_{p,q,w_0,w_1} \leqslant c(n,p,q,\lambda)C_1^{1/p}\|M_\alpha f\|_{p,q,w_0,w_1},$$

which is the desired inequality. This proves the theorem in the case $p < q$.

For the rest of the proof we assume that $0 < q < p < \infty$. In this case we begin by considering the *right hand side* of (2.1). Define $r$ by $1/r = p/q - 1$. By duality there is a nonnegative function $g(s)$ such that

$$\text{(4.2)} \qquad \|M_\alpha f\|^p_{p,q,w_0,w_1} = \int \Big(\int M_\alpha f(x,s)^p w_0(x)\,dx\Big) g(s)^{-1} w_1(s)\,ds,$$

$$\text{(4.3)} \qquad \int g(s)^r w_1(s)\,ds = 1.$$

Choose $\sigma$, $1 < \sigma < \infty$, as in the first case of the proof and put $p_1 = \lambda p$ and $q_1 = \lambda q$, with $\lambda q > 1$ and $\lambda > 1$. Then

$$\frac{1}{r} = \frac{p}{q} - 1 = \frac{p_1}{q_1} - 1$$

and by Lemma 3.5 and (4.3) there exists $v(s)$ such that $g(s) \leqslant v(s)$, $\int v(s)^r w_1(s)\,ds \leqslant c(n,p,q,\lambda)$ and $v(s)^{-1}w_1(s) \in A_{p_1}$ independently of $w_1$ in $A_{q_1}$. Hölder's inequality and (4.2) now give

$$\begin{aligned}\|R_\alpha f\|_{p,q,w_0,w_1} &= \Big(\int \Big(\int R_\alpha f(x,s)^p w_0(x)\,dx\Big)^{p/q} v(s)^{-q/p} v(s)^{q/p} w_1(s)\,ds\Big)^{1/q}\\ &\leqslant \Big(\int \Big(\int R_\alpha f(x,s)^p w_0(x)\,dx\Big) v(s)^{-1} w_1(s)\,ds\Big)^{1/p} \Big(\int v(s)^r w_1(s)\,ds\Big)^{1/(rp)}\\ &\leqslant c(n,p,q,\lambda) C_2 \Big(\iint M_\alpha f(x,s)^p w_0(x) w_1(s) g(s)^{-1}\,dx\,ds\Big)^{1/p}\\ &= c(n,p,q,\lambda) C_2 \|M_\alpha f\|_{p,q,w_0,w_1},\end{aligned}$$

with $C_2$ independent of $w_0(x)$ and $w_1(s)$ in $A_\infty$. This completes the proof of Theorem 2.1.

Remark. It is an open question under what conditions on the weight $w$ Theorem 2.1 is true for the more general weighted mixed norms

$$\|f\|_{p,q,w} = \Big(\int \Big(\int |f(x,s)|^p w(x,s)\,dx\Big)^{q/p} ds\Big)^{1/q}.$$

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# Uncomplementability of the spaces of norm continuous functions in some spaces of "weakly" continuous functions

by

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**Abstract.** The paper deals with the complementability problem for the spaces of norm continuous functions (from compact spaces to Banach spaces) in some spaces of weaker-than-norm (e.g., weakly or weak*) continuous functions. The results obtained are fairly general and strongly support the conjecture that complementability can occur only if the spaces in question coincide.

**Introduction and main results.** Throughout, we let $K$ denote an *infinite* compact Hausdorff space, $X$ a Banach space, and $\tau$ a linear Hausdorff topology on $X$ which is *weaker* than the norm topology. Then $C(K;X)$, the Banach space of all (norm) continuous functions from $K$ into $X$, is obviously a closed linear subspace of $C(K;X,\tau)$, the Banach space of all $\tau$-continuous (norm) bounded functions from $K$ to $X$. (Of course, both spaces are endowed with the sup-norms.) This paper is concerned with the following

CONJECTURE. $C(K;X)$ is not complemented in $C(K;X,\tau)$ unless $C(K;X) = C(K;X,\tau)$.

As yet, we have been unable to verify this conjecture in general. Our main result in this direction is the following

THEOREM 1. *If $X$ contains a $\tau$-convergent sequence which is not norm convergent, then $C(K;X)$ is not complemented in $C(K;X,\tau)$.*

This, in particular, covers the two most important cases.

COROLLARY 1. *If $X$ does not have the Schur property, then $C(K;X)$ is uncomplemented in $C(K;X,w)$, the space of all weakly continuous functions from $K$ to $X$.*

COROLLARY 2. *If $X$ is infinite-dimensional, then $C(K;X^*)$ is uncomplemented in $C(K;X^*,w^*)$, the space of all weak* continuous functions from $K$ to $X^*$.*

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